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LIMIT LAWS ON EXTREMES OF NONHOMOGENEOUS GAUSSIAN RANDOM FIELDS

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Abstract

In this paper, by using the tail asymptotics derived by Dębicki *et al.* (2016), we prove the Gumbel limit laws for the maximum of a class of nonhomogeneous Gaussian random fields. As an application of the main results, we derive the Gumbel limit law for Shepp statistics of fractional Brownian motion and Gaussian integrated processes.

Keywords: Extremes; Gumbel limit law; nonhomogeneous Gaussian random field; Shepp statistic; fractional Brownian motion

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1. Introduction

The studies on the Gumbel limit law for Gaussian processes have a long history and can date back to Pickands (1969). Suppose that $\{X(t)\colon t \in [0,\infty)\}$ is a stationary Gaussian process with the covariance function $r(t)$ satisfying the following condition:

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha), \qquad t \to 0, \quad \text{and} \quad r(t) < 1, \qquad t > 0,$$

with $\alpha \in (0, 2]$. It is well known (see, e.g. Pickands (1969) and Leadbetter *et al.* (1983)) that if the so-called Berman's condition,

$$r(t)\ln t \to 0 \quad \text{as } t \to \infty,$$

holds, then the Gumbel limit law

$$\mathbb{P}\Big(a_T\Big(\sup_{0\le t\le T} X(t) - b_T\Big) \le x\Big) \to \exp(-\mathrm{e}^{-x}) \tag{1.1}$$

holds for any $x \in \mathbb{R}$, as $T \to \infty$, where

$$a_T = \sqrt{2\ln T}, \qquad b_T = \sqrt{2\ln T} + \frac{\ln[(2\pi)^{-1/2}\mathcal{H}_\alpha(2\ln T)^{-1/2+1/\alpha}]}{\sqrt{2\ln T}}.$$

Here $\mathcal{H}_\alpha$ denotes the Pickands constant given by

$$\mathcal{H}_\alpha = \lim_{\lambda\to\infty} \frac{\mathcal{H}_\alpha[0,\lambda]}{\lambda} \in (0,\infty)$$

with

$$\mathcal{H}_\alpha[0,\lambda] = \mathbb{E}\exp\Big(\max_{t\in[0,\lambda]} \sqrt{2}B_\alpha(t) - t^\alpha\Big)$$

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and B_α a fractional Brownian motion (FBM) with Hurst parameter $\alpha/2 \in (0, 1]$, that is, a zero-mean Gaussian process with stationary increments such that $\mathbb{E}B_\alpha^2(t) = |t|^\alpha$. To derive the Gumbel limit law (1.1), the following well-known Pickands asymptotics (see, e.g. Pickands (1969), Berman (1974), and Leadbetter *et al.* (1983)) play a crucial role, i.e.

$$\mathbb{P}\Big(\sup_{t\in[0,T]} X(t) > u\Big) = T\mathcal{H}_\alpha u^{2/\alpha}\Psi(u)(1+o(1)) \quad \text{as } u \to \infty$$

for some fixed $T \in (0, \infty)$, where $\Psi(\cdot)$ denotes the tail distribution of a standard normal random variable. For some recent work on the tail asymptotics for extremes, we refer the reader to Chan and Lai (2006), Dębicki *et al.* (2015c), Cheng and Xiao (2016a), (2016b), and the references therein.

The investigation of (1.1) for Gaussian processes and general stochastic processes has received a lot of attention. Mittal and Ylvisaker (1975) extended (1.1) to the strongly dependent Gaussian case; Hülser (1990) investigated (1.1) for the locally stationary Gaussian case, which was recently further extended to Gaussian random fields on manifolds by Qiao and Polonic (2017). We refer the reader to McCormick (1980), Konstant and Piterbarg (1993), and Piterbarg (1996) for further extensions to Gaussian processes and fields; Leadbetter and Rootzén (1982) and Albin (1990) for stationary non-Gaussian processes. For more related extensions, see Dębicki *et al.* (2015b) and the reference therein.

In many applied fields, the Gumbel limit laws for extremes of Gaussian processes play a very important role. In approximation theory, Seleznjev (1991), (1996), Hülser (1999), and Hülser *et al.* (2003) applied the Gumbel limit law for Gaussian processes to investigate the deviation processes of some piecewise linear interpolation problems. In nonparametric statistics, the absolute deviations of many types of density estimators obey the Gumbel limit law; see, e.g. Bickel and Rosenblatt (1973) and Giné *et al.* (2004). In applied statistics, there are also many confidence intervals and bands, which are constructed based on the Gumbel limit law of the estimators, since extremes themselves are also types of very important estimators; see, e.g. Giné and Nickl (2010). For some recent studies on applications of Gumbel limit laws; see Sharpnack and Arias-Castro (2016) and Qiao and Polonik (2016).

Define

$$X(s,t) = Y(s+t) - Y(s), \tag{1.2}$$

where $\{Y(t),\ t \geq 0\}$ is a Gaussian process. The process $\xi_S(t) = \sup_{0\leq s\leq S} X(s,t)$ is referred to as the Shepp statistics in many recent works. Zholud (2008) studied the maximum of the process $\xi_S(t)$ and established the Gumbel limit law when $Y(t)$ is a Brownian motion. Hashorva and Tan (2013) and Tan and Yang (2015) extended the result to FBM. See Piterbarg (2001), Hülser and Piterbarg (2004b), and Hashorva *et al.* (2013) for related work on the FBM.

In this paper we generalize model (1.2) and impose directly some restrictions on the Gaussian random fields. We first consider the Gumbel limit law for the process $\varsigma_T(s) = \sup_{0\leq t\leq T} X(s,t)$ for some fixed $T > 0$, where $X(s,t)$ is a type of nonhomogeneous Gaussian random field. Then we use the obtained results to derive the Gumbel limit law for Shepp statistics. Noting that $\varsigma_T(t)$ is no longer a Gaussian process, we cannot derive the Gumbel limit laws from the Gaussian case directly. However, $\varsigma_T(t)$ also does not satisfy the conditions imposed on general stochastic processes, such as those given by Leadbetter and Rootzén (1982) and Albin (1990). We will follow the method used in Leadbetter *et al.* (1983, Chapter 12). The tail asymptotic result of extremes of the field $X(s,t)$ is a key tool, which was derived by Dębicki *et al.* (2016).

The rest of the paper is organized as follows. In Section 2 we give some tail asymptotic results from Dębicki *et al.* (2016). In Section 3 we state the main results of the paper, and in

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Section 4 we present two applications. The technical proofs are gathered in Section 5, while in Appendix A we give two auxiliary results.

2. Preliminaries

In this section, we present the tail asymptotic result provided by Dębicki *et al.* (2016). Suppose that $\{X(s,t),\ (s,t) \in [0,\infty)\times[0,T]\}$ with fixed T is a centered Gaussian random field with variance function and correlation function given by $\sigma^2(s,t)$ and $r(s,t,s',t')$, respectively. Suppose the following assumptions hold.

Assumption 2.1. *There exists some positive function $\sigma(t)$ which attains its unique maximum on $[0,T]$ at fixed T, and, furthermore,*

$$\sigma(s,t)=\sigma(t), \quad \textit{for all } (s,t)\in[0,\infty)\times[0,T], \qquad \sigma(t)=1-b(T-t)^{\beta}(1+o(1)), \quad t\uparrow T,$$

holds for some $\beta, b>0$.

Assumption 2.2. *There exist constants $a_1>0, a_2>0, a_3\neq 0$, and $\alpha_1,\alpha_2\in(0,2]$ such that*

$$r(s,t,s',t')=1-(|a_1(s-s')|^{\alpha_1}+|a_2(t-t')+a_3(s-s')|^{\alpha_2})(1+o(1))$$

holds uniformly with respect to $s,s'\in[0,L]$ with some constant $L>0$ as $|s-s'|\to 0$, $\min(t,t')\uparrow T$, and, furthermore, there exists some constant $\delta_0\in(0,T)$ such that

$$r(s,t,s',t')<1 \quad \textit{for any } s,s'\in[0,L]$$

satisfying $s\neq s'$ and $t,t'\in[\delta_0,T]$.

Assumption 2.3. *There exist positive constants γ_1,γ_2,γ, and $\mathcal{C}$ such that*

$$\mathbb{E}(X(s,t)-X(s',t'))^2\le \mathcal{C}(|t-t'|^{\gamma}+|s-s'|^{\gamma})$$

holds for all $t,t'\in[\gamma_1,T]$, $s,s'\in[0,L]$ satisfying $|s-s'|<\gamma_2$.

To state the tail asymptotics for the maximum of the field $X(s,t)$ under Assumptions 2.1–2.3, we need the so-called Piterbarg constants and Pickands–Piterbarg constants, respectively. The Piterbarg constant $\mathcal{P}_{\alpha}^{b}$ with constant $b>0$ is defined as

$$\mathcal{P}_{\alpha}^{b}=\lim_{\lambda\to\infty}\mathbb{E}\exp\Big(\max_{t\in[0,\lambda]}\sqrt{2}B_{\alpha}(t)-(1+b)|t|^{\alpha}\Big)\in(0,\infty).$$

For some constants $a_1>0, a_2>0, a_3\neq 0$, and $b>0$, let

$$Y(s,t)=\widetilde{B}_{\alpha}(a_1 s)+B_{\alpha}(a_2 t-a_3 s), \qquad \sigma_Y^2(s,t)=\operatorname{var}(Y(s,t))$$

and

$$\mathcal{H}_Y^b[\lambda_1,\lambda_2]=\mathbb{E}\exp\Big(\max_{(s,t)\in[0,\lambda_1]\times[0,\lambda_2]}\sqrt{2}Y(s,t)-\sigma_Y^2(s,t)-b|t|^{\alpha}\Big),$$

where $\widetilde{B}_{\alpha}$ and B_{α} are two independent FBMs. The Pickands–Piterbarg constant is defined as

$$\mathcal{M}_{Y,\alpha}^{b}=\lim_{\lambda_1\to\infty}\lim_{\lambda_2\to\infty}\frac{1}{\lambda_1}\mathcal{H}_Y^b[\lambda_1,\lambda_2].$$

Under the above assumptions, Dębicki *et al.* (2016) derived the following result.

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Theorem 2.1. *Let $\{X(s,t),(s,t)\in[0,\infty)\times[0,T]\}$ with fixed T be a centered Gaussian random field with almost sure continuous sample paths. Suppose that Assumptions 2.1–2.3 are satisfied with the parameters mentioned therein, we have, as $u\to\infty$,*

$$\mathbb{P}\Big(\sup_{(s,t)\in[0,L]\times[0,T]} X(s,t)>u\Big)=L\mu(u)(1+o(1)),$$

where

(i) *for $\beta>\max\{\alpha_1,\alpha_2\}$,*

$$\mu(u)=\Gamma\Big(\frac{1}{\beta}+1\Big)\prod_{k=1}^{2}(a_k\mathcal{H}_{\alpha_k})b^{-1/\beta}u^{2/\alpha_1+2/\alpha_2-2/\beta}\Psi(u);$$

(ii) *for $\beta=\alpha_2=\alpha_1$,*

$$\mu(u)=\mathcal{M}^{b}_{Y,\alpha_1}u^{2/\alpha_1}\Psi(u);$$

(iii) *for $\beta=\alpha_2>\alpha_1$,*

$$\mu(u)=a_1a_2\mathcal{P}_{\alpha_2}^{ba_2^{-\alpha_2}}\mathcal{H}_{\alpha_1}u^{2/\alpha_1}\Psi(u);$$

(iv) *for $\beta<\alpha_2=\alpha_1$,*

$$\mu(u)=(a_1^{\alpha_1}+|a_3|^{\alpha_1})^{1/\alpha_1}\mathcal{H}_{\alpha_1}u^{2/\alpha_1}\Psi(u);$$

(v) *for $\beta<\alpha_2$ and $\alpha_1<\alpha_2$,*

$$\mu(u)=a_1\mathcal{H}_{\alpha_1}u^{2/\alpha_1}\Psi(u);$$

(vi) *for $\beta=\alpha_1>\alpha_2$,*

$$\mu(u)=a_1\mathcal{P}_{\alpha_1}^{b(|a_3|/a_1a_2)^{\alpha_1}}\mathcal{H}_{\alpha_2}u^{2/\alpha_2}\Psi(u);$$

(vii) *for $\beta<\alpha_1$ and $\alpha_2<\alpha_1$,*

$$\mu(u)=|a_3|\mathcal{H}_{\alpha_2}u^{2/\alpha_2}\Psi(u).$$

This result is very powerful since it can be used to derive the exact tail asymptotics for many type of statistics, such as Shepp statistics for Gaussian processes, Brownian bridge and FBM, maximum loss, and span of Gaussian processes; see Dębicki *et al.* (2016) for details.

3. Main result

Note that the Assumptions 2.1 and 2.2 are local conditions. To derive the Gumbel limit law, we need to impose the following Berman-type weak dependence condition, which is a global condition.

Assumption 3.1. *Assume that, for $c=1+\varepsilon\,\mathbf{1}(\beta\geq\max\{\alpha_1,\alpha_2\})$ with some constant $\varepsilon>0$, the function*

$$\delta(v):=\sup\{|r(s,t,s',t')|,|s-s'|\geq v,s,s'\in[0,\infty),t,t'\in[0,T]\}$$

is such that

$$\lim_{v\to\infty}\delta(v)(\ln v)^c=0,$$

where $\mathbf{1}(\cdot)$ denotes the indicator function.

We now state the main result.

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Theorem 3.1. *Let* $\{X(s,t),\ (s,t) \in [0,\infty) \times [0,T]\}$ *with fixed* T *be a centered Gaussian random field with almost sure continuous sample paths. Suppose that Assumptions 2.1–3.1 are satisfied with the parameters mentioned therein. In addition, assume that* $\{X(s,t),\ (s,t) \in [0,\infty) \times [0,T]\}$ *is homogeneous with respect to the first factor* s*. Then*

$$\lim_{S\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\Big(a_S \Big(\sup_{(s,t)\in[0,S]\times[0,T]} X(s,t) - b_S \Big) \le x \Big) - \exp(-\mathrm{e}^{-x}) \right| = 0,$$

where $a_S = \sqrt{2\ln S}$, $b_S = a_S + a_S^{-1}\omega_S$ *with*

$$\omega_S = \ln\left((2\pi)^{-1/2} \Gamma\left(\frac{1}{\beta}+1\right) \prod_{k=1}^{2} (a_k \mathcal{H}_{\alpha_k}) b^{-1/\beta} a_S^{2/\alpha_1+2/\alpha_2-2/\beta-1} \right) \quad \textit{for } \beta > \max\{\alpha_1,\alpha_2\},$$

$$\omega_S = \ln((2\pi)^{-1/2} \mathcal{M}^b_{Y,\alpha_1} a_S^{2/\alpha_1-1}) \quad \textit{for } \beta = \alpha_2 = \alpha_1,$$

$$\omega_S = \ln((2\pi)^{-1/2} a_1 a_2 \mathcal{P}_{\alpha_2}^{b a_2^{-\alpha_2}} \mathcal{H}_{\alpha_1} a_S^{2/\alpha_1-1}) \quad \textit{for } \beta = \alpha_2 > \alpha_1,$$

$$\omega_S = \ln((2\pi)^{-1/2} (a_1^{\alpha_1} + |a_3|^{\alpha_1})^{1/\alpha_1} \mathcal{H}_{\alpha_1} a_S^{2/\alpha_1-1}) \quad \textit{for } \beta < \alpha_2 = \alpha_1,$$

$$\omega_S = \ln((2\pi)^{-1/2} a_1 \mathcal{H}_{\alpha_1} a_S^{2/\alpha_1-1}) \quad \textit{for } \beta < \alpha_2 \textit{ and } \alpha_1 < \alpha_2,$$

$$\omega_S = \ln((2\pi)^{-1/2} a_1 \mathcal{P}_{\alpha_1}^{b(|a_3|/a_1 a_2)^{\alpha_1}} \mathcal{H}_{\alpha_2} a_S^{2/\alpha_2-1}) \quad \textit{for } \beta = \alpha_1 > \alpha_2,$$

$$\omega_S = \ln((2\pi)^{-1/2} |a_3| \mathcal{H}_{\alpha_2} a_S^{2/\alpha_2-1}) \quad \textit{for } \beta < \alpha_1 \textit{ and } \alpha_2 < \alpha_1.$$

Remark 3.1. Assumption 3.1 is a weakly dependent condition. If $\lim_{v\to\infty} \delta(v)(\ln v)^c = d > 0$ then the field $X(s,t)$ will possess some strongly dependent property with respect to the first parameter. In this case, the limit distribution will no longer be a Gumbel distribution; see Mittal and Ylvisaker (1975) and Tan *et al.* (2012) for some related results about strongly dependent Gaussian processes.

4. Applications

In this section we give two applications of our main results. We derive the exact tail asymptotics and Gumbel limit laws for Shepp statistics. The obtained results are of independent interest.

Throughout this section, let $\{X(t),\ t \ge 0\}$ be a centered Gaussian process and define

$$Z(s,t) = X(s+t) - X(s), \qquad (s,t) \in [0,\infty) \times [0,T] \quad \text{for some fixed } T > 0.$$

The Shepp statistic $\sup_{0\le s\le S} Z(s,t)$, which was introduced by Shepp (see Shepp (1966), (1971)), plays a very important role in statistics. Other important results for the Shepp statistic can be found in Cressie (1980), Deheuvels and Devroye (1987), Siegmund and Venkatraman (1995), Dumbgen and Spokoiny (2001), and Kabluchko (2011). The limit properties of extremes of Shepp statistics when $X(t)$ is a FBM have been studied by Zholud (2008), Hashorva and Tan (2013), Tan and Yang (2015), and Tan and Chen (2016). Applying Theorem 3.1, we study the limit properties of extremes of Shepp statistics for a more general Gaussian process $X(t)$, which is a stationary Gaussian process or nonstationary Gaussian process with stationary increments.

4.1. The stationary case

Let $\{X(t),\ t \ge 0\}$ be a centered stationary Gaussian process. Suppose that the covariance function r_X of $\{X(t),\ t \ge 0\}$ satisfies the following conditions.

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Assumption 4.1. *Assume that $r_X(t)$ attains its minimum on $[0, T]$ at the unique point T.*

Assumption 4.2. *There exist positive constants α_1, a_1, a_2, and $\alpha_2 \in (0, 2)$ such that*

$$r_X(t) = r_X(T) + a_1|t - T|^{\alpha_1}(1 + o(1)), \qquad t \to T,$$
$$r_X(t) = 1 - a_2 t^{\alpha_2}(1 + o(1)), \qquad t \to 0.$$

Assumption 4.3. *Assume that $r_X(s) < 1$ for $s > 0$.*

For simplicity, write $\rho_T = \sqrt{2(1 - r_X(T))}$ and $b_i = a_i/\rho_T^2$, $i = 1, 2$.

Proposition 4.1. *Let $Z(s, t)$ be defined as above. Suppose that $r_X(t)$ satisfies Assumptions 4.1–4.3. In addition, suppose that $r_X(t)$ is twice continuously differentiable on $[\tau, \infty)$ for some $\tau > 0$ and the limit of the double derivative $\lim_{t\to T} |\ddot{r}_X(t)| \in (0, \infty)$. Furthermore, if $\ddot{r}_X(t)(\ln t)^c = o(1)$ with $c = 1 + \varepsilon\, \mathbf{1}(\alpha_1 \geq \alpha_2)$ and some constant $\varepsilon > 0$ as $t \to \infty$, then*

$$\lim_{S\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\Big(a_S\Big(\sup_{(s,t)\in[0,S]\times[0,T]} Z(s, t) - b_S\Big) \leq x\Big) - \exp\{-e^{-x}\}\right| = 0,$$

where $a_S = \rho_T\sqrt{2 \ln S}$, $b_S = a_S + a_S^{-1}\omega_S$ with

$$\omega_S = \ln\left(\Gamma\left(\frac{1}{\alpha_1} + 1\right)\mathcal{H}_{\alpha_2}^2 b_2^{2/\alpha_2} b_1^{-1/\alpha_1}(2\pi)^{-1/2} a_S^{4/\alpha_2 - 2/\alpha_1 - 1}\right) \quad \text{for } \alpha_1 > \alpha_2,$$
$$\omega_S = \ln(\mathcal{M}_{Y,\alpha_1}^{b_1}(2\pi)^{-1/2} a_S^{2/\alpha_2 - 1}) \quad \text{for } \alpha_1 = \alpha_2,$$

with

$$Y = Y(s, t) = \widetilde{B}_{\alpha_2}(b_2^{1/\alpha_2} s) + B_{\alpha_2}(b_2^{1/\alpha_2} t - b_2^{1/\alpha_2} s),$$
$$\omega_S = \ln((2b_2)^{1/\alpha_2}\mathcal{H}_{\alpha_2}(2\pi)^{-1/2} a_S^{2/\alpha_2 - 1}) \quad \text{for } \alpha_1 < \alpha_2.$$

Example 4.1. The Ornstein–Uhlenbeck process with covariance function $r_X(t) = e^{-|t|^\alpha}$ and the generalized Cauchy model with covariance function $r_X(t) = (1 + |t|^\alpha)^{-\beta}$ with $\alpha \in (0, 2)$ and $\beta > 0$ satisfy the conditions of Proposition 4.1.

4.2. The nonstationary case

Let $\{X(t),\ t \geq 0\}$ be a centered nonstationary Gaussian process with stationary increments and variance function $\sigma_X^2(t)$ with almost surely continuous sample paths. Recall that $X(t)$ is said to have stationary increments if the law of the process $\{X(t + t_0) - X(t_0),\ t \in \mathbb{R}\}$ does not depend on the choice of t_0. To study the maximum of $Z(s, t)$, we only need to impose some conditions on the variogram $\gamma(t) = \mathbb{E}(X(t) - X(0))^2$ of X. Note that for this case, the variogram is $\gamma_X(t) = \sigma_X^2(t)$. Suppose that the variance function $\sigma_X^2(t)$ of $\{X(t), t \geq 0\}$ satisfies the following conditions.

Assumption 4.4. *Assume that $\sigma_X(t)$ attains its maximum on $[0, T]$ at the unique point T, and, furthermore,*

$$\sigma_X(t) = 1 - b(T - t)^\beta(1 + o(1)), \qquad t \uparrow T$$

holds for some $\beta, b > 0$.

Assumption 4.5. *Assume that $\sigma_X^2(t)$ is twice continuously differentiable on $[\tau, \infty)$ for $\tau > 0$ with limit of the double derivative $\lim_{t\to T} |\ddot{\sigma}_X^2(t)| \in (0, \infty)$ and, furthermore,*

$$\sigma_X^2(t) = (at)^\alpha(1 + o(1)), \qquad t \to 0$$

holds for some $\alpha \in (0, 2]$, $a > 0$.

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Assumption 4.6. *Assume that* $\ddot{\sigma}_X^2(t)(\ln t)^c \to 0$ *with* $c = 1 + \varepsilon\,\mathbf{1}(\beta \geq \alpha)$ *and some constant* $\varepsilon > 0$ *as* $t \to \infty$.

Proposition 4.2. *Let* $Z(s,t)$ *be defined as above. Suppose that* $\sigma_X(t)$ *satisfies Assumptions 4.4 and 4.5. We have, for some constant* $L > 0$,

$$\mathbb{P}\Big(\sup_{(s,t)\in[0,L]\times[0,T]} Z(s,t) > u\Big) = L\mu(u)(1+o(1))$$

where

$$\mu(u) = 2^{-2/\alpha}\Gamma\Big(\frac{1}{\beta}+1\Big)a^2\mathcal{H}_\alpha^2 b^{-1/\beta}u^{4/\alpha-2/\beta}\Psi(u) \quad \textit{for } \alpha < \beta,$$
$$\mu(u) = \mathcal{M}_Y^b u^{2/\alpha}\Psi(u) \quad \textit{for } \alpha = \beta$$

with

$$Y = Y(s,t) = \widetilde{B}_\alpha(2^{-1/\alpha}as) + B_\alpha(2^{-1/\alpha}at - 2^{-1/\alpha}as),$$
$$\mu(u) = a\mathcal{H}_\alpha u^{2/\alpha}\Psi(u) \quad \textit{for } \alpha > \beta.$$

Furthermore, if Assumption 4.6 holds then

$$\lim_{S\to\infty}\sup_{x\in\mathbb{R}}\Big|\mathbb{P}\Big(a_S\Big(\sup_{(s,t)\in[0,S]\times[0,T]} Z(s,t) - b_S\Big) \leq x\Big) - \exp\{-\mathrm{e}^{-x}\}\Big| = 0,$$

where $a_S = \sqrt{2\ln S}$, *and* $b_S = a_S + a_S^{-1}\omega_S$ *with*

$$\omega_S = \ln(2^{-2/\alpha}\Gamma\Big(\frac{1}{\beta}+1\Big)a^2\mathcal{H}_\alpha^2 b^{-1/\beta}(2\pi)^{-1/2}a_S^{4/\alpha-2/\beta-1}) \quad \textit{for } \alpha < \beta,$$
$$\omega_S = \ln(\mathcal{M}_Y^b(2\pi)^{-1/2}a_S^{2/\alpha-1}) \quad \textit{for } \alpha = \beta,$$
$$\omega_S = \ln(a\mathcal{H}_\alpha(2\pi)^{-1/2}a_S^{2/\alpha-1}) \quad \textit{for } \alpha > \beta.$$

We illustrate Proposition 4.2 by the following two examples on the FBM and Gaussian integrated process.

Example 4.2. Let $B_{H_i}(t)$, $i = 1, 2, \ldots, n$ be a sequence of independent FBMs with Hurst index $H_i \in (0,1)$ and λ_i a positive sequence satisfying $\sum_{i=1}^n \lambda_i^2 = 1$. Since, given $H = H_1 = H_2$, we have $\lambda_1 B_{H_1}(t) + \lambda_2 B_{H_2}(t) \overset{\mathrm{D}}{=} \sqrt{\lambda_1^2+\lambda_2^2}B_H(t)$, we suppose that

$$H := H_1 < H_2 < \cdots < H_n.$$

Let $X(t) = \sum_{i=1}^n \lambda_i T^{-1/2}B_{H_i}(t)$ and $Z(s,t)$ be defined as above. We have, for some constant $L > 0$,

$$\mathbb{P}\Big(\sup_{(s,t)\in[0,L]\times[0,T]} Z(s,t) > u\Big) = L\mu(u)(1+o(1)) \quad \text{as } u \to \infty,$$

where

$$\mu(u) = 2^{-1/H}\mathcal{H}_{2H}^2\lambda_1^{2/H}\Big(\sum_{i=1}^n \lambda_i^2 H_i\Big)^{-1}u^{2/H-2}\Psi(u) \quad \text{for } H \in \big(0, \tfrac{1}{2}\big),$$
$$\mu(u) = \mathcal{M}_{Y,1}^{1/2}u^2\Psi(u) \quad \text{for } H = \tfrac{1}{2}$$

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with

$$Y = Y(s,t) = \widetilde{B}_1(2^{-1}s) + B_1(2^{-1}(t-s)),$$
$$\mu(u) = \lambda_1^{1/H}\mathcal{H}_{2H}u^{1/H}\Psi(u) \quad \text{for } H \in \left(\tfrac{1}{2}, 1\right),$$

and

$$\lim_{S\to\infty}\sup_{x\in\mathbb{R}}\left|\mathbb{P}\Big(a_S\Big(\sup_{(s,t)\in[0,S]\times[0,T]} Z(s,t) - b_S\Big) \le x\Big) - \exp\{-\mathrm{e}^{-x}\}\right| = 0,$$

where $a_S = \sqrt{2\ln S}$, and $b_S = a_S + a_S^{-1}\omega_S$ with

$$\omega_S = \ln\left((2\pi)^{-1/2}2^{-1/H}\mathcal{H}_{2H}^2\lambda_1^{2/H}\left(\sum_{i=1}^n \lambda_i^2 H_i\right)^{-1} a_S^{2/H-3}\right) \quad \text{for } H \in \left(0, \tfrac{1}{2}\right),$$
$$\omega_S = \ln(2\pi)^{-1/2}\mathcal{M}_{Y,1}^{1/2}a_S) \quad \text{for any } H = \tfrac{1}{2},$$
$$\omega_S = \ln((2\pi)^{-1/2}\lambda_1^{1/H}\mathcal{H}_{2H}a_S^{1/H-1}) \quad \text{for } H \in \left(\tfrac{1}{2}, 1\right).$$

Next, we consider the Gaussian integrated process. For related studies, we refer the reader to Hüsler and Piterbarg (2004a).

Example 4.3. Let $\{\zeta(t),\ t \ge 0\}$ be a centered stationary Gaussian process with variance 1 and suppose the covariance function $r_\zeta(t)$ of $\{\zeta(t),\ t \ge 0\}$ satisfies the following conditions.

Assumption 4.7. *Assume that* $r_\zeta(t) \in C([0,\infty))$ *and* $\int_0^t r_\zeta(s)\,\mathrm{d}s > 0$ *for* $t \in (0,T]$.

Assumption 4.8. *Assume that* $r_\zeta(t) = 1 - t^\theta(1+o(1))$ *as* $t \to 0^+$ *with* $\theta \in (0,2]$.

Assumption 4.9. *Assume that* $r_\zeta(t)\ln t = o(1)$ *as* $t \to \infty$.

Define Gaussian integrated processes as $X(t) = \int_0^t \zeta(s)\,\mathrm{d}s$ and let $Z(s,t)$ be defined as above. If Assumptions 4.7 and 4.8 are satisfied, we have for some constant $L > 0$,

$$\mathbb{P}\Big(\sup_{(s,t)\in[0,L]\times[0,T]} Z(s,t) > u\Big) = L\frac{1}{\sqrt{\pi}}u\Psi(u)(1+o(1)) \quad \text{as } u \to \infty.$$

If, furthermore, Assumption 4.9 holds, we have

$$\lim_{S\to\infty}\sup_{x\in\mathbb{R}}\left|\mathbb{P}\Big(a_S\Big(\sup_{(s,t)\in[0,S]\times[0,T]} Z(s,t) - b_S\Big) \le x\Big) - \exp\{-\mathrm{e}^{-x}\}\right| = 0, \tag{4.1}$$

where $a_S = \sqrt{2\ln S}$, and $b_S = a_S + a_S^{-1}\ln((2)^{-1/2}\pi^{-1})$.

5. Proofs

We need the following lemmas to prove Theorem 3.1. For simplicity, write $u = u_S(x) = a_S^{-1}x + b_S$ in the following part.

Lemma 5.1. *Let* $\delta_u = u^{-2/\beta}(\ln u)^{2/\beta}$. *Under the conditions of Theorem 3.1, we have for some constant* $S_0 > 0$,

$$\frac{|\mathbb{P}(\sup_{(s,t)\in[0,S_0]\times[0,T]} X(s,t) \le u) - \mathbb{P}(\sup_{(s,t)\in[0,S_0]\times[T-\delta_u,T]} X(s,t) \le u)|}{\mathbb{P}(\sup_{(s,t)\in[0,S_0]\times[0,T]} X(s,t) > u)} \to 0$$

as $u \to \infty$.

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Proof. See the proof of Theorem 2.2 of Dębicki *et al.* (2016). □

For given $\epsilon > 0$, we divide the interval $[0, S]$ into intervals of length 1, and split each of them into subintervals I_j^ϵ, I_j of length ϵ, $1-\epsilon$, $j = 1, 2, \ldots, \lfloor S \rfloor$, respectively, where $\lfloor x \rfloor$ denotes the integral part of x. It can be easily seen that a possible remaining interval with length smaller than 1 plays no role in our consideration. We denote this interval by J.

Lemma 5.2. *Under the conditions of Theorem 3.1, we have*

$$\left|\mathbb{P}\Big(\sup_{(s,t)\in[0,S]\times[T-\delta_u,T]} X(s,t) \le u\Big) - \mathbb{P}\Big(\sup_{(s,t)\in\cup I_j\times[T-\delta_u,T]} X(s,t) \le u\Big)\right| \to 0$$

as $u \to \infty$ *and* $\epsilon \to 0$.

Proof. By applying Theorem 2.1 and Lemma 5.1, we have

$$\begin{aligned}
&\left|\mathbb{P}\Big(\sup_{(s,t)\in[0,S]\times[T-\delta_u,T]} X(s,t) \le u\Big) - \mathbb{P}\Big(\sup_{(s,t)\in\cup I_j\times[T-\delta_u,T]} X(s,t) \le u\Big)\right| \\
&\quad\le \mathbb{P}\Big(\sup_{(s,t)\in(\cup I_j^\epsilon\cup J)\times[T-\delta_u,T]} X(s,t) > u\Big) \\
&\quad\le \sum_{j=1}^{\lfloor S\rfloor} \mathbb{P}\Big(\sup_{(s,t)\in I_j^\epsilon\times[T-\delta_u,T]} X(s,t) > u\Big) + \mathbb{P}\Big(\sup_{(s,t)\in J\times[T-\delta_u,T]} X(s,t) > u\Big) \\
&\quad\sim \sum_{j=1}^{\lfloor S\rfloor} \mathbb{P}\Big(\sup_{(s,t)\in I_j^\epsilon\times[0,T]} X(s,t) > u\Big) + \mathbb{P}\Big(\sup_{(s,t)\in J\times[0,T]} X(s,t) > u\Big) \\
&\quad\le (\lfloor S\rfloor\epsilon + 1)\mu(u).
\end{aligned}$$

Noting that by the definitions of a_S and b_S, we have $S\mu(u) = O(1)$ as $u \to \infty$, thus, the result follows by letting $\epsilon \to 0$. □

In the following, let $q_i = \mathrm{d}u^{-2/\alpha_i}$ for some $d > 0$.

Lemma 5.3. *Under the conditions of Theorem 3.1, we have, for any* $j = 1, 2, \ldots, \lfloor S \rfloor$,

$$\left|\mathbb{P}\Big(\sup_{(s,t)\in I_j\times[T-\delta_u,T]} X(s,t) \le u\Big) - \mathbb{P}\Big(\sup_{(kq_1,lq_2)\in I_j\times[T-\delta_u,T]} X(kq_1,lq_2) \le u\Big)\right| \le K\rho(d)\mu(u)$$

as $u \to \infty$, *where* $\rho(d) \to 0$ *as* $d \to 0$.

Proof. Without loss of generality, we only show the $j = 1$ case.

(i) *The* $\beta > \max(\alpha_1, \alpha_2)$ *case.* For simplicity, we only consider the $\alpha_1 = \alpha_2 =: \alpha$ case. Choose first a constant $\alpha_0 \in (\alpha, \beta)$ and denote

$$\Delta_{ij} = \Delta_i \times \Delta_j, \qquad \Delta_{ij}^T = \Delta_i \times (T - \Delta_j),$$

with $\Delta_i = [iu^{-2/\alpha_0}, (i+1)u^{-2/\alpha_0}]$. Furthermore, set

$$N_1(u) = \lfloor (1-\epsilon)u^{2/\alpha_0} \rfloor + 1, \qquad N_2(u) = \lfloor (\ln u)^{2/\beta} u^{2/\alpha_0 - 2/\beta} \rfloor + 1.$$

For any $\varepsilon \in (0, 1)$, let $\{\eta_{\pm\varepsilon}(s,t),\ (s,t) \in [0,\infty)^2\}$ be centered homogeneous Gaussian random fields with covariance functions

$$r_{\pm\varepsilon}(s,t) = \exp(-(1\pm\varepsilon)^\alpha(|a_1 s|^\alpha + |a_2 t + a_3 s|^\alpha)), \qquad (s,t) \in [0,\infty)^2.$$

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From the proof of Theorem 2.2(i) in Dębicki *et al.* (2016), it is easy to show that (letting $q = du^{-2/\alpha}$)

$$\begin{aligned}
&\sum_{i=0}^{N_1(u)}\sum_{j=0}^{N_2(u)} \mathbb{P}\Big(\sup_{(s,t)\in\Delta_{ij}} \eta_{+\varepsilon}(s, T-t) > u_{j-}\Big) \\
&\quad\ge \sum_{i=0}^{N_1(u)}\sum_{j=0}^{N_2(u)} \mathbb{P}\Big(\sup_{(s,t)\in\Delta_{ij}} \frac{X(s, T-t)}{\sigma(s, T-t)} > u_{j-}\Big) \\
&\quad\ge \sum_{i=0}^{N_1(u)}\sum_{j=0}^{N_2(u)} \mathbb{P}\Big(\sup_{(s,t)\in\Delta_{ij}^T} X(s,t) > u\Big) \\
&\quad\ge \mathbb{P}\Big(\sup_{(s,t)\in I_1\times[T-\delta_u,T]} X(s,t) > u\Big) \\
&\quad\ge \mathbb{P}\Big(\sup_{(kq,lq)\in I_1\times[T-\delta_u,T]} X(kq,lq) > u\Big) \\
&\quad\ge \sum_{i=0}^{N_1(u)-1}\sum_{j=0}^{N_2(u)-1} \mathbb{P}\Big(\sup_{(kq,lq)\in\Delta_{ij}^T} X(kq,lq) > u\Big) - \Sigma_1(u) \\
&\quad\ge \sum_{i=0}^{N_1(u)-1}\sum_{j=0}^{N_2(u)-1} \mathbb{P}\Big(\sup_{(kq,lq)\in\Delta_{ij}} \frac{X(kq, T-lq)}{\sigma(kq, T-lq)} > u_{j+}\Big) - \Sigma_1(u) \\
&\quad\ge \sum_{i=0}^{N_1(u)-1}\sum_{j=0}^{N_2(u)-1} \mathbb{P}\Big(\sup_{(kq,lq)\in\Delta_{ij}} \eta_{-\varepsilon}(kq, T-lq) > u_{j+}\Big) - \Sigma_1(u), \qquad (5.1)
\end{aligned}$$

where

$$u_{j-} = u(1 + b(1-\varepsilon)(ju^{-2/\alpha_0})^\beta), \qquad u_{j+} = u(1 + b(1+\varepsilon)((j+1)u^{-2/\alpha_0})^\beta),$$

and

$$\Sigma_1(u) = \sum_{0\le i,\, i'\le N_1(u)-1}\ \sum_{0\le j,\, j'\le N_2(u)-1} \mathbb{P}\Big(\sup_{(s,t)\in\Delta_{ij}^T} X(s,t) > u, \sup_{(s,t)\in\Delta_{i'j'}^T} X(s,t) > u\Big).$$

We can also obtain the following results from the above mentioned paper:

$$\Sigma_1(u) = o(\mu(u)) \quad \text{as } u\to\infty \qquad (5.2)$$

and

$$\begin{aligned}
&\sum_{i=0}^{N_1(u)}\sum_{j=0}^{N_2(u)} \mathbb{P}\Big(\sup_{(s,t)\in\Delta_{ij}} \eta_{+\varepsilon}(s, T-t) > u_{j-}\Big) \\
&\quad\sim \sum_{i=0}^{N_1(u)-1}\sum_{j=0}^{N_2(u)-1} \mathbb{P}\Big(\sup_{(s,t)\in\Delta_{ij}} \eta_{-\varepsilon}(s, T-t) > u_{j+}\Big) \sim \mu(u) \qquad (5.3)
\end{aligned}$$

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as $u \to \infty$ and $\varepsilon \to 0$. For the homogeneous Gaussian random fields $\eta_{\pm\varepsilon}(s,t)$, by Lemma A.2, we use the following estimate:

$$\Big|\mathbb{P}\Big(\sup_{(s,t)\in\Delta_{ij}} \eta_{\pm\varepsilon}(s, T-t) > u\Big) - \mathbb{P}\Big(\sup_{(kq,lq)\in\Delta_{ij}} \eta_{\pm\varepsilon}(kq, T-lq) > u\Big)\Big| \le \rho(d)a_1a_2u^{4/\alpha-4/\alpha_0}\Psi(u)(1+g(u)),$$

where $\rho(d) \to 0$ as $d \to 0$ and $g(u) \to 0$ as $u \to \infty$. Denote $G(u) = 1 + \sup_{v\ge u}|g(u)| \to 1$ as $u \to \infty$. Then $u/u_{j\pm} \to 1$ as $u \to \infty$ uniformly in j and also

$$\Big|\mathbb{P}\Big(\sup_{(s,t)\in\Delta_{ij}} \eta_{\pm\varepsilon}(s, T-t) > u_{j\pm}\Big) - \mathbb{P}\Big(\sup_{(kq,lq)\in\Delta_{ij}} \eta_{\pm\varepsilon}(kq, T-lq) > u_{j\pm}\Big)\Big| \le \rho(d)a_1a_2u_{j\pm}^{4/\alpha-4/\alpha_0}\Psi(u_{j\pm})G(u).$$

Thus, there exists $K > 0$ such that

$$\begin{aligned}\Big|\sum_{i=0}^{N_1(u)-1}\sum_{j=0}^{N_2(u)-1} &\mathbb{P}\Big(\sup_{(s,t)\in\Delta_{ij}} \eta_{-\varepsilon}(s, T-t) > u_{j+}\Big)\\ &- \sum_{i=0}^{N_1(u)-1}\sum_{j=0}^{N_2(u)-1} \mathbb{P}\Big(\sup_{(kq,lq)\in\Delta_{ij}} \eta_{-\varepsilon}(kq, T-lq) > u_{j+}\Big)\Big|\\ &\le K\rho(d)u^{4/\alpha-2/\beta}\Psi(u). \end{aligned} \tag{5.4}$$

Now, from (5.1)–(5.4), it follows that

$$\begin{aligned}\Big|\mathbb{P}\Big(\sup_{(s,t)\in I_1\times[T-\delta_u,T]} X(s,t) > u\Big) &- \mathbb{P}\Big(\sup_{(kq,lq)\in I_1\times[T-\delta_u,T]} X(kq,lq) > u\Big)\Big|\\ &\le \sum_{i=0}^{N_1(u)}\sum_{j=0}^{N_2(u)} \mathbb{P}\Big(\sup_{(s,t)\in\Delta_{ij}} \eta_{+\varepsilon}(s, T-t) > u_{j-}\Big)\\ &\quad - \sum_{i=0}^{N_1(u)-1}\sum_{j=0}^{N_2(u)-1} \mathbb{P}\Big(\sup_{(kq,lq)\in\Delta_{ij}} \eta_{-\varepsilon}(kq, T-lq) > u_{j+}\Big) + \Sigma_1(u)\\ &\sim \sum_{i=0}^{N_1(u)-1}\sum_{j=0}^{N_2(u)-1} \mathbb{P}\Big(\sup_{(s,t)\in\Delta_{ij}} \eta_{-\varepsilon}(s, T-t) > u_{j+}\Big)\\ &\quad - \sum_{i=0}^{N_1(u)-1}\sum_{j=0}^{N_2(u)-1} \mathbb{P}\Big(\sup_{(kq,lq)\in\Delta_{ij}} \eta_{-\varepsilon}(kq, T-lq) > u_{j+}\Big)\\ &\le K\rho(d)u^{4/\alpha-2/\beta}\Psi(u).\end{aligned}$$

(ii) *The $\beta = \alpha_1 = \alpha_2$ case.* For simplicity, set $\alpha = \alpha_1 = \alpha_2$. Let S_0 and T_0 be two positive constants and define

$$\begin{aligned}\widehat{\Delta}_i &= [iS_0u^{-2/\alpha}, (i+1)S_0u^{-2/\alpha}], \qquad i = 0, 1, \cdots, \widehat{N}_1(u),\\ \widetilde{\Delta}_j &= [jT_0u^{-2/\alpha}, (j+1)T_0u^{-2/\alpha}], \qquad j = 0, 1, \ldots, \widetilde{N}_2(u),\\ \overline{\Delta}_{ij} &= \widehat{\Delta}_i \times \widetilde{\Delta}_j, \qquad \overline{\Delta}_{ij}^T = \widehat{\Delta}_i \times (T - \widetilde{\Delta}_j),\end{aligned}$$

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where

$$\widehat{N}_1(u) = \left\lfloor \frac{1-\epsilon}{S_0} u^{2/\alpha} \right\rfloor + 1, \qquad \widetilde{N}_2(u) = \left\lfloor \frac{(\ln u)^{2/\beta}}{T_0} u^{2/\alpha} \right\rfloor + 1.$$

From the proof of Theorem 2.2(ii) of Dębicki *et al.* (2016) again, it is easy to show that (letting $q = du^{-2/\alpha}$)

$$\begin{aligned}\Sigma_2(u) + \sum_{i=0}^{\widehat{N}_1(u)} \mathbb{P}\Big(\sup_{(s,t)\in\overline{\Delta}_{i0}^T} X(s,t) > u \Big) &\geq \mathbb{P}\Big(\sup_{(s,t)\in I_1\times[T-\delta_u,T]} X(s,t) > u \Big) \\ &\geq \mathbb{P}\Big(\sup_{(kq,lq)\in I_1\times[T-\delta_u,T]} X(kq,lq) > u \Big) \\ &\geq \sum_{i=0}^{\widehat{N}_1(u)-1} \mathbb{P}\Big(\sup_{(kq,lq)\in\overline{\Delta}_{i0}^T} X(kq,lq) > u \Big) - \Sigma_3(u),\end{aligned}$$

where

$$\Sigma_2(u) = \sum_{i=0}^{\widehat{N}_1(u)} \sum_{j=1}^{\widetilde{N}_2(u)} \mathbb{P}\Big(\sup_{(s,t)\in\overline{\Delta}_{ij}^T} X(s,t) > u \Big) = o(\mu(u)),$$

$$\Sigma_3(u) = \sum_{0<i<i'<\widehat{N}_1(u)-1} \mathbb{P}\Big(\sup_{(s,t)\in\overline{\Delta}_{i0}^T} X(s,t) > u, \sup_{(s,t)\in\overline{\Delta}_{i'0}^T} X(s,t) > u \Big) = o(\mu(u))$$

as $u \to \infty$. We can also obtain the following results by Lemma 2.1 of Dębicki *et al.* (2016):

$$\mathbb{P}\Big(\sup_{(s,t)\in\overline{\Delta}_{i0}^T} X(s,t) > u \Big) \sim \mathbb{P}\Big(\sup_{(s,t)\in\overline{\Delta}_{i0}} \frac{\widetilde{\eta}(s,t)}{1+bt^\beta} > u \Big) \sim \mathcal{H}_{Y_1}^b[S_0,T_0]\Psi(u)$$

and

$$\begin{aligned}\sum_{i=0}^{\widehat{N}_1(u)} \mathbb{P}\Big(\sup_{(s,t)\in\overline{\Delta}_{i0}^T} X(s,t) > u \Big) &\sim \sum_{i=0}^{\widehat{N}_1(u)-1} \mathbb{P}\Big(\sup_{(s,t)\in\overline{\Delta}_{i0}^T} X(s,t) > u \Big) \\ &\sim \frac{(1-\epsilon)}{S_0} u^{\frac{2}{\alpha}} \mathcal{H}_{Y_1}^b[S_0,T_0]\Psi(u) \quad \text{as } u\to\infty,\end{aligned}$$

where $\{\widetilde{\eta}(s,t),\ (s,t) \in [0,\infty)^2\}$ is a centered homogeneous Gaussian random field with covariance function

$$r(s,t) = \exp(-(|a_1 s|^\alpha + |a_2 t - a_3 s|^\alpha)), \qquad (s,t)\in[0,\infty)^2.$$

Since $X(s,t)$ is homogeneous with respect to s, we have

$$\begin{aligned}0 &\leq \mathbb{P}\Big(\sup_{(s,t)\in I_1\times[T-\delta_u,T]} X(s,t) > u \Big) - \mathbb{P}\Big(\sup_{(kq,lq)\in I_1\times[T-\delta_u,T]} X(kq,lq) > u \Big) \\ &\leq \Sigma_2(u) + \sum_{i=0}^{\widehat{N}_1(u)} \mathbb{P}\Big(\sup_{(s,t)\in\overline{\Delta}_{i0}^T} X(s,t) > u \Big) \\ &\quad - \sum_{i=0}^{\widehat{N}_1(u)-1} \mathbb{P}\Big(\sup_{(kq,lq)\in\overline{\Delta}_{i0}^T} X(kq,lq) > u \Big) + \Sigma_3(u)\end{aligned}$$

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$$
\begin{aligned}
&= \Sigma_2(u) + \widehat{N}_1(u)\mathbb{P}\Big(\sup_{(s,t)\in\overline{\Delta}_{00}^T} X(s,t) > u\Big) \\
&\quad - (\widehat{N}_1(u)-1)\mathbb{P}\Big(\sup_{(kq,lq)\in\overline{\Delta}_{00}^T} X(kq,lq) > u\Big) + \Sigma_3(u) \\
&\le (\widehat{N}_1(u)-1)\mathbb{P}\Big(\sup_{(s,t)\in\overline{\Delta}_{00}} \frac{\widetilde{\eta}(s,t)}{1+bt^\beta} > u, \sup_{(kq,lq)\in\overline{\Delta}_{00}} \frac{\widetilde{\eta}(kq,lq)}{1+b(lq)^\beta} \le u\Big) \\
&\quad + \mathcal{H}_{Y_1}^b[S_0,T_0]\Psi(u) + o(\mu(u)).
\end{aligned}
$$

For the constants $a_1 > 0, a_2 > 0, a_3 \neq 0$, and $b > 0$, let (as in Section 2)

$$
Y(s,t) = \widetilde{B}_\alpha(a_1 s) + B_\alpha(a_2 t - a_3 s), \qquad \sigma_Y^2(s,t) = \mathrm{var}(Y(s,t)),
$$

and

$$
\mathcal{H}_Y^b[\lambda_1,\lambda_2](d) = \mathbb{E}\exp\Big(\max_{(kd,ld)\in[0,d\lambda_1]\times[0,d\lambda_2]} \sqrt{2}Y(kd,ld) - \sigma_Y^2(kd,ld) - b|ld|^\beta\Big) \in (0,\infty),
$$

where $\widetilde{B}_\alpha$ and B_α are two independent FBMs. By the same arguments as in the proof of Lemma 6.1 of Dębicki *et al.* (2016), we can show that

$$
\mathcal{M}_{Y,\alpha}^b(d) = \lim_{\lambda_1\to\infty}\lim_{\lambda_2\to\infty}\frac{1}{d\lambda_1}\mathcal{H}_Y^b[\lambda_1,\lambda_2](d) \in (0,\infty).
$$

Following the arguments of Lemma 12.2.7 of Leadbetter *et al.* (1983), we can show that $\lim_{d\to 0}\mathcal{M}_{Y,\alpha}^b(d) = \mathcal{M}_{Y,\alpha}^b$. Now, following the arguments of Lemma 6.1 of Dębicki *et al.* (2016) (see also the proof of Lemma 6.1 of Piterbarg (1996)), we have

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{(s,t)\in\overline{\Delta}_{00}} \frac{\widetilde{\eta}(s,t)}{1+bt^\beta} > u, \sup_{(kq,lq)\in\overline{\Delta}_{00}} \frac{\widetilde{\eta}(kq,lq)}{1+b(lq)^\beta} \le u\Big) \\
&\quad = \Psi(u)\int_0^{+\infty} \mathrm{e}^w \mathbb{P}\Big(\sup_{(s,t)\in[0,S_0]\times[0,T_0]} [\sqrt{2}Y(s,t) - \sigma_Y^2(s,t) - b|t|^\beta] > w, \\
&\qquad\qquad \sup_{(kq,lq)\in[0,S_0]\times[0,T_0]} [\sqrt{2}Y(kq,lq) - \sigma_Y^2(kd,ld) - b|lq|^\beta] \le w\Big) \\
&\qquad\qquad \times \mathrm{d}w(1+o(1)) \\
&\quad = \Psi(u)\Big(\mathcal{H}_Y^b\Big[\frac{S_0}{d},\frac{T_0}{d}\Big](d) - \mathcal{H}_Y^b[S_0,T_0]\Big)(1+o(1)) \quad \text{as } u\to\infty.
\end{aligned}
$$

Now, we can conclude that

$$
\begin{aligned}
&0 \le \mathbb{P}\Big(\sup_{(s,t)\in I_1\times[T-\delta_u,T]} X(s,t) > u\Big) - \mathbb{P}\Big(\sup_{(kq,lq)\in I_1\times[T-\delta_u,T]} X(kq,lq) > u\Big) \\
&\le (1-\epsilon)u^{2/\alpha}\Psi(u)\Big(\frac{\mathcal{H}_Y^b[S_0/d,T_0/d](d)}{S_0} - \frac{\mathcal{H}_Y^b[S_0,T_0]}{S_0}\Big) + \mathcal{H}_{Y_1}^b[S_0,T_0]\Psi(u) + o(\mu(u)) \\
&\le K(\mathcal{M}_{Y,\alpha}^b(d) - \mathcal{M}_{Y,\alpha}^b)\mu(u) \\
&=: K\rho(d)\mu(u),
\end{aligned}
$$

where $\rho(d)\to 0$ as $d\to 0$.

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(iii) *The $\beta = \alpha_2 > \alpha_1$ case.* This case can be proved as case (ii) by some obvious changes as follows. Let S_0 and T_0 be two positive constants and define

$$\widehat{\Delta}_i = [iS_0u^{-2/\alpha_1}, (i+1)S_0u^{-2/\alpha_1}], \qquad i = 0, 1, \ldots, \widehat{N}_1(u),$$
$$\widetilde{\Delta}_j = [jT_0u^{-2/\alpha_2}, (j+1)T_0u^{-2/\alpha_2}], \qquad j = 0, 1, \ldots, \widetilde{N}_2(u),$$
$$\overline{\Delta}_{ij} = \widehat{\Delta}_i \times \widetilde{\Delta}_j, \qquad \overline{\Delta}^T_{ij} = \widehat{\Delta}_i \times (T - \widetilde{\Delta}_j),$$

where

$$\widehat{N}_1(u) = \left\lfloor \frac{1-\epsilon}{S_0} u^{2/\alpha_1} \right\rfloor + 1, \qquad \widetilde{N}_2(u) = \left\lfloor \frac{(\ln u)^{2/\beta}}{T_0} u^{2/\alpha_2} \right\rfloor + 1.$$

Let $q_1 = du^{-2/\alpha_1}$ and $q_2 = du^{-2/\alpha_2}$, then repeating the proof of case (ii) by replacing kq and lq by kq_1 and lq_2, we obtain the desired result.

(iv) *The $\beta < \alpha_2 = \alpha_1$ case.* For simplicity let $\alpha := \alpha_2 = \alpha_1$ and $q = du^{-2/\alpha}$. Let us consider the Gaussian process $\{X(s, T),\ s \geq 0\}$. It is easy to check that $\{X(s, T),\ s \geq 0\}$ is a standard stationary Gaussian process, i.e. with mean 0 and variance 1. For the covariance function of $\{X(s, T),\ s \geq 0\}$, it holds that

$$r(s, T, s', T) = 1 - (a_1^\alpha + |a_3|^\alpha)|s - s'|^\alpha(1 + o(1))$$

uniformly with respect to $s, s' \in [0, S_0]$ as $|s - s'| \to 0$. For some constant $a > 0$, let

$$\mathcal{H}^a_\alpha[0, \lambda] = \mathbb{E}\exp\Big(\max_{ak\in[0,a\lambda]} \sqrt{2}B_\alpha(ak) - (ak)^\alpha\Big)$$

and define

$$\mathcal{H}_\alpha(a) = \lim_{\lambda\to\infty} \frac{\mathcal{H}^a_\alpha[0, \lambda]}{a\lambda} \in (0, +\infty).$$

Note that $\lim_{a\to 0} \mathcal{H}_\alpha(a) = \mathcal{H}_\alpha$; see, e.g. Leadbetter *et al.* (1983). So by Lemmas A.1 and A.2 (for the one-dimensional case), we have

$$\mathbb{P}\Big(\sup_{s\in I_1} X(s, T) > u\Big) = (1-\epsilon)(a_1^\alpha + |a_3|^\alpha)^{1/\alpha}\mathcal{H}_\alpha u^{2/\alpha}\Psi(u)(1 + o(1)), \tag{5.5}$$

$$\mathbb{P}\Big(\sup_{kq\in I_1} X(kq, T) > u\Big) = (1-\epsilon)(a_1^\alpha + |a_3|^\alpha)^{1/\alpha}\mathcal{H}_\alpha(d)u^{2/\alpha}\Psi(u)(1 + o(1)), \tag{5.6}$$

and

$$\Big|\mathbb{P}\Big(\sup_{s\in I_1} X(s, T) > u\Big) - \mathbb{P}\Big(\sup_{kq\in I_1} X(kq, T) > u\Big)\Big| \leq K\rho(d)u^{2/\alpha}\Psi(u) \quad \text{as } u \to \infty, \tag{5.7}$$

where $\rho(d) = \mathcal{H}_\alpha(d) - \mathcal{H}_\alpha$. By repeating the proof of Theorem 2.2(iv) of Dębicki *et al.* (2016), it is easy to show that

$$\mathbb{P}\Big(\sup_{(kq,lq)\in I_1\times[T-\delta_u,T]} X(kq, lq) > u\Big) = (1-\epsilon)(a_1^\alpha + |a_3|^\alpha)^{1/\alpha}\mathcal{H}_\alpha(d)u^{2/\alpha}\Psi(u)(1 + o(1)). \tag{5.8}$$

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Write

$$\begin{aligned}
&\Big|\mathbb{P}\Big(\sup_{(s,t)\in I_1\times[T-\delta_u,T]} X(s,t) > u\Big) - \mathbb{P}\Big(\sup_{(kq,lq)\in I_1\times[T-\delta_u,T]} X(kq,lq) > u\Big)\Big| \\
&\quad\le \Big|\mathbb{P}\Big(\sup_{(s,t)\in I_1\times[T-\delta_u,T]} X(s,t) > u\Big) - \mathbb{P}\Big(\sup_{s\in I_1} X(s,T) > u\Big)\Big| \\
&\qquad + \Big|\mathbb{P}\Big(\sup_{s\in I_1} X(s,T) > u\Big) - \mathbb{P}\Big(\sup_{kq\in I_1} X(kq,T) > u\Big)\Big| \\
&\qquad + \Big|\mathbb{P}\Big(\sup_{kq\in I_1} X(kq,T) > u\Big) - \mathbb{P}\Big(\sup_{(kq,lq)\in I_1\times[T-\delta_u,T]} X(kq_1,lq_2) > u\Big)\Big| \\
&\quad=: M_1 + M_2 + M_3, \qquad (5.9)
\end{aligned}$$

where $M_1 = o(\mu(u))$ by case (iv) of Theorem 2.1 and (5.5), $M_2 = K\rho(d)u^{2/\alpha}\Psi(u)$ by (5.7), and $M_3 = o(\mu(u))$ by (5.6) and (5.8) as $u \to \infty$.

(v) *The $\beta < \alpha_2$ and $\alpha_1 < \alpha_2$ cases.* The proof is the same as that of case (iv).

(vi) *The $\beta = \alpha_1 > \alpha_2$, $\beta < \alpha_1$, and $\alpha_2 < \alpha_1$ cases.* These two cases can be proved by the same arguments as for cases (iii) and (v) after some time scaling as in Dębicki *et al.* (2016), so we omit the details. □

Lemma 5.4. *Under the conditions of Theorem 3.1, we have*

$$\begin{aligned}
&\Big|\mathbb{P}\Big(\sup_{(s,t)\in\cup I_j\times[T-\delta_u,T]} X(s,t) \le u\Big) - \mathbb{P}\Big(\sup_{(kq_1,lq_2)\in\cup I_j\times[T-\delta_u,T]} X(kq_1,lq_2) \le u\Big)\Big| \\
&\quad\le K\rho(d)S\mu(u) \quad \text{as } u\to\infty. \qquad (5.10)
\end{aligned}$$

Proof. By Lemma 5.3, we have

$$\begin{aligned}
&\Big|\mathbb{P}\Big(\sup_{(s,t)\in\cup I_j\times[T-\delta_u,T]} X(s,t) \le u\Big) - \mathbb{P}\Big(\sup_{(kq_1,lq_2)\in\cup I_j\times[T-\delta_u,T]} X(kq_1,lq_2) \le u\Big)\Big| \\
&\quad\le S\max_j \Big|\mathbb{P}\Big(\sup_{(s,t)\in I_j\times[T-\delta_u,T]} X(s,t) \le u\Big) \\
&\qquad - \mathbb{P}\Big(\sup_{(kq_1,lq_2)\in I_j\times[T-\delta_u,T]} X(kq_1,lq_2) \le u\Big)\Big| \\
&\quad\le K\rho(d)S\mu(u),
\end{aligned}$$

which completes the proof. □

Lemma 5.5. *Under the conditions of Theorem 3.1, we have*

$$\begin{aligned}
&\Big|\mathbb{P}\Big(\sup_{(kq_1,lq_2)\in\cup I_j\times[T-\delta_u,T]} X(kq_1,lq_2) \le u\Big) - \prod_{j=1}^{\lfloor S\rfloor}\mathbb{P}\Big(\sup_{(kq_1,lq_2)\in I_j\times[T-\delta_u,T]} X(kq_1,lq_2) \le u\Big)\Big| \\
&\quad\to 0 \quad \text{as } u\to\infty.
\end{aligned}$$

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Proof. Applying Berman's inequality (see, e.g. Piterbarg (1996)), we have

$$\begin{aligned}
&\left|\mathbb{P}\Big(\sup_{(kq_1,lq_2)\in\cup I_j\times[T-\delta_u,T]} X(kq_1,lq_2)\le u\Big)-\prod_{j=1}^{\lfloor S\rfloor}\mathbb{P}\Big(\sup_{(kq_1,lq_2)\in I_j\times[T-\delta_u,T]} X(kq_1,lq_2)\le u\Big)\right| \\
&\quad=\left|\mathbb{P}\Big(\sup_{(kq_1,lq_2)\in\cup I_j\times[T-\delta_u,T]}\frac{X(kq_1,lq_2)}{\sigma(lq_2)}\le\frac{u}{\sigma(lq_2)}\Big)\right. \\
&\qquad\left.-\prod_{j=1}^{\lfloor S\rfloor}\mathbb{P}\Big(\sup_{(kq_1,lq_2)\in I_j\times[T-\delta_u,T]}\frac{X(kq_1,lq_2)}{\sigma(lq_2)}\le\frac{u}{\sigma(lq_2)}\Big)\right| \\
&\quad\le\sum_{j\ne j'}\sum_{\substack{(kq_1,lq_2)\in I_j\times[T-\delta_u,T]\\(k'q_1,l'q_2)\in I_{j'}\times[T-\delta_u,T]}}|r(kq_1,lq_2,k'q_1,l'q_2)|\exp\Big(-\frac{(\sigma^{-2}(lq_2)+\sigma^{-2}(l'q_2))u^2}{2(1+r(kq_1,lq_2,k'q_1,l'q_2))}\Big) \\
&\quad\le\sum_{j\ne j'}\sum_{\substack{(kq_1,lq_2)\in I_j\times[T-\delta_u,T]\\(k'q_1,l'q_2)\in I_{j'}\times[T-\delta_u,T]}}|r(kq_1,lq_2,k'q_1,l'q_2)|\exp\Big(-\frac{u^2}{1+r(kq_1,lq_2,k'q_1,l'q_2)}\Big).
\end{aligned}$$

Since $|kq_1-k'q_1|\ge\epsilon$ by definition, $r(kq_1,lq_2,k'q_1,l'q_2)\le\delta<1$. Set $\gamma<(1-\delta)/(1+\delta)$ and split the last sum into two parts W_1 and W_2 with $|kq_1-k'q_1|<S^\gamma$ and $|kq_1-k'q_1|\ge S^\gamma$, respectively. For the first sum, there are $S^{1+\gamma}/q_2^2$ combinations of two points $kq_1,k'q_1\in\cup_j I_j$. Together with the lq_2 combinations, there are $(S^{1+\gamma}/q_1^2)(\delta_u/q_2^2)$ terms in the sum W_1. Note that

$$S\mu(u)=O(1),\qquad u\to\infty,$$

which implies that for case (i),

$$u^2=2\ln S+\Big(\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-\frac{2}{\beta}-1\Big)\ln\ln S+O(1),$$

for cases (ii)–(v),

$$u^2=2\ln S+\Big(\frac{2}{\alpha_1}-1\Big)\ln\ln S+O(1),$$

for cases (vi) and (vii),

$$u^2=2\ln S+\Big(\frac{2}{\alpha_2}-1\Big)\ln\ln S+O(1).$$

Thus, W_1 is bounded by

$$\begin{aligned}
&\delta\frac{S^{1+\gamma}\delta^2(u)}{q_1^2q_2^2}\exp\Big(-\frac{u^2}{1+\delta}\Big) \\
&\quad\le\delta\exp\Big((1+\gamma)\ln S+\Big(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\Big)\ln\ln S-\frac{2(1+o(1))}{1+\delta}\ln S\Big) \\
&\quad=\delta\exp\Big((\ln S)\Big[(1+\gamma)-\frac{2(1+o(1))}{1+\delta}+\frac{(1/\alpha_1+1/\alpha_2)\ln\ln S}{\ln S}\Big]\Big) \\
&\quad\to 0\quad\text{as } S\to\infty,
\end{aligned}$$

since $1+\gamma<2/(1+\delta)$ by the choice of γ.

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For the second sum W_2 with $|kq_1 - k'q_1| \geq S^\gamma$, we use the fact that

$$\sup_{|kq_1 - k'q_1| \geq S^\gamma} r(kq_1, lq_2, k'q_1, l'q_2)(\ln S)^c = o(1) \quad \text{as } S \to \infty.$$

In this case, there are $(S/q_1)^2$ many combinations of two points $kq_1, k'q_1 \in \cup_i \boldsymbol{I}_i$. Hence, W_2 is bounded by

$$\begin{aligned} R(S) &:= \frac{o(1)}{(\ln S)^c} \frac{S^2}{q_1^2} \frac{\delta^2(u)}{q_2^2} \exp\left(-\frac{u^2}{1 + o(1)/\ln S}\right) \\ &\leq C \exp\left(2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln\ln S + \frac{4}{\beta} \ln\ln\ln S \right. \\ &\qquad \left. - (c-1) \ln\ln S - \frac{u^2}{1 + o(1)/\ln S}\right). \end{aligned}$$

For case (i), by Assumption 3.1, $c > 1$, we have

$$\begin{aligned} R(S) &\leq C \exp\left(2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln\ln S + \frac{4}{\beta} \ln\ln\ln S - (c-1) \ln\ln S \right. \\ &\qquad \left. - \frac{(1 + o(1))}{1 + o(1)/\ln S} \left[2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln\ln S\right]\right) \\ &\leq C \exp\left(-(c-1) \ln\ln S + \frac{4}{\beta} \ln\ln\ln S + o(1)\right) \to 0 \quad \text{as } S \to \infty, \end{aligned}$$

since $c > 1$.

For cases (ii) and (iii), noting that $c > 1$, we have

$$\begin{aligned} R(S) &\leq C \exp\left(2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln\ln S + \frac{4}{\beta} \ln\ln\ln S - (c-1) \ln\ln S \right. \\ &\qquad \left. - \frac{(1 + o(1))}{1 + o(1)/\ln S} \left[2 \ln S + \left(\frac{2}{\alpha_1} - 1\right) \ln\ln S\right]\right) \\ &\leq C \exp\left(\left(\frac{2}{\alpha_2} - \frac{2}{\beta}\right) \ln\ln S - (c-1) \ln\ln S + \frac{4}{\beta} \ln\ln\ln S + o(1)\right) \\ &\to 0 \quad \text{as } S \to \infty, \end{aligned}$$

since $\beta = \alpha_2$ and $c > 1$.

For cases (iv) and (v), noting that $c = 1$, we have

$$\begin{aligned} R(S) &\leq C \exp\left(2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln\ln S + \frac{4}{\beta} \ln\ln\ln S \right. \\ &\qquad \left. - \frac{(1 + o(1))}{1 + o(1)/\ln S} \left[2 \ln S + \left(\frac{2}{\alpha_1} - 1\right) \ln\ln S\right]\right) \\ &\leq C \exp\left(\left(\frac{2}{\alpha_2} - \frac{2}{\beta}\right) \ln\ln S + \frac{4}{\beta} \ln\ln\ln S + o(1)\right) \\ &\to 0 \quad \text{as } S \to \infty, \end{aligned}$$

since $\beta < \alpha_2$.

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For case (vi), we have

$$\begin{aligned} R(S) &\le C\exp\Bigg(2\ln S + \left(\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-\frac{2}{\beta}-1\right)\ln\ln S + \frac{4}{\beta}\ln\ln\ln S - (c-1)\ln\ln S \\ &\qquad - \frac{(1+o(1))}{1+o(1)/\ln S}\left[2\ln S + \left(\frac{2}{\alpha_2}-1\right)\ln\ln S\right]\Bigg) \\ &\le C\exp\left(\left(\frac{2}{\alpha_1}-\frac{2}{\beta}\right)\ln\ln S - (c-1)\ln\ln S + \frac{4}{\beta}\ln\ln\ln S + o(1)\right) \\ &\to 0 \quad \text{as } S\to\infty, \end{aligned}$$

since $\beta = \alpha_1$ and $c > 1$.

For case (vii), noting that $c = 1$, we have

$$\begin{aligned} R(S) &\le C\exp\Bigg(2\ln S + \left(\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-\frac{2}{\beta}-1\right)\ln\ln S + \frac{4}{\beta}\ln\ln\ln S \\ &\qquad - \frac{(1+o(1))}{1+o(1)/\ln S}\left[2\ln S + \left(\frac{2}{\alpha_2}-1\right)\ln\ln S\right]\Bigg) \\ &\le C\exp\left(\left(\frac{2}{\alpha_1}-\frac{2}{\beta}\right)\ln\ln S + \frac{4}{\beta}\ln\ln\ln S + o(1)\right) \\ &\to 0 \quad \text{as } S\to\infty, \end{aligned}$$

since $\beta < \alpha_1$. The proof of the lemma is complete. □

Proof of Theorem 2.1. Recall that $u = u(x) = a_S^{-1}x + b_S$. By the stationarity of $X(s,t)$ with respect to the first component, Lemma 5.1, Theorem 2.1, and the choice of a_S, b_S, we have

$$\begin{aligned} \prod_{j=1}^{\lfloor S\rfloor}\mathbb{P}\Big(\max_{(s,t)\in I_j\times[T-\delta_u,T]} X(s,t)\le u\Big) &\sim \exp\Big(-\lfloor S\rfloor\mathbb{P}\Big(\max_{(s,t)\in I_1\times[T-\delta_u,T]} X(s,t) > u\Big)\Big) \\ &\sim \exp(-\lfloor S\rfloor(1-\epsilon)\mu(u)) \\ &\to \exp(-\mathrm{e}^{-x}), \qquad \epsilon\downarrow 0,\ S\to\infty. \end{aligned}$$

Furthermore, by Lemmas 5.1–5.5, it holds that, as $S\to\infty$,

$$\begin{aligned} \mathbb{P}\Big(\max_{(s,t)\in[0,S]\times[0,T]} X(s,t)\le u\Big) &\sim \mathbb{P}\Big(\max_{(s,t)\in\cup_j I_j\times[T-\delta_u,T]} X(s,t)\le u\Big) \\ &\sim \mathbb{P}\Big(\max_{(kq_1,lq_2)\in\cup_j I_j} X(kq_1,lq_2)\le u\Big) \\ &\sim \prod_{j=1}^{\lfloor S\rfloor}\mathbb{P}\Big(\max_{(kq_1,lq_2)\in I_j\times[T-\delta_u,T]} X(kq_1,lq_2)\le u\Big). \end{aligned}$$

Therefore, the claim follows. □

Proof of Proposition 4.1. In Dębicki *et al.* (2016), it was shown that the standard deviation function of Z satisfies Assumption 2.1 and the correlation function of Z satisfies Assumption 2.2. It was also shown that Assumption 2.3 holds for Z. So, in order to prove this proposition, it

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suffices to show that Assumption 3.1 holds. For the correlation function $r_Z(s,t,s',t')$ of Z, we have

$$r_Z(s,t,s',t') = r_X(|s+t-s'-t'|) - r_X(|s-s'-t'|) - r_X(|s+t-s'|) + r_X(|s-s'|).$$

Since $r_X(t)$ is twice continuously differentiable in $(0,\infty)$, we have

$$|r_X(|s+t-s'-t'|) - r_X(|s-s'-t'|) - r_X(|s+t-s'|) + r_X(|s-s'|)| \le C\ddot{r}_X(s-s')$$

for $t, t' \in [0,T]$ as $s-s' \to \infty$. Now using the condition that $\ddot{r}_X(t)(\ln t)^c \to 0$ as $t \to \infty$, we show that Assumption 3.1 holds. □

Proof of Proposition 4.2. We check that Assumptions 2.1–2.3 and 3.1 hold. Using the stationarity of the increments of $X(t)$ and Assumption 4.4, it follows that the variance $\sigma_Z^2(s,t)$ of $Z(s,t)$ attains its maximum on $[0,T]$ at the unique point T, and, furthermore,

$$\sigma_Z(s,t) = \sigma_X(t) = 1 - b(T-t)^\beta(1+o(1)), \qquad t \uparrow T$$

holds for some $\beta, b > 0$.

Note that for the process $X(t)$ with stationary increments

$$\operatorname{cov}(X(t), X(s)) = \tfrac{1}{2}[\sigma_X^2(t) + \sigma_X^2(s) - \sigma_X^2(|t-s|)].$$

Thus, using the stationarity of the increments of $X(t)$ again, we have, for the correlation function of $Z(s,t)$,

$$r_Z(s,t,s',t') = \frac{1}{2\sigma_X(t)\sigma_X(t')}[-\sigma_X^2(|s+t-s'-t'|) + \sigma_X^2(|s-s'-t'|) + \sigma_X^2(|s-s'+t'|) - \sigma_X^2(|s-s'|)].$$

It follows from Assumption 4.5 that

$$r_Z(s,t,s',t') = 1 - \tfrac{1}{2}[(a|s+t-s'-t'|)^\alpha + (a|s-s'|)^\alpha](1+o(1))$$

as $t, t \to T$ and $|s-s'| \to 0$. Assumption 2.3 holds obviously. Thus, by Theorem 2.1, the first assertion of Proposition 4.2 holds.

By Taylor expansions, it is straightforward to verify that

$$|r_Z(s,t,s',t')| \le C\ddot{\sigma}_X^2(|s-s'|) \quad \text{as } |s-s'| \to \infty,$$

which combined with Assumption 4.6 implies Assumption 3.1. Thus, by Theorem 3.1, the second assertion holds. □

Appendix A

Let $\{\xi(\boldsymbol{t}) : \boldsymbol{t} \ge \mathbf{0}\}$ denote a two-dimensional homogeneous Gaussian field with covariance function

$$r_\xi(\boldsymbol{t}) = \operatorname{cov}(\xi(\boldsymbol{t}), \xi(\mathbf{0})).$$

Assume that the covariance function satisfies the following conditions.

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Assumption A.1. *There exists a nondegenerate matrix* $\mathbb{C}$ *such that*

$$r_\xi(\mathbb{C}\boldsymbol{t}) = 1 - |t_1|^{\alpha_1} - |t_2|^{\alpha_2} + o(|t_1|^{\alpha_1} + |t_2|^{\alpha_2}) \quad \text{as } \boldsymbol{t} \to 0$$

with $\alpha_i \in (0, 2]$.

Assumption A.2. *Assume that* $r_\xi(\boldsymbol{t}) < 1$ *for* $\boldsymbol{t} \neq \boldsymbol{0}$.

To state two key lemmas, we recall the following type of Pickands constant. For constant $a > 0$, *let*

$$\mathcal{H}_\alpha^a[0, \lambda] = \mathbb{E}\exp\Big(\max_{ak\in[0,a\lambda]} \sqrt{2}B_\alpha(ak) - (ak)^\alpha\Big)$$

and define

$$\mathcal{H}_\alpha(a) = \lim_{\lambda\to\infty} \frac{\mathcal{H}_\alpha^a[0, \lambda]}{a\lambda}.$$

We need the following results for the proofs of our main results.

Lemma A.1. *Let* $q_i = du^{-2/\alpha_i}$ *for some* $d > 0$ *and assume that Assumptions A.1 and A.2 hold. Then, for any fixed rectangle* $\boldsymbol{I}_h = [0, h_1] \times [0, h_2]$, *we have*

$$\mathbb{P}\Big(\max_{\boldsymbol{t}\in\boldsymbol{I}_h} \xi(\boldsymbol{t}) > u\Big) = h_1h_2\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}|\det\mathbb{C}^{-1}|u^{2/\alpha_1+2/\alpha_2}\Psi(u)(1 + o(1))$$

and

$$\mathbb{P}\Big(\max_{\boldsymbol{kq}\in\boldsymbol{I}_h} \xi(\boldsymbol{kq}) > u\Big) = h_1h_2\mathcal{H}_{\alpha_1}(d)\mathcal{H}_{\alpha_2}(d)|\det\mathbb{C}^{-1}|u^{2/\alpha_1+2/\alpha_2}\Psi(u)(1 + o(1))$$

as $u \to \infty$. *The results also hold for the cases* $h_1 = u^{-2/\alpha'}$ *and* $h_2 = u^{-2/\alpha''}$ *for* $\alpha' > \alpha_1$ *and* $\alpha'' > \alpha_2$, *respectively.*

Proof. The first and second assertions can be proved following the proof of Lemma 7.1 of Piterbarg (1996) with some obvious changes; see also the proof of Lemma 1 of Dębicki *et al.* (2015a). The third assertion follows from the proofs of the former two by using the double sums method; see the proof of Theorem 7.2 of Piterbarg (1996). □

Lemma A.2. *Let* $q_i = du^{-2/\alpha_i}$ *for some* $d > 0$ *and choose two constants* $\alpha' > \alpha_1$ *and* $\alpha'' > \alpha_2$. *Assume that Assumptions A.1 and A.2 hold. Then, for the rectangle* $\boldsymbol{I} = [0, u^{-2/\alpha'}] \times [0, u^{-2/\alpha''}]$, *we have*

$$\mathbb{P}\Big(\max_{\boldsymbol{kq}\in\boldsymbol{I}} \xi(\boldsymbol{kq}) \le u\Big) - \mathbb{P}\Big(\max_{\boldsymbol{t}\in\boldsymbol{I}} \xi(\boldsymbol{t}) \le u\Big) \le |\det\mathbb{C}^{-1}|\rho(d)u^{2/\alpha_1+2/\alpha_2-2/\alpha'-2/\alpha''}\Psi(u),$$

where $\rho(d) = \mathcal{H}_{\alpha_1}(d)\mathcal{H}_{\alpha_2}(d) - \mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2} \to 0$ *as* $d \to 0$.

Proof. This is an immediate consequence of Lemma 6.1. □

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