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Semilocal Convergence Theorem for a Newton-like Method

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Abstract. The semilocal convergence of a third-order Newton-like method for solving nonlinear equations is considered. Under a weak condition (the so-called γ-condition) on the derivative of the nonlinear operator, we establish a new semilocal convergence theorem for the Newton-like method and also provide an error estimate. Some numerical examples show the applicability and efficiency of our result, in comparison to other semilocal convergence theorems.

AMS subject classifications: 65F10, 65F50, 65H10

Key words: Newton-like method, nonlinear equation, Newton-Kantorovich theorem, γ-condition, error estimate.

1. Introduction

Finding solutions of the system

$$F(x)=0 \tag{1.1}$$

is widely studied theoretically and in many applications, where F is a nonlinear operator often defined on an open convex Ω of a Banach space X with values in a Banach space Y. Iterative numerical methods are often used and the most famous is the Newton method where

$$x_{n+1}=x_n-F'(x_n)^{-1}F(x_n),\quad n\geqslant 0,\quad x_0\in\Omega, \tag{1.2}$$

which converges quadratically under reasonable conditions.

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Many researchers have focused on the convergence properties of the Newton method. One famous result is the well-known Kantorovich theorem [4], which guarantees the convergence of the sequence $\{x_n\}$ to a solution $\{x^*\}$ under mild conditions. Subsequently, many Kantorovich-type convergence theorems were obtained — cf. Refs. [1–3, 5, 6]. Thus in considering the computing complexity, Smale presented the α-theory at the 20th International Conference of Mathematicians (cf. Ref. [5]); and Wang [6] proposed a weak condition (the so-called γ—condition) on the derivative of the nonlinear operator, which improved Smale's theorem.

On the other hand, many Newton-like methods have also been proposed to improve the convergence [7–11, 18–28]. For example, Frontini presented a Newton-like method where

$$\begin{cases} y_n = x_n - f'(x_n)^{-1} f(x_n), \\ x_{n+1} = x_n - f'\left(\frac{x_n+y_n}{2}\right)^{-1} f(x_n), \end{cases} \tag{1.3}$$

where f is a real or complex function; and Weerakoon & Fernando presented a Newton-like method with third order where

$$\begin{cases} y_n = x_n - f'(x_n)^{-1} f(x_n), \\ x_{n+1} = x_n - 2\left[f'(x_n) + f'(y_n)\right]^{-1} f(x_n). \end{cases} \tag{1.4}$$

They and others [12–17] extended these methods to Banach space, and also established relevant Newton-Kantorovich theorems under the γ-condition.

Homeier [18] presented an iterative method

$$x_{n+1} = x_n - f(x_n) \sum_{r=1}^{m} \omega_r \frac{1}{f'(x_n - \tau_r f(x_n)/f'(x_n))},$$

where f is a real smooth function with a simple zero and ω_r, τ_r satisfy $\sum_{r=1}^{m} \omega_r = 1$, $\sum_{r=1}^{m} \omega_r \tau_r = 1/2$. He proved that the sequence $\{x_n\}$ generated by this method converges to the solution x^* cubically. For $m = 2$, $\omega_1 = \omega_2 = 1/2$ and $\tau_1 = 1 - \tau_2 = 0$ we have

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - f(x_n)/f'(x_n))}\right). \tag{1.5}$$

We generalise this Newton-like method for solving Eq. (1.1) in Banach space. If $F(x)$ is Fréchet differentiable in Ω and $F'(x)^{-1}$ exists, the iteration (1.5) can be reformulated as

$$\begin{cases} y_n = x_n - F'(x_n)^{-1} F(x_n), \\ x_{n+1} = x_n - \frac{1}{2}\left[F'(x_n)^{-1} + F'(y_n)^{-1}\right] F(x_n). \end{cases} \tag{1.6}$$

We establish a semilocal convergence theorem for this Newton-like method (1.6) under the γ-condition, using the majorising principle. In Section 2, we introduce the majorising sequences and give some of their properties in presenting our new semilocal convergence theorem where the derivative is assumed to satisfy the γ-condition. In Section 3, some numerical examples are discussed where the conditions of our theorem are satisfied but the hypotheses of other such theorems in Gutiérrez [19], Ezquerro [20], Argyros [21] and Wu [22] are not. Section 4 briefly summarises our results.

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2. Main Results

Our Newton-Kantorovich convergence theorem under the γ-condition is first considered. We define

$$g(t)=\beta-t+\frac{\gamma t^2}{1-\gamma t},\qquad t\in[0,1/\gamma)\,,$$

where β and γ are real numbers. This function has two real roots when $\alpha=\beta\gamma\leqslant 3-2\sqrt{2}$ — viz.

$$t^*=\left(1+\alpha-\sqrt{(1+\alpha)^2-8\alpha}\right)/(4\gamma),$$
$$t^{**}=\left(1+\alpha+\sqrt{(1+\alpha)^2-8\alpha}\right)/(4\gamma),$$

which satisfy

$$\beta\leqslant t^*\leqslant\left(1+\frac{1}{\sqrt{2}}\right)\beta\leqslant\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\gamma}\leqslant t^{**}\leqslant\frac{1}{2\gamma}\,.$$

Suppose $t_0=0$, and the sequences $\{s_n\}$ and $\{t_n\}$ are generated as follows:

$$\begin{cases} s_n=t_n-g'(t_n)^{-1}g(t_n),\\ t_{n+1}=t_n-\frac{1}{2}\big(g'(s_n)^{-1}+g'(t_n)^{-1}\big)g(t_n). \end{cases}\tag{2.1}$$

Definition 2.1 (cf. Ref. [15]). If $F(x)$ has a continuous second Fréchet derivative in the closed ball $\overline{S(x_0,\rho)}$ with $x_0\in\Omega$, and $F'(x_0)^{-1}$ exists and satisfies the inequalities

$$\left\|F'(x_0)^{-1}F(x_0)\right\|\leqslant\beta,$$
$$\left\|F'(x_0)^{-1}F''(x)\right\|\leqslant\frac{2\gamma}{(1-\gamma\|x-x_0\|)^3}=g''\big(\|x-x_0\|\big),\qquad x\in\Omega,$$
$$\|x-x_0\|\leqslant\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\gamma},$$

then $F(x)$ is said to satisfy the γ-condition.

Lemma 2.1. *Suppose the sequences $\{t_n\}$ and $\{s_n\}$ are generated by the formula* (2.1). *Then when $\alpha\leqslant 3-2\sqrt{2}$ the sequences $\{t_n\}$, $\{s_n\}$ increase and converge to t^*, and*

$$0\leqslant t_n\leqslant s_n\leqslant t_{n+1}<t^*.$$

Proof. Let us denote

$$U(x)=x-\frac{g(x)}{g'(x)},$$
$$V(x)=x-\frac{1}{2}\left(\frac{1}{g'(x)}+\frac{1}{g'(U(x))}\right)g(x)\,.$$

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When $t \in [0, t^*)$, we know that $g(t) > 0$, $g'(t) < 0$, $g''(t) > 0$, and $g''(t)$ increases monotonically with respect to t. Then we get

$$U'(x) = \frac{g(x)g''(x)}{g'(x)^2} > 0\,,$$

hence $U(x)$ increases strictly on $[0, t^*)$ such that $x < U(x) < U(t^*) = t^*$ when $x \in [0, t^*)$. From Eqs. (2.1), we have $s_n = U(t_n)$ and $t_0 = 0$, and the inequality $0 \leqslant t_n \leqslant s_n < t^*$ follows by induction. Since

$$\begin{aligned} V(x) - U(x) &= -\frac{1}{2}\big(g'(x)^{-1} + g'(U(x))^{-1}\big)g(x) + \frac{g(x)}{g'(x)} \\ &= \frac{1}{2}\big(g'(x)^{-1} - g'(U(x))^{-1}\big)g(x) \\ &= \frac{g(x)}{2g'(x)g'(U(x))}\big(g'(U(x)) - g'(x)\big) \\ &= \frac{g(x)}{2g'(x)g'(U(x))}g''(\xi)\big(U(x) - x\big) \geqslant 0\,, \quad \xi \in \big(x, U(x)\big)\,, \end{aligned}$$

we have $U(x) \leqslant V(x)$ such that $t_n \leqslant s_n = U(t_n) \leqslant V(t_n) = t_{n+1} \leqslant s_{n+1} < t^*$, and the lemma follows by induction. □

Lemma 2.2. *Suppose F satisfies the γ–condition, and $\|x - x_0\| < (1 - 1/\sqrt{2})/\gamma$. Then $F'(x)^{-1}$ exists, and*

1. $\|F'(x)^{-1}F'(x_0)\| \leqslant -g'(\|x - x_0\|)^{-1}$,
2. $\|\frac{1}{2}(F'(x)^{-1} + F'(y)^{-1})F'(x_0)\| \leqslant -\frac{1}{2}(g'(\|x - x_0\|)^{-1} + g'(\|y - x_0\|)^{-1})$.

Proof. From the γ-condition,

$$\begin{aligned} \big\|F'(x_0)^{-1}F'(x) - I\big\| &= \big\|F'(x_0)^{-1}\big(F'(x) - F'(x_0)\big)\big\| \\ &= \left\|F'(x_0)^{-1}\int_0^1 F''\big(x_0 + t(x - x_0)\big)\mathrm{d}t(x - x_0)\right\| \\ &\leqslant \int_0^1 g''\big(t\|x - x_0\|\big)dt\|x - x_0\| \\ &= g'\big(\|x - x_0\|\big) - g'(0) = g'\big(\|x - x_0\|\big) + 1 < 1\,. \end{aligned}$$

From the Banach lemma, $F'(x)^{-1}$ exists and

$$\big\|F'(x)^{-1}F'(x_0)\big\| \leqslant -g'\big(\|x - x_0\|\big)^{-1}\,. \tag{2.2}$$

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Similarly, we obtain

$$\begin{aligned}\left\|F'(x_0)^{-1}F'(y)-I\right\| &= \left\|F'(x_0)^{-1}\big(F'(y)-F'(x_0)\big)\right\| \\ &= \left\|F'(x_0)^{-1}\int_0^1 F''\big(x_0+t(y-x_0)\big)\mathrm{d}t(y-x_0)\right\| \\ &\leqslant \int_0^1 g''\big(t\|y-x_0\|\big)dt\|y-x_0\| \\ &= g'\big(\|y-x_0\|\big)-g'(0) = g'\big(\|y-x_0\|\big)+1<1\,.\end{aligned}$$

As before for inequality (2.2), we also have

$$\left\|F'(y)^{-1}F'(x_0)\right\| \leqslant -g'\big(\|y-x_0\|\big)^{-1}, \tag{2.3}$$

and from inequalities (2.2) and (2.3) we readily get

$$\begin{aligned}\left\|\frac{1}{2}\big(F'(x)^{-1}+F'(y)^{-1}\big)F'(x_0)\right\| &\leqslant \frac{1}{2}\Big(\left\|F'(x)^{-1}F'(x_0)\right\|+\left\|F'(y)^{-1}F'(x_0)\right\|\Big) \\ &\leqslant -\frac{1}{2}\Big(g'\big(\|x-x_0\|\big)^{-1}+g'\big(\|y-x_0\|\big)^{-1}\Big)\,,\end{aligned}$$

to complete the proof. □

Lemma 2.3. *Suppose X and Y are Banach spaces and Ω is an open convex subset of X, such that $F:\Omega\subset X\to Y$ has third-order continuous Fréchet derivatives. If the sequences $\{x_n\}$ and $\{y_n\}$ are generated by the formula* (1.6)*, then for all natural numbers n we have*

$$\begin{aligned}F(x_{n+1}) &= \int_0^1 F''\big(y_n+t(x_{n+1}-y_n)\big)(1-t)\mathrm{d}t(x_{n+1}-y_n)^2 \\ &\quad+\frac{1}{2}\int_0^1 F''\big(x_n+t(y_n-x_n)\big)\mathrm{d}t(y_n-x_n)^2 \\ &\quad-\int_0^1\int_0^1 F''\big(y_n-s(1-t)(y_n-x_n)\big)(1-t)\mathrm{d}s\mathrm{d}t(y_n-x_n)^2\,.\end{aligned}$$

Proof. From Eq. (1.5),

$$\begin{aligned}F(x_{n+1}) &= F(x_{n+1})-F(y_n)-F'(y_n)(x_{n+1}-y_n)+F(y_n)+F'(y_n)(x_{n+1}-y_n) \\ &= \int_0^1 F''\big(y_n+t(x_{n+1}-y_n)\big)(1-t)\mathrm{d}t(x_{n+1}-y_n)^2+F(y_n)+F'(y_n)(x_{n+1}-y_n)\,,\end{aligned}$$

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and

$$\begin{aligned}
&F(y_n)+F'(y_n)(x_{n+1}-y_n)\\
=&F(y_n)-F(x_n)-F'(y_n)(y_n-x_n)+F(x_n)+F'(y_n)(y_n-x_n)+F'(y_n)(x_{n+1}-y_n)\\
=&\int_0^1\big(F'(x_n+t(y_n-x_n))-F'(y_n)\big)\,\mathrm{d}t\,(y_n-x_n)-F'(x_n)(y_n-x_n)+F'(y_n)(y_n-x_n)\\
&+F'(y_n)(x_{n+1}-y_n)\\
=&-\int_0^1\int_0^1 F''\big(y_n-s(1-t)(y_n-x_n)\big)(1-t)\,\mathrm{d}s\mathrm{d}t\,(y_n-x_n)^2\\
&+\frac{1}{2}\big(F'(y_n)-F'(x_n)\big)(y_n-x_n)\\
=&-\int_0^1\int_0^1 F''\big(y_n-s(1-t)(y_n-x_n)\big)(1-t)\,\mathrm{d}s\mathrm{d}t\,(y_n-x_n)^2\\
&+\frac{1}{2}\int_0^1 F''\big(x_n+t(y_n-x_n)\big)\mathrm{d}t(y_n-x_n)^2\,,
\end{aligned}$$

which completes the proof. □

Theorem 2.1. *Suppose that X and Y are Banach spaces and $\Omega\subset X$ is an open convex subset of the Banach space X, such that $F:\Omega\subset X\to Y$ satisfies the γ-condition. When $\alpha\leqslant 3-2\sqrt{2}$ and $\overline{S(x_0,t^*)}\subset\Omega$, then the sequence $\{x_n\}$ generated by (1.6) is well defined with $x_n\in\overline{S(x_0,t^*)}$, and it converges to the unique solution x^* in $S(x_0,(1-1/\sqrt{2})/\gamma)$ and $\|x_n-x^*\|\leqslant t^*-t_n$.*

Proof. By induction, we can prove the following inequalities:

$$\begin{aligned}
&\|x_n-x_0\|\leqslant t_n\,,\\
&\big\|F'(x_n)^{-1}F'(x_0)\big\|\leqslant -F'(t_n)^{-1}\,,\\
&\|y_n-x_n\|\leqslant s_n-t_n\,,\\
&\|y_n-x_0\|\leqslant s_n\,,\\
&\left\|\frac{1}{2}\big(F'(x_n)^{-1}+F'(y_n)^{-1}\big)F'(x_0)\right\|\leqslant -\frac{1}{2}\big(g'(s_n)^{-1}+g'(t_n)^{-1}\big)\,,\\
&\|x_{n+1}-y_n\|\leqslant t_{n+1}-s_n\,.
\end{aligned}$$

Indeed, from Lemma 2.1 we have $t_n<t^*$ for any natural number n. When $n=0$, the above inequalities obviously hold, so suppose they hold when $n\geqslant 0$. Then

$$\|x_{n+1}-x_0\|\leqslant\|x_{n+1}-y_n\|+\|y_n-x_0\|\leqslant t_{n+1}-s_n+s_n=t_{n+1}\,,$$

and from Lemma 2.2 we get

$$\big\|F'(x_{n+1})^{-1}F'(x_0)\big\|\leqslant -g'\big(\|x_{n+1}-x_0\|\big)^{-1}\leqslant -g'(t_{n+1})^{-1}\,.$$

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From Lemma 2.1, $-g'(t)$ and $g''(t)$ are positive and increase monotonically on the interval $[0,(1-1/\sqrt{2})/\gamma)$, and from Lemma 2.3 we obtain

$$\begin{aligned}\left\|F'(x_0)^{-1}F(x_{n+1})\right\| \leqslant & \int_0^1 g''\big(s_n+t(t_{n+1}-s_n)\big)(1-t)\mathrm{d}t(t_{n+1}-s_n)^2\\ & +\frac{1}{2}\int_0^1 g''\big(t_n+t(s_n-t_n)\big)\mathrm{d}t(s_n-t_n)^2\\ & +\int_0^1\int_0^1 g''\big(s_n+s(1-t)(s_n-t_n)\big)(1-t)\mathrm{d}s\mathrm{d}t(s_n-t_n)^2\\ =&-g'(s_n)(t_{n+1}-s_n)+g(t_{n+1})-g(s_n)+\frac{1}{2}\big(g'(s_n)-g'(t_n)\big)(s_n-t_n)\\ &+g(s_n)-g(t_n)-g'(s_n)(s_n-t_n)\\ =&-g'(s_n)(t_{n+1}-t_n)+\frac{1}{2}\big(g'(s_n)-g'(t_n)\big)(s_n-t_n)+g(t_{n+1})-g(t_n)\\ =&g(t_{n+1}).\end{aligned}$$

Then

$$\begin{aligned}\|y_{n+1}-x_{n+1}\| =&\left\|-F'(x_{n+1})^{-1}F(x_{n+1})\right\|\\ \leqslant&\left\|F'(x_{n+1})^{-1}F'(x_0)\right\|\left\|F'(x_0)^{-1}F(x_{n+1})\right\|\\ \leqslant&-g'(t_{n+1})^{-1}g(t_{n+1})=s_{n+1}-t_{n+1},\end{aligned}$$

and from Lemma 2.2 we get

$$\left\|F'(y_{n+1})^{-1}F'(x_0)\right\|\leqslant -g'\big(\|y_{n+1}-x_0\|\big)^{-1}\leqslant -g'(s_{n+1})^{-1}$$

such that

$$\left\|\frac{1}{2}\big(F'(x_{n+1})^{-1}+F'(y_{n+1})^{-1}\big)F'(x_0)\right\|\leqslant-\frac{1}{2}\big(g'(s_{n+1})^{-1}+g'(t_{n+1})^{-1}\big).$$

Moreover, we have

$$\begin{aligned}\|x_{n+2}-y_{n+1}\|\leqslant&\frac{1}{2}\left\|F'(x_{n+1})^{-1}F(x_{n+1})-F'(y_{n+1})^{-1}F(x_{n+1})\right\|\\ =&\frac{1}{2}\left\|F'(y_{n+1})^{-1}\big(F'(y_{n+1})-F'(x_{n+1})\big)F'(x_{n+1})^{-1}F(x_{n+1})\right\|\\ =&\frac{1}{2}\Big\|F'(y_{n+1})^{-1}F'(x_0)F'(x_0)^{-1}\int_0^1 F''(x_{n+1}+t(y_{n+1}-x_{n+1}))\mathrm{d}t\\ &\times(y_{n+1}-x_{n+1})F'(x_{n+1})^{-1}F(x_{n+1})\Big\|\\ \leqslant&-\frac{1}{2}g'(s_{n+1})^{-1}\int_0^1 g''\big(t\|y_{n+1}-x_{n+1}\|\big)\mathrm{d}t\|y_{n+1}-x_{n+1}\|\big(-g'(t_{n+1})^{-1}g(t_{n+1})\big)\\ \leqslant&\frac{1}{2}\big(g'(t_{n+1})^{-1}-g'(s_{n+1})^{-1}\big)g(t_{n+1})=t_{n+2}-s_{n+1},\end{aligned}$$

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hence the sequence $\{x_n\}$ generated by (1.6) is well defined, $x_n \in \overline{S(x_0, t^*)}$, and x_n converges to the solution $x^* \in \overline{S(x_0, t^*)}$ of (1.1).

Now, we prove the uniqueness of the solution. Suppose y^* is also a solution of the nonlinear system (1.1) on $S(x_0, (1-1/\sqrt{2})/\gamma)$, and $g'(t) < 0$ when $t \in [0, (1-1/\sqrt{2})/\gamma)$. Then

$$\begin{aligned}
&\left\| F'(x_0)^{-1} \int_0^1 F'\big(x^* + t(y^* - x^*)\big)\mathrm{d}t - I \right\| \\
&= \left\| F'(x_0)^{-1} \int_0^1 \big[F'\big(x^* + t(y^* - x^*)\big) - F'(x_0)\big]\mathrm{d}t \right\| \\
&= \left\| F'(x_0)^{-1} \int_0^1 \int_0^1 F''\big(x_0 + s\big(x^* - x_0 + t(y^* - x^*)\big)\big)\mathrm{d}s\mathrm{d}t\big(x^* - x_0 + t(y^* - x^*)\big) \right\| \\
&\leqslant \int_0^1 \int_0^1 g''\big(s\big\|x^* - x_0 + t(y^* - x^*)\big\|\big)\mathrm{d}s\mathrm{d}t\big\|x^* - x_0 + t(y^* - x^*)\big\| \\
&= \int_0^1 g'\big(\big\|x^* - x_0 + t(y^* - x^*)\big\|\big)\mathrm{d}t - g'(0) \\
&= \int_0^1 g'\big(\big\|x^* - x_0 + t(y^* - x^*)\big\|\big)\mathrm{d}t + 1 < 1\,.
\end{aligned}$$

From the Banach lemma, the inverse of $\int_0^1 F'(x^* + t(y^* - x^*))\mathrm{d}t$ exists such that

$$F(y^*) - F(x^*) = \int_0^1 F'\big(x^* + t(y^* - x^*)\big)\mathrm{d}t(y^* - x^*),$$

hence we get $y^* = x^*$. This completes the proof of the uniqueness of the solution.

Furthermore, when $m > n$ we have

$$\|x_m - x_n\| \leqslant \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+1} - x_n\| \leqslant t_m - t_n\,, \tag{2.4}$$

and when $m \to \infty$ we have

$$\|x_n - x^*\| \leqslant t^* - t_n\,,$$

which completes the proof. □

3. Numerical Examples

Example 3.1. Let $X = [-1, 1]$, $x_0 = 0$ and the function $F : X \to Y$ be

$$F(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{5}{6}x + \frac{3}{20}\,. \tag{3.1}$$

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Then we have $\beta = 9/50$, $\gamma = 4/5$, $\alpha = \beta\gamma = 36/250 < 3-2\sqrt{2}$ and $t^* < 1$. Moreover,

$$\frac{2\gamma}{(1-\gamma|x-x_0|)^3} \geqslant 2\gamma \geqslant \left|F'(x_0)^{-1}F''(x)\right|,$$

so the sequence $\{x_n\}$ generated by (1.6) is well defined and $\{x_n\}$ converges to the solution x^* of Eq. (3.1). We chose

$$\beta = \frac{6}{5}, \quad \eta = \frac{9}{50}, \quad M = \frac{4}{3}, \quad N = 1, \quad K = \frac{19}{8}, \quad h = K\beta\eta > \frac{1}{2},$$

so the hypotheses of Theorem 1 in Ref. [22] are not satisfied; and also

$$a = \frac{9}{50} \quad \text{and} \quad b_2 = \frac{24}{15}$$

such that $ab_2 = 0.288 > (14-6\sqrt{6})/25$, so the hypotheses of the theorem in Ref. [19] are not satisfied.

Example 3.2. Let $X = Y = R$, $x_0 = 0$ and the function $F : X \to Y$ be

$$F(x) = \cos x - 5x + 7.$$

For $\beta = 8/5$, $\gamma = 1/10$, $\alpha = \beta\gamma = 0.16 < 3-2\sqrt{2}$ and

$$\left\|F'(x_0)^{-1}F''(x)\right\| \leqslant \left\|F'(x_0)^{-1}F''(x_0)\right\| \leqslant \frac{2\gamma}{(1-\gamma\|x-x_0\|)^3},$$
$$\|x-x_0\| \leqslant \left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\gamma},$$

the hypotheses of Theorem 2.1 are satisfied, so the sequence $\{x_n\}$ generated by (1.6) is well defined and converges to the solution of the equation. We chose

$$a = \frac{8}{5} \quad \text{and} \quad b_2 = \frac{1}{5},$$

such that $ab_2 = 8/25 = 0.32 > (14-\sqrt{6})/25$, so the hypotheses of the theorem in Ref. [19] are not satisfied and therefore cannot guarantee the convergence of the sequence $\{x_n\}$; and also

$$\beta = \frac{1}{5} \quad \text{and} \quad \eta = \frac{8}{5}, \quad L = 1,$$

such that $\alpha_0 = L\beta\eta > (4-\sqrt{6})/5$, so the hypotheses of the theorem in Ref. [20] are not satisfied. Similarly, with

$$L_0 = \frac{1}{5}, \quad K = \frac{1}{5}, \quad \eta = \frac{8}{5}, \quad \eta_0 = \frac{1}{5}, \quad M = \frac{6}{5},$$

and $(M + 4L_0 + \sqrt{M^2 + 8L_0M})\eta > 4$, the hypotheses of the theorem in Ref. [21] are not satisfied.

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Example 3.3. Let $X = Y = R^2$ and $F : X \to Y$ be

$$F(x) = \begin{pmatrix} g(x) \\ h(x) \end{pmatrix} = \begin{pmatrix} uv - 1 \\ uv + u - 2v \end{pmatrix}, \quad \forall\ x = (u, v)^T \in X\,.$$

We have

$$F'(x) = \begin{pmatrix} v & u \\ v+1 & u-2 \end{pmatrix},$$

$$F'(x)^{-1} = -\frac{1}{u+2v}\begin{pmatrix} u-2 & -u \\ -v-1 & v \end{pmatrix},$$

$$F''(x) = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ \hline 0 & 1 \\ 1 & 0 \end{array}\right),$$

such that

$$F'(x)^{-1}F(x) = -\frac{1}{u+2v}\begin{pmatrix} -u^2 - u + 2 \\ -2v^2 + v + 1 \end{pmatrix}.$$

We chose $x_0 = (u_0, v_0)^T = (1.1, 1.1)^T$, and took the norm $\|x\| = \max\{|u|, |v|\}$ for $x = (u, v)$ in R^2 and the norm $\|A\| = \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\}$ for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then we define the norm of a bilinear operator B on R^2 by

$$\|B\| = \sup_{\|u\|=1} \max_i \sum_{j=1}^{2} \left| \sum_{k=1}^{2} b_i^{jk} u_k \right|,$$

where $u = (u_1, u_2)^T$ and

$$B = \left(\begin{array}{cc} b_1^{11} & b_1^{12} \\ b_1^{21} & b_1^{22} \\ \hline b_2^{11} & b_2^{12} \\ b_2^{21} & b_2^{22} \end{array}\right).$$

We obtain $\beta = 16/165 \approx 0.09696$, $\gamma = 32/33 \approx 0.9696$, and

$$\left\|F'(x_0)^{-1}F(x_0)\right\| = \left\|\left(\frac{3.1}{22}, \frac{3.2}{33}\right)\right\| = 0.09696 = \beta\,,$$

$$\left\|F'(x_0)^{-1}F''(x)\right\| = \left\|F'(x_0)^{-1}F''(x_0)\right\| = 2\gamma < \frac{2\gamma}{(1-\gamma\|x - x_0\|)^3}\,,$$

hence $\alpha = \beta\gamma < 3 - 2\sqrt{2}$ and the hypotheses of Theorem 2.1 are satisfied. From our Newton-like method, we obtain the solution after two steps, as the results in Table 1 show (x^* is the exact solution of the equation). We chose another initial point $x_0 = (u_0, v_0)^T = (1.2, 1.2)^T$, to obtain $\beta = 0.1889$, $\gamma = 0.8333$, and $\alpha = \beta\gamma < 3 - 2\sqrt{2}$. The results are shown in Table 2. Comparing the results in Table 1 and Table 2, the numerical stability of our method is evident.

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Table 1: Results for Example 3.3 using our approach with the $x_0 = (u_0, v_0)^T = (1.1, 1.1)^T$.

n	$x_n^{(1)}$	$x_n^{(2)}$	$y_n^{(1)}$	$y_n^{(2)}$	$\|x_n - x^*\|$
0	1.1000	1.1000	1.0060	1.0030	0.1000
1	1+1.1219e-05	1+6.0972e-06	1+4.9567e-11	1+2.4784e-11	1.2194e-5
2	1.0000	1.0000			0.0000

Table 2: Results for Example 3.3 using our approach with the initial point $x_0 = (u_0, v_0)^T = (1.2, 1.2)^T$.

n	$x_n^{(1)}$	$x_n^{(2)}$	$y_n^{(1)}$	$y_n^{(2)}$	$\|x_n - x^*\|$
0	1.2000	1.2000	1.0222	1.0111	0.2000
1	1+1.6221e-04	1+8.1103e-05	1+8.7693e-9	1+4.3847e-9	1.6221e-4
2	1.0000	1.0000			0.0000

Example 3.4. Consider the two-dimensional nonlinear convection-diffusion equations

$$\begin{cases} -(u_{xx} + u_{yy}) + q_1 u_x + q_2 u_y = -e^u, & \text{for } (x, y) \in \Omega, \\ u(x, y) = 0, & \text{for } (x, y) \in \partial\Omega, \end{cases}$$

where $\Omega = (0, 1) \times (0, 1)$ has boundary $\partial\Omega$ and q_1, q_2 are positive constants that measure the magnitude of the convective terms. Applying the centred finite difference scheme on an equidistant discretisation grid with the stepsize $h = 1/(N + 1)$, we obtain the system of nonlinear equations $F(x) = 0$ where

$$F(x) = Mx + h^2\Phi(x) = 0,$$

N is a prescribed positive integer,

$$M = T_x \otimes I + I \otimes T_y \quad \text{and} \quad \Phi(x) = (e^{x_1}, e^{x_2}, \cdots, e^{x_n})^T$$

involve the tridiagonal matrices

$$T_x = \text{tridiag}(-1, -Re_1, 2, -1 + Re_1),$$
$$T_y = \text{tridiag}(-1, -Re_2, 2, -1 + Re_2),$$

$Re_j = (1/2)q_j h$, $j = 1, 2$ is such that $Re = \max\{Re_1, Re_2\}$ is the "mesh Reynolds number", $\otimes$ is the Kronecker product symbol and $n = N \times N$. The positive constant $q_1 = 600$, $N = 30$, $q_2 = 1/h$, the initial guess $x_0 = 0$, and the stopping criterion for the Newton-like iteration is

$$\frac{\|F(x_k)\|_2}{\|F(x_0)\|_2} \le 10^{-14}.$$

We readily compute

$$\beta = \left\|F'(x_0)^{-1}F(x_0)\right\| = 0.1049,$$
$$\gamma = \frac{1}{2}\left\|F'(x_0)^{-1}F''(x_0)\right\| \le \frac{1}{2}\left\|F'(x_0)^{-1}\right\|\left\|F''(x_0)\right\| = 0.5452,$$

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Table3:

n	$\|F(x_k)\|_2/\|F(x_0)\|_2$
0	1.0000
1	4.4167e-10
2	1.4251e-15

Table 4:

n	$\|F(x_k)\|_2/\|F(x_0)\|_2$
0	1.0000
1	7.9425e-8
2	4.0292e-17

hence $\alpha = \beta\gamma < 3-2\sqrt{2}$ such that the γ-condition is satisfied. We chose another initial point $x_0 = diag(0.1)$, to obtain $\beta = 0.1049$, $\gamma = 0.5452$, and $\alpha = \beta\gamma < 3-2\sqrt{2}$. According to the results in Section 2, the sequence $\{x_k\}$ generated by the third-order Newton-like method therefore converges to $x^* = 0$. The results are displayed in Tables 3 and 4.

4. Conclusion

We have considered the semilocal convergence of a third-order Newton-like method, and proved a relevant Newton-Kantorovich convergence theorem under the γ-condition in Banach space. An error estimate has been provided, and some examples demonstrate the applicability and efficiency of our theorem.

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