Macroeconomic Dynamics, **20**, 2016, 1640–1651. Printed in the United States of America.
doi:10.1017/S1365100514000947

GROWTH-MAXIMIZING PUBLIC DEBT UNDER CHANGING DEMOGRAPHICS

Nikola Bokan
Croatian National Bank

Andrew Hughes Hallett
George Mason University

Svend E. Hougaard Jensen
Copenhagen Business School

This paper develops an overlapping-generations model to study the growth-maximizing level of public debt under conditions of demograhic change. It is shown that the optimal debt level depends on a positive marginal productivity of public capital. In general, it also depends on the demographic parameters, but not if the government is not allowed to borrow to cover revenue shortfalls for current age-related spending. In that context, balanced budget rules are not an approriate form of fiscal rule. The implication is that a government facing demograhic change or demands for more welfare spending will have to adjust its fiscal plans to accommodate those changes, most likely downward, if growth is to be preserved. An advantage of this model is that it allows us to determine in advance the way in which fiscal policies need to adjust as demographic parameters change.

Keywords: Fiscal Rules, Public Debt, Demographic Changes

1. INTRODUCTION

Public debt and aging populations are the two of the most pressing longer-term problems facing all developed OECD economies. Recent debates between the conflicting policy recommendations of austerity vs. growth as strategies for reducing fiscal imbalances have generated a lot of heat, but not much light. This paper evaluates an alternative strategy: optimal debt targets, set to maximize growth and expressed as a function of population parameters and age-related spending. This allows us to assess whether the projected demographic changes call for even greater cuts in public debt than those already made necessary by the sovereign debt crisis, especially in Southern Europe.

We have argued elsewhere that debt targets are superior to deficit targets for both theoretical and practical reasons [Hughes Hallett and Jensen (2012)]. Clearly, if a

Address correspondence to: Svend E. Hougaard Jensen, Department of Economics, Copenhagen Business School, Porcelaenshaven 16A, DK-2000 Frederiksberg, Denmark; e-mail: shj.eco@cbs.dk.

© 2015 Cambridge University Press 1365-1005/15

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press

debt-targeting rule is preferred, the question then arises: What public debt level, or debt-to-GDP ratio, should be targeted? Is there such a thing as an optimal level of public debt? The answer to this may involve complicated trade-offs, such as how concerns about intergenerational equity should be balanced against economic performance and long-term fiscal sustainability [Auerbach (2009)].

The problem of choosing a target for public debt also involves the question of whether to account for implicit (future) liabilities. Typically, the government liabilities entering the calculation of public debt only include explicit liabilities. However, if the implicit liabilities originating from aging populations are ignored, the debt criterion will ignore the future revenues required to avoid default despite the need to cover the benefits promised to existing workers/beneficiaries. This is the case for extending debt-targeting rules to account for predictable demographic changes.

Several papers, including Kotlikoff (2006) and Davig et al. (2010), have emphasized the fiscal "overhang" posed by the uncovered expected financial liabilities associated with public pension schemes and likely health and social support costs in most OECD economies. A recent paper by the International Monetary Fund (2009) has put this problem into perspective by showing that the financial stress caused by the great financial crash of 2007–2010 was probably only about 10% of that likely to be caused by future age-related spending in economies with a shrinking labor force. Against that, if fiscal sustainability is the objective, it makes sense to search for fiscal rules capable of ensuring sustainable public finances given aging populations, shrinking labor forces and greater implicit liabilities.

This paper makes three contributions. First, drawing on the idea of fiscal space, we offer a formal evaluation of the optimal debt level around which the economy needs to stabilize.[1] Second, using an overlapping-generations model ad modum Yakita (2008), we study how the optimal debt levels vary as a function of the population parameters. Third, we show how fiscal rules designed to stabilize the economy around that debt level need to vary with the population age, life expectancy, birth rate, and rising welfare expenditures.

We find that balanced budget rules, the key to Europe's fiscal compact, are not appropriate. Instead, forward-looking fiscal rules are needed that allow for the liabilities implied by population parameters and age-related spending. The implication is that a government facing demograhic change or the need for higher social spending will have to adjust (and most probably restrict) its fiscal plans to accommodate those changes. A key theme of this paper is the need to make debt control forward-looking by designing a rule where the instrument of fiscal policy reacts, not only to changes in the existing level of debt, but also to anticipated changes in future liabilities caused by changing demographics.

In principle, a debt-targeting rule focused on the existing level of debt produces some forward-looking behavior because the debt stock is persistent: being off target today means being off target tomorrow and hence more correction today. However, such a rule will not allow for the additional burden caused by changes in demographics unless it is adjusted to do so, because these liabilities still lie in the

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press

future. And a fixed deficit limit rule, such as a balanced budget rule, completely fails to account for predictable future liabilities in either sense.

To fix that omission, we add these features using an overlapping-generations model to show the effect of population change on fiscal balances or debt. To our knowledge, there are only two models that determine optimal levels of debt in the literature: those of Aschauer (2000) and Aiyagari and McGrattan (1998). Building on those models, we examine public debt under balanced growth, and where optimal government debt is defined to be that level which maximizes growth.[2]

Given this, we then ask whether, and in what circumstances, deteriorating demographics will affect the net fiscal position; and, second, whether it is acceptable to allow larger debt burdens, or whether greater tax or spending austerity is always needed in such cases.

2. THE MODEL

The model that follows is based on Yakita's (2008) overlapping-generations model with population aging, and with public capital accumulation as the driving force behind economic growth. Specifically, we assume an overlapping-generations economy populated by homogenous individuals, symmetric firms, and a government. Individuals are assumed to have a lifespan divided into a working and a retirement period. In each period, a generation working in that period forms the young cohort. Their working lifetime is postulated to be of fixed length, whereas the retirement period is of uncertain length. At the end of each period a certain number of individuals die. We assume for simplicity that the individual is alive at the beginning of the next period with probability $(1-\lambda) \in (0, 1)$. This probability is the same for every individual. Individuals who survive into the second period of life are retired, and the generation of retirees forms the old cohort. With the population of young and working agents equal to N_t, the total population in period t is equal to $N_t + (1 - \lambda)N_{t-1}$.

2.1. Firms

We assume a large number of symmetric firms which produce a homogenous product by combining the services of private capital and labor. The production technology for each firm j is described by a constant-returns-to-scale production function with labor augmenting productivity,

$$Y_{j,t} = AK_{j,t}^{\alpha}(h_t L_{j,t})^{1-\alpha}, \tag{1}$$

where $Y_{j,t}$, $K_{j,t}$, $L_{j,t}$ denote respectively the output level, private capital stock, and labor inputs in firm j for period t and where $A > 1$ and h_t denote constants representing scale effects (TFP) and labor productivity that is not firm-specific, and $0 < \alpha < 1$. Assuming perfect competition in both the goods and private factor

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press

markets, and denoting r_t and w_t as the rental rate of private capital and wage rate, the first-order conditions of firm $j's$ profit maximization are

$$r_t = A\alpha \left(\frac{K_{j,t}}{L_{j,t}}\right)^{\alpha-1} (h_t)^{1-\alpha}, \tag{2}$$

$$w_t = A\,(1-\alpha) \left(\frac{K_{j,t}}{L_{j,t}}\right)^{\alpha} (h_t)^{1-\alpha}. \tag{3}$$

According to these two conditions, the marginal product of each factor of production, labor and private capital, is set equal to its price.

In line with other papers, including Kalaitzidakis and Kalyvitis (2004) and Yakita (2008), we assume that labor productivity is given by

$$h_t = \frac{K_t^{\beta} G_t^{1-\beta}}{L_t}, \tag{4}$$

where $0 < \beta < 1$, and where $K_t = \sum_j K_{j,t}$ stands for aggregate private capital stock, G_t the stock of public capital, and $L_t = \sum_j L_{j,t}$ the aggregate labor input. This specification of labor productivity implies the existence of positive externalities from aggregate public and private capital, G_t and K_t, onto production, with the result that the per capita labor input itself (i.e., stripped of productivity increases) is unity. This is the standard specification: the models of Arrow (1962), Aiyagari and McGrattan (1998), and Aschauer (2000) all arrive at the same formulation.

Last, the first-order conditions given in (2) and (3) imply that the private capital-to-labor ratio is the same across all firms. Therefore, at equilibrium, we have that $\frac{K_{j,t}}{L_{j,t}} = \frac{K_t}{L_t}$, which then allows us to rewrite (4) in aggregate terms,

$$Y_t = AK_t^{\alpha}(h_t L_t)^{1-\alpha} = AK_t^{\alpha+\beta(1-\alpha)} G_t^{(1-\beta)(1-\alpha)}, \tag{5}$$

where $Y_t = \sum_j Y_{jt}$. Defining $\omega = \alpha + \beta(1-\alpha)$, we rewrite the *aggregate* production function as

$$Y_t = AK_t^{\omega} G_t^{1-\omega}, \tag{6}$$

and the corresponding first-order conditions for private capital and labor inputs, like (2) and (3), as

$$r_t = A\alpha \left(\frac{K_t}{G_t}\right)^{\omega-1}, \tag{7}$$

$$w_t = A\,(1-\alpha) \left(\frac{K_t}{G_t}\right)^{\omega} \left(\frac{G_t}{L_t}\right). \tag{8}$$

To simplify the exposition and expressions that follow, we assume that neither private nor public capital depreciates over time. We define the public-to-private capital ratio to be $X_t = G_t/K_t$.

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press

2.2. Households

In this economy, young agents consume, work, and have children. If they survive into old age they retire and enjoy the savings made while young. To analyze the decisions in each period, we assume that the representative agent of generation t has preferences represented by the lifetime utility function

$$lnc_t + (1-\lambda)\rho \ln d_{t+1} + \varepsilon \ln n_t, \tag{9}$$

where c_t and d_{t+1} denote consumption in periods t and $t+1$; n_t the number of children; ε the priority (weight) with which having a certain number of children enters the utility function; and λ the probability that the individual will die before the next period, and $\rho \in (0, 1)$ is a time discount factor.

Each young agent is endowed with one unit of time for his working period (a unit per capita input). Young agents earn labor income w_t, which is allocated between consumption, savings, and tax payments determined by the rate of income tax, θ_t. They also receive a subsidy for the child-rearing time, on which taxes are also paid. For simplicity, we assume that all taxes are paid at the same rate as income tax and postulate the subsidy to be a constant fraction, ρ_w, of the wage rate w_t. The budget constraint of the young and working agents of generation t is therefore

$$(1-\theta_t)\left[w_t(1-zn_t) + \rho_w w_t z n_t\right] = c_t + s_t, \tag{10}$$

where $z > 0$ denotes rearing time per child, θ_t is the tax rate on labor, the subsidy, and capital incomes, and s_t denotes savings (purchases of annuity assets). In what follows it will be convenient to operate on the basis that $(1 - zn_t) > 0$.[3] We note that policy makers may well choose $\rho_w \to 0$.

As in Blanchard (1985), we assume the existence of an actuarially fair representative insurance company operating in the perfectly competitive market for insurance, which collects the savings from private agents and invests them in private capital and/or purchases of government bonds. This company repays the returns to the agents who survive into old age. Older individuals (if alive) will therefore receive an actuarially fair payment from their purchases of annuity assets, s_t, equal to $\frac{1+r_{t+1}}{(1-\lambda)}s_t$. Taking into account that income taxes have to be paid on those earnings, we can write the second period of life budget constraint as

$$\frac{1 + (1-\theta_{t+1})r_{t+1}}{(1-\lambda)}s_t = d_{t+1}. \tag{11}$$

The problem of the young individual is then to choose consumption while working, purchases of annuity assets/savings, and a number of children, all to maximize lifetime utility subject to the budget constraints (10) and (11). After rearranging the first-order conditions obtained with respect to c_t, s_t and n_t for the young agent, we obtain the conditions

$$\frac{d_{t+1}}{c_t} = \rho\left[1 + (1-\theta_{t+1})r_{t+1}\right] \tag{12}$$

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press

and

$$\frac{n_t}{c_t} = \frac{\varepsilon}{(1-\theta_t)(1-\rho_w)w_t z}. \tag{13}$$

Using those two conditions, we obtain optimal solutions for s_t and n_t:

$$s_t = \frac{(1-\lambda)\rho}{1+(1-\lambda)\rho+\varepsilon}(1-\theta_t)w_t \tag{14}$$

and

$$n_t = \frac{\varepsilon}{z(1-\rho_w)\left[1+(1-\lambda)\rho+\varepsilon\right]} \equiv n > 0. \tag{15}$$

Two conclusions follow. First, savings are a constant fraction of after-tax wage income. Second, the number of children will be constant over time, but increasing in ρ_w. Consequently, an implication of the model is that the economy will continue to survive, $n_t \geq 1$, so long as

$$\varepsilon \geq \frac{z(1-\rho_w)\left[1+(1-\lambda)\rho\right]}{1-z(1-\rho_w)}. \tag{16}$$

We assume this condition on ε to hold, given the parameter restrictions in the model. Societies do not *choose* to liquidate themselves voluntarily.

Notice that we do not assume $n_t \geq 1$ here. The assumption is instead on ε, the preference of young agents for children. The implication is that the model is not designed to deal with cases where $n_t < 1$, since then the population would shrink and there would be no steady state. The fact is that, with immigration, population growth is positive (if small in some places) almost everywhere, as data from the *World Development Indicators* show. Revealed preference therefore justifies (16) as an appropriate restiction.[4]

2.3. Government

The government collects tax receipts from wages, subsidies, and returns on savings. Specifically, in each period, it levies taxes at a rate of θ_t on wages w_t, subsidies, and savings s_{t-1}. It also issues public debt b_t and invests the proceeds in public capital G_t. The government's budget constraint is

$$\begin{aligned} b_{t+1} = (1+r_t)\,b_t + (G_{t+1}-G_t) + \rho_w w_t z n_t N_t - \theta_t(w_t L_t + \rho_w w_t z n_t N_t \\ + r_t s_{t-1} N_{t-1}). \end{aligned} \tag{17}$$

We assume that interest payments and public consumption are financed via taxes on wages, subsidies received, and savings income, whereas public debt is issued to finance public capital formation (the "golden rule" of public finance). Those assumptions imply that the following two conditions hold:

$$r_t b_t + \rho_w w_t z n_t N_t = \theta_t(w_t L_t + \rho_w w_t z n_t N_t + r_t s_{t-1} N_{t-1}) \tag{18}$$

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press

and

$$b_t = G_t. \tag{19}$$

3. SOLVING FOR THE OPTIMAL PUBLIC-TO-PRIVATE CAPITAL RATIO

The actuarially fair insurance company, and hence young workers, can invest in both private and public capital. Equilibrium in capital markets requires that

$$s_t N_t = K_{t+1} + G_{t+1}. \tag{20}$$

We can use conditions (18) and (19), and combine them with wages and capital rents, (2) and (3), to derive an expression for the income tax rate needed to satisfy (18):

$$\theta_t = 1 - \frac{1}{\alpha X + \rho_w (1-\alpha) zn/(1-zn) + 1}. \tag{21}$$

Using the equilibrium condition for capital markets in (20), the solution for s_t in (14), and the fact that $L_t = (1 - zn)N_t$ (that labor supply equals the time endowment less child rearing time), we get

$$\frac{(1-\lambda)\rho}{1+(1-\lambda)\rho+\varepsilon}(1-\theta_t) w_t N_t = K_{t+1} + G_{t+1}. \tag{22}$$

This equation determines the dynamic relationship between wages, working population, and public and private capital. Next we define the balanced growth path as a situation in which all variables grow at a constant rate. Let the aggregate growth rate be defined as

$$\frac{G_{t+1}}{G_t} = \frac{K_{t+1}}{K_t} = \frac{Y_{t+1}}{Y_t} = \gamma n \equiv \gamma^A, \tag{23}$$

where n is the constant growth rate of population. This implies that the ratio of public to private capital is constant in the steady state. It also implies that the tax rate θ_t and the interest rate r_t will be constant.

Note that higher subsidy rates and taxes lead to higher fertility, n, and higher debt, d^*, via (28), because ρ_w raises n by (15) and d^* by (28); and because ρ_w raises θ_t by (21).[5] But high interest rates reduce investment, as we might expect, because they raise θ_t and lower $K_t + G_t$, by (18) and (22).

Next we define $\tilde{C} = \frac{(1-\lambda)\rho}{1+(1-\lambda)\rho+\varepsilon}$ and then use (22), with (3), (18), and (21), to obtain a relationship from which we can derive a closed form expression for the economy's growth rate γ^A:

$$\tilde{C}\left(\frac{1}{\alpha X + \rho_w(1-\alpha)\frac{zn}{1-zn} + 1}\right) A(1-\alpha)\left(\frac{K_t}{G_t}\right)^{\omega} \frac{G_t}{L_t} N_t = K_{t+1} + G_{t+1}. \tag{24}$$

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press

This allows us to solve for the economy's aggregate growth rate (see Appendix A for the full solution):

$$\gamma^A = \frac{A\tilde{C}\,\frac{1-\alpha}{1-zn}}{X^{\omega-1}(X+1)(\alpha X+\tilde{C}+1)}. \tag{25}$$

Next, to derive the public-to-private capital ratio that maximizes the aggregate growth rate along the balanced growth, we take the first derivative of (25) with respect to the public-to-private capital ratio X, set it equal to zero, and solve for X. The result is a general solution of the form[6]

$$X_{1,2} = \frac{-\omega\left(\rho_w\frac{(1-\alpha)zn}{1-zn}+1+\alpha\right) \pm \sqrt{\omega^2\left(\rho_w\frac{(1-\alpha)zn}{1-zn}+1+\alpha\right)^2 - 4\alpha(1+\omega)(\omega-1)\left(\rho_w\frac{(1-\alpha)zn}{1-zn}+1\right)}}{2\alpha(1+\omega)}. \tag{26}$$

With our parameter restrictions, it is easy to show that the positive solution to this equation is also positive:[7]

$$X^* = \frac{-\omega\left(\rho_w\frac{(1-\alpha)zn}{1-zn}+1+\alpha\right) + \sqrt{\omega^2\left(\rho_w\frac{(1-\alpha)zn}{1-zn}+1+\alpha\right)^2 - 4\alpha(1+\omega)(\omega-1)\left(\rho_w\frac{(1-\alpha)zn}{1-zn}+1\right)}}{2\alpha(1+\omega)} > 0. \tag{27}$$

Given (6), the optimal debt-to-GDP ratio in this model is now

$$d^* = \frac{1}{A}(X^{*\omega}). \tag{28}$$

From (27) it follows that if the subsidy $\rho_w \to 0$, the optimal debt level becomes independent of any demographic parameters. Otherwise, given that $\omega<1$, the debt burden d^* will increase with ρ_w.[8]

4. POLICY IMPLICATIONS

We have noted that, if $\rho_w \to 0$, the optimal debt ratio becomes independent of the age-related parameters. Thus, if a government is not allowed to borrow to cover revenue shortfalls for age-related spending, the debt burden will not vary with demographic change. Otherwise, it will do so.

The policy lessons are as follows:

First, what matters is not how a government chooses to spend its resources, but how it chooses to finance that spending (whether it is tax-financed or bond-financed).

Second, the subsidies described in the preceding section can be interpreted in a number of different ways to cover the different types of age-related spending and/or social support in the OECD economies. The most important alternatives are the following:

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press

(a) Subsidies to raising children. For example, the United Kingdom provides child benefit payments that were taxed at standard rates until the 1980s and again from 2012. In fact, any grant that is either taxed or means-tested can be written as (10) under uniform grant/means test rates. Most tax codes, including that in the United States, offer a tax-free element per child that, in this context, is equivalent to a *net* income supplement of $\rho_w(1-\theta_t)wzn$ taxed at the standard rate (where ρ_w is set to make this expression equal to the tax saving). That then implies (10).
(b) Subsidies to education. The United States, for example, taxes certain training grants and fee waivers. Subsidized loans or means-tested tax reductions on fees operate the same way as child benefits, except that z now refers to time spent in education (childrearing by society, rather than by parents).
(c) Education more generally, where state spending per pupil is related to the average wage, but taxes (levied at standard rates on steady state earnings) fund that spending. z is now the proportion of the young population in state-funded schools.
(d) Health care costs, where caregivers are paid through a state subsidy, or where, as in the United States, those costs contain hidden subsidies (ρ_w).
(e) Sick pay. Most European Union countries pay a fraction of the wage for time off sick, but tax it as income.
(f) State pensions. These are taxed as income, where ρ_w is the pension replacement rate, and where $[(1-\lambda)/n]$ replaces zn from the government budget constaint (17) onward, n having been set in the previous generation.

Third, ρ_w itself may vary: (i) Debt becomes independent of age-related spending if $\rho_w \to 0$. (ii) We might expect d^* to increase with ρ_w, and it approaches its maximum value as ρ_w approaches 1. It is easy to check that this is indeed the case. (iii) We need to preserve the incentive to work; i.e., make the subsidy smaller than wages, $w(1-zn) > \rho_w wzn$. This bounds ρ_w from above: $\rho_w < \min\{(1-zn)/zn, 1\}$, which limits the debt burden, because the latter increases with ρ_w.

This leads to two additional implications: First, the optimal debt level depends, in general, on demographic parameters. However, this is not the case if governments are forbidden to borrow to cover current shortfalls for age-related/social spending. Second, these results are useful because they open the door to determining how a country's best fiscal position and debt burden are affected (increased or decreased) by demographic changes: for example, aging or the baby-boomer problem (variations in ρ); increasing longevity, falls in the death rate, or alterations in the retirement age (variations in λ); or changes in the demand for medical benefits, social support, or education spending (variations in ρ_w).

NOTES

1. The optimal debt level depends on the marginal productivity of public capital, which is defined as public spending on investment projects that (a) are productive, (b) have an identifiable rate of return, and (c) have a longer investment horizon than consumption expenditures. It includes education therefore. Conceptually this is clear-cut, but in practice there may be problems in measuring this separately from public consumption [Checherita et al. (2013)].

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press

2. There are empirical representations, however: Reinhart and Rogoff (2010) and Checherita and Rother (2012). Also, there are others where debt is defined to be unproductive: Greiner (2010). Aschauer (2000) considers a social optimum by maximizing the future stream of consumption spending, but finds that it makes little numerical difference. To keep things simple, we do not pursue that option. Auerbach (2009) contains a fuller discussion of the long-term objectives for public debt.

3. In fact, $(1-znt) > 0$ is guaranteed if the interval between the lower bound on ε in (16) and the upper bound on ε when $(1-znt) = 0$ is nonempty.This interval is always nonempty if $z < 1$, a restriction that has to hold because all agents cannot spend all of their allotted lifetime rearing children and still survive. The parameter restrictions of this model therefore guarantee that $(1-znt) > 0$.

4. Population growth may also be improved by mortality reductions (λ) as well as fertility decisions (n, ε).

5. For higher taxes, (15) and (18) show that $\partial n/\partial \rho_w.\partial \rho_w/\partial \theta > 0$. The intuition: the higher the tax rate, the more budget constraint (18) is eased, and the more will be spent on child allowances (among other things). More children then follow.

6. For a detailed derivation, see again Appendix A.

7. See Appendix B. We are only interested in the positive solution, because the capital stock ratio must remain nonnegative.

8. Note that if public capital becomes unproductive relative to private capital, then $(1 - \omega) \to 0$ and X^* and the optimal level of debt, d^* in (28), tend to zero, a result in line with Greiner (2010) and Checherita et al. (2013).

REFERENCES

Aiyagari, S. Rao and Ellen R. McGrattan (1998) The optimum quantity of debt. *Journal of Monetary Economics* 42, 447–469.

Arrow, Kenneth J. (1962) The economic implications of learning by doing. *Review of Economic Studies* 29, 155–173.

Aschauer, David A. (2000) Do states optimise? Public capital and economic growth. *Annals of Regional Science* 34, 343–363.

Auerbach, Alan (2009) Long-term objectives for government debt. *FinanzArchiv* 65, 472–501.

Blanchard, Olivier (1985) Debt, deficits, and finite horizons. *Journal of Political Economy* 93, 223–247.

Checherita, Christina, Andrew Hughes Hallett, and Philipp Rother (2013) Fiscal sustainability using growth-maximising debt targets. *Applied Economics* 46, 638–647.

Checherita, Cristina and Philipp Rother (2012) The impact of high and growing government debt on economic growth: An empirical investigation in the euro area. *European Economic Review* 56, 1392–1405.

Davig, T., E. Leeper, and T. Walker (2010) Unfunded liabilities and uncertain fiscal financing. *Journal of Monetary Economics* 57, 600–619.

Greiner, Alfred (2010) Economic growth, public debt and welfare: Comparing three budgetary rules. *German Economic Review* 12, 205–222.

Hughes Hallett, Andrew and Svend E. Hougaard Jensen (2012) Fiscal governance in the Eurozone: Institutions vs. rules. *Journal of European Public Policy* 19, 646–664.

International Monetary Fund (2009) Fiscal Implications of the Global and Financial Crisis. IMF Position Note SPN/09/13, 9 June.

Kalaitzidakis, Pantelis and Sarantis Kalyvitis (2004) On the macroeconomic implications of maintenance in public capital. *Journal of Public Economics* 88, 695–712.

Kotlikoff, L. (2006) Is the US bankrupt? *Federal Reserve Bank of St. Louis Economic Review* 88, 235–249.

Reinhart, Carmen M. and Kenneth S. Rogoff (2010) Growth in a time of debt. *American Economic Review* 100, 573–578.

Yakita, Akira (2008) Aging and public capital accumulation. *International Tax and Public Finance* 15, 582–598.

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press

APPENDIX A: THE SOLUTION TO THE OPTIMAL LEVEL OF DEBT PROBLEM

We start from (22), with (21), (3), and (18), to obtain an expression from which we can derive a closed form expression for the economy's growth rate γ^A,

$$\tilde{C}\left(\frac{1}{\alpha X+\rho_w(1-\alpha)\frac{zn}{1-zn}+1}\right)A(1-\alpha)\left(\frac{K_t}{G_t}\right)^{\omega}\frac{G_t}{L_t}N_t = K_{t+1}+G_{t+1}, \qquad \textbf{(A.1)}$$

where $\tilde{C}=\frac{(1-\lambda)\rho}{1+(1-\lambda)\rho+\varepsilon}$. This is equation (24) in the main text. Next we set $C=\rho_w(1-\alpha)\frac{zn}{1-zn}$, so that (A.1) becomes

$$A\tilde{C}\left(\frac{1}{\alpha X+C+1}\right)\frac{1-\alpha}{1-zn}\frac{1}{X^{\omega}} = \frac{K_{t+1}}{G_{t+1}}\frac{G_{t+1}}{G_t}+\frac{G_{t+1}}{G_t}. \qquad \textbf{(A.2)}$$

Using (23), we can now rewrite (A.2) as

$$A\tilde{C}\left(\frac{1}{\alpha X+C+1}\right)\frac{1-\alpha}{1-zn}\frac{1}{X^{\omega}} = \gamma^A\left(1+\frac{1}{X}\right).$$

This allows us to solve for the economy's aggregate growth rate, γ^A, to get

$$\gamma^A = \frac{A\tilde{C}\frac{1-\alpha}{1-zn}}{X^{\omega-1}(X+1)(\alpha X+C+1)}, \qquad \textbf{(A.3)}$$

which is the growth rate we need to optimize with respect to the public-to-private capital ratio.

To perform this optimization, define $C_3=AC\frac{1-\alpha}{1-zn}$. Take the first-order derivative of γ^A (denoted by $\hat{\gamma}$) with respect to X:

$$\hat{\gamma}(x) = -C_3\frac{(\omega-1)X^{\omega-2}\left[\alpha X^2+(\alpha+C+1)X+C+1\right]+X^{\omega-1}(2\alpha X+\alpha+C+1)}{\left[X^{\omega-1}(X+1)(\alpha X+C+1)\right]^2}.$$

Setting this expression equal to zero, we can write

$$\alpha(\omega-1)X^{\omega}+(\omega-1)(\alpha+C+1)X^{\omega-1}+(\omega-1)(C+1)X^{\omega-2}+2\alpha X^{\omega}+(\alpha+C+1)X^{\omega-1}=0,$$

which results in a quadratic equation to be solved for the optimal public-to-private capital ratio X:

$$\alpha(\omega+1)X^2+\omega(\alpha+C+1)X+(\omega-1)(C+1)=0.$$

The required solution is therefore

$$X_{1,2}=\frac{-\omega\left(\rho_w\frac{(1-\alpha)zn}{1-zn}+1+\alpha\right)\pm\sqrt{\omega^2\left(\rho_w\frac{(1-\alpha)zn}{1-zn}+1+\alpha\right)^2-4\alpha(1+\omega)(\omega-1)\left(\rho_w\frac{(1-\alpha)zn}{1-zn}+1\right)}}{2\alpha(1+\omega)},$$

which is (26) in the main text.

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press

APPENDIX B: THE POSITIVE ROOT

We now demonstrate that our solution for X^* is in fact positive. We start from (27) in the text. Because $\omega > 0, 0 < z < 1, n \geq 1, 0 < \alpha < 1$, it follows that $\omega\left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha\right) > 0$ and $2\alpha(1+\omega) > 0$. Therefore, to get a positive solution, $X^* > 0$, we need the numerator of (27) to be positive. It will be positive if and only if

$$-\omega\left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha\right) + \sqrt{\omega^2\left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha\right)^2 - 4\alpha(1+\omega)(\omega-1)\left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1\right)} > 0.$$

We therefore need to check if this condition is in fact satisfied for the parameter constraints that we have imposed. We require

$$\sqrt{\omega^2\left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha\right)^2 - 4\alpha(1+\omega)(\omega-1)\left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1\right)} > \omega\left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha\right)$$

or

$$\omega^2\left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha\right)^2 - 4\alpha(1+\omega)(\omega-1)\left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1\right) > \omega^2\left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 + \alpha\right)^2$$

or

$$-4\alpha(1+\omega)(\omega-1)\left(\rho_w \frac{(1-\alpha)zn}{1-zn} + 1\right) > 0,$$

which is true because $-4\alpha(1+\omega)(\omega-1) = -4\alpha(\omega^2-1) > 0$ and $\rho_w \frac{(1-\alpha)zn}{1-zn} + 1 > 0$ necessarily hold. Hence X^* in (27) is positive.

https://doi.org/10.1017/S1365100514000947 Published online by Cambridge University Press