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Cohomological annihilators

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1. *Introduction*

The local cohomology theory introduced by Grothendieck (1) is a useful tool for attacking problems in commutative algebra and algebraic geometry. Let A denote a local ring with its unique maximal ideal $\mathfrak{m}$. For an ideal $I \subset A$ and a finitely generated A-module M we consider the local cohomology modules $H^i_I(M)$, $i \in \mathbb{Z}$, of M with respect to I, see Grothendieck (1) for the definition. In particular, the vanishing resp. non-vanishing of the local cohomology modules is of a special interest. For more subtle considerations it is necessary to study the cohomological annihilators, i.e. $\mathfrak{a}^i_I(M) := \operatorname{Ann}_A H^i_I(M)$, $i \in \mathbb{Z}$. In the case of the maximal ideal $I = \mathfrak{m}$ these ideals were used by Roberts (6) to prove the 'New Intersection Theorem' for local rings of prime characteristic. Furthermore, we used this notion (7) in order to show the amiability of local rings possessing a dualizing complex. Note that the amiability of a system of parameters is the key step for Hochster's construction of big Cohen–Macaulay modules for local rings of prime characteristic, see Hochster (3) and (4).

In the present paper we want to continue with the study of the cohomological annihilators. In § 2 we start with a variation of Roberts' theorem (6). We show a relation between the cohomological annihilators of the modules of a complex and the local hypercohomology of this complex, see Theorem 1. Next we give a slight sharpening of the 'lemme d'acyclicité' of Peskine and Szpiro (5), which shows the usefulness of the cohomological annihilators in order to prove the exactness of a complex. In the last chapter we show a relation of the local cohomological annihilators $\mathfrak{a}^i_I(M)$ to certain quotient modules of M, see Theorem 3 for the precise statement. Theorem 3 has a nice application concerning the characterization of regular sequences in the Rees algebra, see (8). This seems to be a good justification to study the cohomological annihilators. We conclude with a result concerning the amiability of local rings, see § 4,

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for the definition. We show that A is amiable if and only if there is an amiable system of parameters, see Corollary 3. It is a noteworthy fact that we do not use the existence of a dualizing complex of A.

2. *A variation of Roberts' theorem*

At the beginning we recall some basic facts on Koszul complexes and local cohomology modules. Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ be a system of elements of the local Noetherian ring A. Let $I \subset A$ denote an ideal such that $\operatorname{Rad} I = \operatorname{Rad} \mathbf{x}A$. For an integer $t \geqslant 1$ we denote by $K^{\cdot}(\mathbf{x}^t; A)$ the Koszul complex of A with respect to $\mathbf{x}^t = \{x_1^t, \ldots, x_n^t\}$. By virtue of the natural maps

$$K^{\cdot}(\mathbf{x}^t; A) \to K^{\cdot}(\mathbf{x}^s; A) \quad \text{for} \quad s \geqslant t$$

the Koszul complexes $K^{\cdot}(\mathbf{x}^t; A)$, $t \geqslant 1$, form a direct system of complexes. Its direct limit is denoted by $K^{\cdot}_{\mathbf{x}}(A) := \varinjlim_t K^{\cdot}(\mathbf{x}^t; A)$. Therefore, we have

$$K^m_{\mathbf{x}}(A) \cong \begin{cases} \bigoplus_{1 \leqslant i_1 < \ldots < i_m \leqslant n} A_{(x_{i_1} \ldots x_{i_m})} & \text{for} \quad 0 \leqslant m \leqslant n \quad \text{and} \\ 0 & \text{otherwise,} \end{cases}$$

where A_x is the localization of A with respect to the multiplicative closed set $\{x^r \mid r \geqslant 0\}$ for $x \in A$. In particular, we note that $K^0_{\mathbf{x}}(A) \cong A$. For a bounded below complex $M^{\cdot}$ of A-modules we define

$$K^{\cdot}_{\mathbf{x}}(M^{\cdot}) := K^{\cdot}_{\mathbf{x}}(A) \otimes_A M^{\cdot}.$$

For a finitely generated A-module M it follows that

$$H^i(K^{\cdot}_{\mathbf{x}}(M)) \cong H^i_I(M) \quad (i \in \mathbb{Z}),$$

see Grothendieck (1), § 2. Let $\mathbf{R}\Gamma_I(\cdot)$ denote the right derived functor of the functor of global sections with support contained in $V(I)$ in the sense of Hartshorne (2). Then it follows that

$$H^i(K^{\cdot}_{\mathbf{x}}(M^{\cdot})) \cong H^i_I(M^{\cdot}) \quad (i \in \mathbb{Z}),$$

for a bounded below complex $M^{\cdot}$ of A-modules whose cohomology modules are finitely generated. Here we define $H^i_I(M^{\cdot}) := H^i(\mathbf{R}\Gamma_I(M^{\cdot}))$, $i \in \mathbb{Z}$. Now we consider a variation of Roberts' theorem.

THEOREM 1. *If $M^{\cdot}: 0 \to M^0 \to M^1 \to \ldots \to M^s \to 0$ is a bounded complex of finitely generated A-modules, then*

$$\mathfrak{a}^0_I(M^n)\, \mathfrak{a}^1_I(M^{n-1}) \ldots \mathfrak{a}^n_I(M^0) \cdot H^n_I(M^{\cdot}) = 0$$

for an integer n, $0 \leqslant n \leqslant s$.

Proof. There is a spectral sequence for computing the hypercocohomology of $K^{\cdot}_{\mathbf{x}}(M^{\cdot})$ with the E_1-term

$$H^p(K^{\cdot}_{\mathbf{x}}(A) \otimes_A M^q) \Rightarrow H^{p+q}_I(M^{\cdot}).$$

By virtue of the above considerations we get

$$H^p_I(M^q) \cong H^p(K^{\cdot}_{\mathbf{x}}(A) \otimes_A M^q).$$

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Therefore $\mathfrak{a}_I^p(M^q)$ annihilates E_1^{pq} and all the (p,q)-terms in any subsequent stage of the spectral sequence. Hence, $H_I^n(M^\cdot)$ possesses a filtration

$$H_I^n(M^\cdot) = F^0 \supset F^1 \supset \ldots \supset F^n \supset F^{n+1} = 0,$$

such that F^i/F^{i+1} is annihilated by $\mathfrak{a}_I^i(M^{n-i})$. But this implies that

$$\mathfrak{a}_I^0(M^n)\,\mathfrak{a}_I^1(M^{n-1}) \ldots \mathfrak{a}_I^n(M^0),$$

annihilates the cohomology module $H_I^n(M^\cdot)$.∎

Of particular interest is the case where the cohomology modules of $M^\cdot$ have a support contained in $V(I)$.

COROLLARY 1. *Let $M^\cdot$ be as before. Suppose that the cohomology modules $H^i(M^\cdot)$, $0 \leqslant i \leqslant s$, have a support contained in $V(I)$. Then it follows*

$$\mathfrak{a}_I^0(M^n)\,\mathfrak{a}_I^1(M^{n-1}) \ldots \mathfrak{a}_I^n(M^0) \cdot H^n(M^\cdot) = 0$$

for an integer $0 \leqslant n \leqslant s$.

Proof. We consider the spectral sequence for computing the hypercohomology of $K_{\mathfrak{x}}^\cdot(M^\cdot)$ with the E_1-term

$$H^q(K_I^p(A) \otimes_A M^\cdot) \Rightarrow H_I^{p+q}(M^\cdot).$$

Because $K_{\mathfrak{x}}^p(A)$ is an A-flat module we get

$$H^q(K_{\mathfrak{x}}^p(A) \otimes_A M^\cdot) \cong K_{\mathfrak{x}}^p(A) \otimes_A H^q(M^\cdot).$$

According to the structure of $K_{\mathfrak{x}}^p(A)$ our additional assumption on the cohomology modules yields

$$K_I^p(A) \otimes_A H^q(M^\cdot) = 0 \quad \text{for all} \quad p \neq 0$$

and

$$K_I^0(A) \otimes_A H^q(M^\cdot) \cong H^q(M^\cdot).$$

So the spectral sequence degenerates, i.e. $H_I^n(M^\cdot) \cong H^n(M^\cdot)$. Therefore, Theorem 1 proves the statement.∎

In the case $x_1, \ldots, x_n$ is a system of parameters of A we have $\operatorname{Rad} \mathfrak{x}A = \mathfrak{m}$. So Corollary 1 gives us a sharpening of Roberts' Theorem (6).

3. *Acyclicity of complexes*

It is easily seen that Theorem 1 resp. Corollary 1 has to do with the acyclicity of complexes. In particular, suppose $H_I^j(M^j) = 0$ for all $i+j < k$ in Corollary 1. Then it follows $H^n(M^\cdot) = 0$ for all $n < k$. Pursuing this point of view further we shall show an improvement of the 'lemme d'acyclicité' of Peskine and Szpiro (5). To this end we write $\operatorname{depth}_I M$ for the largest number n such that there exist elements $x_1, \ldots, x_n \in I$ which form an M-regular sequence. For a finitely generated A-module $M \neq 0$ it is uniquely determined. According to Grothendieck (1), § 3, we have $\operatorname{depth}_I M = \min\{i \in \mathbb{Z} \mid H_I^i(M) \neq 0\}$.

THEOREM 2. *Let $M^\cdot: 0 \to M^0 \to M^1 \to \ldots \to M^s$ be a complex of finitely generated A-modules. Suppose the following conditions are satisfied:*

(*a*) $\operatorname{depth}_I M^i \geqslant s-i$ *for* $0 \leqslant i < s$,

(*b*) $\operatorname{depth}_I H^i(M^\cdot) \leqslant 1$ *or* $H^i(M^\cdot) = 0$ *for* $0 \leqslant i < s-1$, *and*

(*c*) $\operatorname{depth}_I H^{s-1}(M^\cdot) = 0$ *or* $H^{s-1}(M^\cdot) = 0$.

Then $M^\cdot$ is acyclic, i.e. $H^i(M^\cdot) = 0$ for $i < s$.

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Proof. The first assumption implies $H_I^j(M^j) = 0$ for all $i+j < s$. Therefore, Theorem 1 yields $H_I^i(M^\cdot) = 0$ for $i < s$. For computing $H_I^i(M^\cdot)$ there is the following spectral sequence

$$E_2^{pq} = H_I^p(H^q(M^\cdot)) \Rightarrow E^n = H_I^n(M^\cdot).$$

Now we want to show our statement, i.e. $H^n(M^\cdot) = 0$ for $n < s$, by an induction on n. Suppose $H^0(M^\cdot) \neq 0$ and $\operatorname{depth}_I H^0(M^\cdot) = 0$. Then we get $E_2^{0,0} \neq 0$. The subsequent stages of the spectral sequence yield

$$0 = E_r^{-r,r-1} \to E_r^{0,0} \to E_r^{r,-r+1} = 0,$$

because $r \geqslant 2$. That is, $E_\infty^{0,0} \cong E_2^{0,0} \neq 0$, which contradicts $H_I^0(M^\cdot) = 0$. In the case $s \geqslant 2$ an analogous discussion leads $\operatorname{depth}_I H^0(M^\cdot) = 1$ to a contradiction. Now we assume that $H^i(M^\cdot) = 0$ for all $i < n$ and $n < s$. We show $H^n(M^\cdot) = 0$. Suppose we have $H^n(M^\cdot) \neq 0$ and $\operatorname{depth}_I H^n(M^\cdot) = 0$. Then it follows $E_2^{0,n} \neq 0$ and in the same manner as before $E_\infty^{0,n} \cong E_2^{0,n} \neq 0$ which contradicts $H_I^n(M^\cdot) = 0$. If $\operatorname{depth}_I H^n(M^\cdot) = 1$ and $n \leqslant s-2$, the same argument and $H_I^{n+1}(M^\cdot) = 0$ leads to a contradiction. So the complex is acyclic. ∎

One should ask if (*b*) in Theorem 2 is the best possible condition for the acyclicity of $M^\cdot$. The third condition cannot be weakened to $\operatorname{depth}_I H^{s-1}(M^\cdot) \leqslant 1$, as can be seen by simple examples.

COROLLARY 2. *Let* $F^\cdot : 0 \to F^0 \to F^1 \to \ldots \to F^s$ *be a complex of finitely generated free* A*-modules. Suppose* $\operatorname{depth} A \geqslant s$, $\dim_A H^i(F^\cdot) \leqslant 1$, $0 \leqslant i < s-1$, *and* $\dim_A H^{s-1}(F^\cdot) \leqslant 0$, *then* $F^\cdot$ *is acyclic.*

We want to note that there is a very simple proof of Theorem 3 which does not use spectral sequences and local cohomology. This 'elementary' proof can be given simply by counting depth, as was done by Peskine and Szpiro in the case of their 'lemme d'acyclicité' in (5). We prefer this proof because it is closely related to our considerations on cohomological annihilators.

4. *An application to the Koszul cohomology*

As above let A denote a local Noetherian ring. Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ be a system of elements of A and $I \subset A$ an ideal such that $\operatorname{Rad} I = \operatorname{Rad} \mathbf{x}A$. Furthermore let M be a finitely generated A-module. We consider the following quotients

$$(x_1^t, \ldots, x_{i-1}^t)\, M : x_i^t / (x_1^t, \ldots, x_{i-1}^t)\, M,$$

$i = 1, \ldots, n$ and $t \geqslant 1$. We define $\mathfrak{r}_{\mathbf{x}}(M)$ the intersection of the annihilators of all these modules for $i = 1, \ldots, n$ and all $t \geqslant 1$. In the following we want to compare $\mathfrak{r}_{\mathbf{x}}(M)$ with the cohomological annihilators $\mathfrak{a}_I^i(M)$, $i = 0, 1, \ldots, n-1$.

THEOREM 3. *Let* $A, M, \mathbf{x}, I$ *be as before. Then we have the following inclusions*:

(*a*) $\mathfrak{a}_I^0(M)\, \mathfrak{a}_I^1(M) \ldots \mathfrak{a}_I^{n-1}(M) \subseteq \mathfrak{r}_{\mathbf{x}}(M)$, *and*

(*b*) $(\mathfrak{r}_{\mathbf{x}}(M))^k \subseteq \mathfrak{a}_I^0(M) \cap \mathfrak{a}_I^1(M) \cap \ldots \cap \mathfrak{a}_I^{n-1}(M)$ *for* $k = \binom{n}{[n/2]}$.

Proof. Let $K^\cdot(\mathbf{x}^t; M)$ denote the Koszul complex of M with respect to $\mathbf{x}^t = \{x_1^t, \ldots, x_n^t\}$. Because the cohomology modules $H^i(\mathbf{x}^t; M)$, $i \in \mathbb{Z}$, of the Koszul complex $K^\cdot(\mathbf{x}^t; M)$

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are annihilated by $\mathbf{x}^{\mathbf{t}}A$ it follows that the support of $H^i(\mathbf{x}^{\mathbf{t}}; M)$, $i \in \mathbb{Z}$, is contained in $V(I)$. Note that $\operatorname{Rad} I = \operatorname{Rad} \mathbf{x}A$. According to Corollary 1 we get

$$\mathfrak{a}_I^0(M)\,\mathfrak{a}_I^1(M) \dots \mathfrak{a}_I^{n-1}(M).\,H^{n-1}(\mathbf{x}^{\mathbf{t}}; M) = 0,$$

for any n-tuple $\mathbf{t} = (t_1, \dots, t_n)$ of positive integers. Let us abbreviate

$$\mathbf{x}'^{\mathbf{t}'} := \{x_1^{t_1}, \dots, x_{n-1}^{t_{n-1}}\}.$$

For the cohomology modules $H^{n-1}(\mathbf{x}^{\mathbf{t}}; M)$ there is the following short exact sequence

$$0 \to H^{n-2}(\mathbf{x}'^{\mathbf{t}'}; M) \otimes_A A/x_n^{t_n} \to H^{n-1}(\mathbf{x}^{\mathbf{t}}; M) \to (\mathbf{x}'^{\mathbf{t}'}M : x_n^{t_n})/\mathbf{x}'^{\mathbf{t}'}M \to 0.$$

First of all it shows

$$\prod_{j=0}^{n-1} \mathfrak{a}_I^j(M)\,((x_1^t, \dots, x_{n-1}^t)\,M : x_n^t/(x_1^t, \dots, x_{n-1}^t)\,M) = 0,$$

for an integer $t \geqslant 1$. Furthermore, it follows

$$\prod_{j=0}^{n-1} \mathfrak{a}_I^j(M)\,H^{n-2}(\mathbf{x}'^{\mathbf{t}'}; M) \subseteq \bigcap_{t_n \geqslant 1} x_n^{t_n} H^{n-2}(\mathbf{x}'^{\mathbf{t}'}; M) = 0,$$

by the Krull intersection theorem. Following the same line of reasoning we see that

$$\prod_{j=0}^{n-1} \mathfrak{a}_I^j(M)\,((x_1^t, \dots, x_{i-1}^t)\,M : x_i^t/(x_1^t, \dots, x_{i-1}^t)\,M) = 0,$$

for $i = 1, \dots, n$ and all $t \geqslant 1$. This proves the first statement.

For the second we want to show for $k \leqslant n$

$$(\mathfrak{r}_x(M))^{(t)} H^i(x_1^t, \dots, x_k^t; M) = 0,$$

for $0 \leqslant i < k$ and $t \geqslant 1$. To this end we make an induction on k. If $k = 1$ and $i = 0$, then $H^0(x_1^t; M) \cong 0_M : x_1^t$ is annihilated by $\mathfrak{r}_{\mathbf{x}}(M)$. Assume $k \geqslant 2$. We show the statement by an induction on i. For $i = 0$ we have

$$H^0(x_1^t, \dots, x_k^t; M) \subseteq 0_M : x_1^t$$

and the assertion is true. For $i \geqslant 1$ there is a short exact sequence

$$0 \to H^{i-1}(x_1^t, \dots, x_{k-1}^t; M) \otimes_A A/x_k^t A \to H^i(x_1^t, \dots, x_k^t; M) \to 0_{H^i(x_1^t, \dots, x_{k-1}^t; M)} : x_k^t \to 0,$$

$t \geqslant 1$. If $i < k-1$, the induction hypothesis yields the statement. In the case $i = k-1$ we get

$$0_{H^{k-1}(x_1^t, \dots, x_{k-1}^t; M)} : x_k^t \cong (x_1^t, \dots, x_{k-1}^t)\,M : x_k^t/(x_1^t, \dots, x_{k-1}^t)\,M.$$

So the short exact sequence proves the statement on the annihilation. In particular, we have

$$(\mathfrak{r}_x(M))^{(t)} H^i(x_1^t, \dots, x_n^t; M) = 0 \quad (0 \leqslant i < n,\ t \geqslant 1).$$

According to Grothendieck (1), § 2, we get

$$H_I^i(M) \cong \varinjlim_t H^i(x_1^t, \dots, x_n^t; M).$$

Note that $\operatorname{Rad} \mathbf{x}A = \operatorname{Rad} I$. Therefore it follows $(\mathfrak{r}_x(M))^{(t)} H_I^i(M) = 0$, which proves the second inclusion. ∎

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By similar considerations as in the proof of Theorem 3 it follows

$$(\mathrm{Ann}_A M)^{n+1} \subseteq \mathfrak{a}_I^0(M)\,\mathfrak{a}_I^1(M) \dots \mathfrak{a}_I^n(M) \subseteq \mathrm{Ann}_A M,$$

see (7) for the case $I = \mathfrak{m}$ and $M = A$.

Note that $\mathbf{x} = \{x_1, \dots, x_d\}$ is called a system of parameters of the local ring A, if $d = \dim A$ and $\mathrm{Rad}\,\mathbf{x}A = \mathfrak{m}$. We call a system of parameters $\mathbf{x}$ of A amiable, if there is an element $c \in \mathfrak{r}_{\mathbf{x}}(A)$ which is not nilpotent. The local ring A is called amiable, if there is an element $c \in A$ which is not nilpotent and makes every system of parameters of A amiable. The study of amiable systems of parameters was initiated by Hochster (3) and (4) in order to construct big Cohen–Macaulay modules for local rings with prime characteristic. In (7) we proved the amiability of a local ring A admitting a dualizing complex. Furthermore, there are local rings A for which there is no amiable system of parameters, see (7), Example and Proposition 5. Using our previous considerations we want to give a further characterization of the amiability of a local ring.

COROLLARY 3. *Let A denote a local ring. Then the following conditions are equivalent*:

(i) *A possesses an amiable system of parameters, and*

(ii) *A is amiable, i.e. every system of parameters of A is amiable.*

Proof. We have to show (i) $\Rightarrow$ (ii). Let $\mathbf{x} = \{x_1, \dots, x_d\}$, $d = \dim A$, be an amiable system of parameters. By the definition there is an element of $\mathfrak{r}_{\mathbf{x}}(A)$ which is not nilpotent. According to Theorem 3(b) there is an element in $\mathfrak{a}_m^0(A) \dots \mathfrak{a}_m^{d-1}(A)$ which is not nilpotent. By virtue of Theorem 3(a) it follows that every system of parameter is amiable. ∎

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