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FAIR GAMBLER'S RUIN STOCHASTICALLY MAXIMIZES PLAYING TIME

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Abstract

For the gambler's ruin problem with two players starting with the same amount of money, we show the playing time is stochastically maximized when the games are fair.

Keywords: Stochastic inequality; playing time; gambler's ruin

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1. Introduction

For a simple random walk $S_n = \sum_{i=1}^{n} X_i$, where each X_i is 1 with probability p and -1 with probability $1-p$, we are interested in $T(p)$, the time until the random walk hits either k or $-k$, where k is a specified positive integer. It should be noted that $T(p)$ is the duration of the gambler's ruin when both players start with k. In [11] it was recently shown, by a direct computation, that $E[T(p)]$ is maximized when $p = 1/2$. In this paper we prove the stronger result that $T(p)$ is stochastically maximized when $p = 1/2$.

The gambler's ruin problem is one of the oldest problems in probability. As described in [10], the computation of each player's chance of winning was carried out by Pascal and Fermat and first published, together with four other problems, in [4], the first book published on probability theory. The study of the distribution of the duration of the game started with [7] and [8] and extended into modern times with [2], [5], [6], and the references therein. Applications of the problem are found in many modern settings, e.g. in computer science [3] and business [1].

2. Main result

In this section we state and prove our main result.

Theorem 1. *For any* $0 < p < 1$ *we have* $T(p) \leq_{st} T(1/2)$.

To prove the theorem we need some lemmas. Let $T_0(p)$ denote the first time (after time 0) when the walk revisits the state 0, and let

$$A_p = \{T_0(p) < T(p)\}$$

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be the event that the walk revisits 0, having started at 0, before the game ends. We can write

$$T(p) = Z(p) + \sum_{i=1}^{N(p)-1} Y_i(p),$$

where

$$Z(p) \sim \left(T(p)|A_p^c\right), \qquad Y_i(p) \sim \left(T_0(p)|A_p\right), \qquad N(p) \sim \text{Geometric}\left(P\left(A_p^c\right)\right)$$

are all independent variables. We will prove the following lemma, from which Theorem 1 follows immediately.

Lemma 1. *For any* $0 < p < 1$ *we have* $Y_i(p) \leq_{st} Y_i(1/2)$, $Z(p) \leq_{st} Z(1/2)$, *and* $N(p) \leq_{st} N(1/2)$.

To prove this lemma, let $u_i(p)$ be the chance that the walk goes up next when it is at level i given that it returns to 0 before the game ends, and let $v_i(p)$ be the chance that the walk goes up next when it is at level i given that the game ends before the walk returns to zero.

Lemma 2. *For* $0 < p < 1$ *and* $1 \leq i \leq k-2$ *we have* $u_i(p) \leq u_i\left(\frac{1}{2}\right)$, $v_i(p) \geq v_i\left(\frac{1}{2}\right)$, *and* $P\left(A_p^c\right) \geq P\left(A_{1/2}^c\right)$.

Proof of Lemma 2. Letting

$$P_i(p) = \begin{cases} \frac{1-\left(\frac{1-p}{p}\right)^i}{1-\left(\frac{1-p}{p}\right)^k}, & p \neq \frac{1}{2}, \\ i/k, & p = \frac{1}{2}, \end{cases}$$

from [9, Equation (4.14), p. 235] be the chance that a gambler reaches k before 0 starting with i, we have

$$u_i(p) = \frac{p\left(1 - P_{i+1}(p)\right)}{1 - P_i(p)}.$$

Making the substitution $n = k - i$, we have

$$u_i(1/2) = \frac{n-1}{2n}.$$

For $p \neq 1/2$, when we make the substitution $x = (1-p)/p$, we have

$$1 - P_i(p) = 1 - \frac{1-x^i}{1-x^k} = \frac{x^i - x^k}{1-x^k}$$

and $p = 1/(1+x)$, and we get

$$u_i(p) = \frac{1}{1+x}\frac{x^{i+1} - x^k}{x^i - x^k} = \frac{x^n - x}{(1+x)(x^n - 1)}.$$

We will thus have $u_i(p) \leq u_i(1/2)$ whenever

$$f(x) \equiv x^n - x - \frac{n-1}{2n}(x+1)\left(x^n - 1\right) \begin{cases} \leq 0 & \text{if } x > 1, \\ \geq 0 & \text{if } x < 1. \end{cases} \tag{1}$$

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Since

$$f'(x) = -\frac{(n+1)\left(x - x^{n+1} + n(x-1)x^n\right)}{2nx}$$

implies $f'(1) = 0$ and

$$f''(x) = \frac{1}{2}(n-1)(n+1)(1-x)x^{n-2} \begin{cases} < 0 & \text{if } x > 1, \\ > 0 & \text{if } x < 1, \end{cases}$$

this means $f(x)$ is monotone decreasing with $f(1) = 0$, and therefore (1) holds, implying $u_i(p) \le u_i(1/2)$.

Next, looking at the walk reflected across the origin gives $v_i(p) = 1 - u_{k-i}(1-p)$, and thus, by the reasoning above, $v_i(p) \ge v_i(1/2)$.

Finally, we have

$$P\left(A_p^c\right) = pP_1(p) + (1-p)P_1(1-p)$$

and thus

$$P\left(A_{1/2}^c\right) = 1/k,$$

and for $p \neq 1/2$, making the substitution $x = (1-p)/p$ gives

$$P\left(A_p^c\right) = \frac{1}{1+x}\frac{1-x}{1-x^k} + \frac{x}{1+x}\frac{1-x^{-1}}{1-x^{-k}} = \frac{(x-1)\left(x^k+1\right)}{(x+1)\left(x^k-1\right)}.$$

So we will have $P\left(A_p^c\right) \ge 1/k$ whenever

$$f(x) \equiv k(x-1)\left(x^k+1\right) - (x+1)\left(x^k-1\right) \begin{cases} \ge 0 & \text{if } x > 1, \\ \le 0 & \text{if } x < 1. \end{cases} \tag{2}$$

Since

$$f'(x) = -\frac{(k+1)\left((k(x-1)-x)x^k + x\right)}{x}$$

implies $f'(1) = 0$ and

$$f''(x) = k(k-1)(k+1)(x-1)x^{k-2} \begin{cases} > 0 & \text{if } x > 1, \\ < 0 & \text{if } x < 1, \end{cases}$$

this means $f(x)$ is monotone increasing with $f(1) = 0$ and therefore (2) holds, implying $P\left(A_p^c\right) \ge P\left(A_{1/2}^c\right)$. □

Proof of Lemma 1. We first construct two coupled random walks, both conditional on a revisit to the origin before the game ends, but where the first walk uses any $0 < p < 1$ and the second uses $p = 1/2$. We will let $Y(p)$ and $Y(1/2)$ respectively denote the first time the origin is revisited for each of the two conditioned walks. We let the first walk take the first step, and since the second walk is symmetric and has the same length distribution regardless of its first step, we let its first step be the same as that of the first walk. Suppose both the walks first step above the origin. Whenever the walks are at different levels, we let them step independently. When the walks are at the same level, by Lemma 2 the chance of a step away from the origin is

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at least as large for the second walk as it is for the first walk, so we can couple the steps so that the second walk always ends up no closer to the origin than the first walk. This means that in this case we will have $Y(p) \leq Y(1/2)$ almost surely. In the event that the walks first step below the origin, the same reasoning applies, because the distance to the origin for the first walk is the same as if the step had been above the origin, but with $1-p$ substituted for p. This shows that $Y(p) \leq_{st} Y(1/2)$.

A similar construction using Lemma 2 gives $Z(p) \leq_{st} Z(1/2)$, and $N(p) \leq_{st} N(1/2)$ follows from Lemma 2 since $P\big(A_p^c\big) \geq P\big(A_{1/2}^c\big)$. □

Proof of Theorem 1. By Lemma 2, we can couple $T(1/2)$ together with $T(p)$ on the same probability space so that $T(1/2) \geq T(p)$ almost surely, and the theorem follows. □

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