Random walks arising from a Fibonacci's-rabbits scenario

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1. *Introduction*

At the beginning of the 13th century, Fibonacci posed a question concerning the potential growth of a rabbit population under ideal circumstances [1]. A version of this question is as follows:

> 'Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month, and have a gestation period of one month, so that at the end of her second month a female produces a pair of rabbits. Assuming that our rabbits never die, and that each female always produces exactly one mixed pair (one male, one female) from the end of her second month onwards, how many pairs will there be in one year?'

The answer to this problem is F_{13}, and in general there will be F_{n+1} pairs after n months, where the nth Fibonacci number F_n is defined by way of the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$, with $F_1 = F_2 = 1$. It ought to be borne in mind that several variations of this question appear in the literature. Some, for example, lead to the answer $F_{12} = 144$.

We now extend the above scenario somewhat by supposing that there are two adjacent rabbit populations, P_1 and P_2 say, each occupying a large plot of grassland. Let us assume that P_1 and P_2 comprise distinct species and that no interbreeding takes place. The population P_1 is established by introducing one mixed pair of newly-born rabbits. Then, exactly one month later, P_2 is established in a similar manner. In this particularly simple case, P_1 and P_2 consist of F_{n+1} and F_n pairs, respectively, after n months.

At the beginning of each month, from the second month onwards, a skirmish takes place between P_1 and P_2, resulting in one of the populations gaining a piece of grassland at the other's expense. We will assume that the area of each piece of land either won or lost is always the same, and that the probability of P_1 gaining this unit of land is equal to the current proportion of the total rabbit population belonging to P_1.

The problem of calculating the expected gain in units of land by P_1 after n months may be couched in terms of a one-dimensional random walk on the integers $\mathbb{Z}$. This might typically be regarded as a discrete stochastic process defined by a sequence of random variables $\{S_n\}$, the nth term of which is given by

$$S_n = S_0 + \sum_{k=1}^{n} T_k,$$

where $S_0 \in \mathbb{Z}$ is the starting position and $\{T_n\}$ is a sequence of independent random variables, each term of which takes only values in $\{1, -1\}$. If the

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terms in $\{T_n\}$ are also identically distributed, so that, for some fixed p satisfying $0 < p < 1$, $\mathrm{P}(T_k = 1) = p$ and $\mathrm{P}(T_k = -1) = 1 - p$ for each $k \in \mathbb{N}$, $\{S_n\}$ is termed a *simple* random walk. Further details of such processes are to be found in [2]. Note, from this definition of a simple walk, that the random walks arising from the Fibonacci's-rabbits scenario described above are non-simple.

In the following section we generalise the situation somewhat by allowing the initial sizes of P_1 and P_2 each to be greater than one mixed pair. This will require the use of a class of generalised Fibonacci numbers known as the *gibonacci* numbers [3]. The nth gibonacci number G_n is defined by way of the recurrence $G_n = aF_{n-2} + bF_{n-1}$, where $G_n = G_{n-1} + G_{n-2}$ for $n \geqslant 3$, with $G_1 = a$ and $G_2 = b$ for some $a, b \in \mathbb{N}$. It is a straightforward matter to show that $G_n = aF_{n-2} + bF_{n-1}$ for $n \geqslant 1$, remembering that $F_0 = 0$ and $F_{-1} = 1$.

Both exact and asymptotic expressions for the expected gain in units of land by P_1 after n months will be derived for the general scenario. We then proceed to show that this particular family of non-simple random walks may be well approximated by simple ones.

2. *A non-simple random walk*

The non-simple Fibonacci random walk described in Section 1 may be simulated by way of the following procedure. We have an urn initially containing two discs of identical shape and size. One disc is black and the other is white. A marker is initially at position 0, and its first step is determined by picking out a disc at random. If the disc is black then the marker moves one unit in the positive direction to arrive at position 1, while if the disc is white it moves one unit in the negative direction to arrive at position −1. The disc is then put back in the urn.

In order to decide the direction of the second step, we add a black disc to the urn, so that it now contains one white disc and two black discs, and choose a disc from the urn. More generally, for the nth step, where $n \geqslant 2$, we add a further F_{n-2} white and F_{n-1} black discs to the urn. The direction is subsequently chosen by picking a disc at random from the urn, which now contains F_n white and F_{n+1} black counters. Let W_n denote the random variable giving the position of the marker after n steps.

Incidentally, it is worth noting that any simple random walk $\{S_n\}$ satisfies both

$$\mathrm{P}(S_n = j + q \mid S_0 = p + q) = \mathrm{P}(S_n = j \mid S_0 = p)$$

and

$$\mathrm{P}(S_{n+m} = j \mid S_m = p) = \mathrm{P}(S_n = j \mid S_0 = p).$$

These properties are known as *spatial* and *temporal homogeneity*, respectively [2]. The random walk $\{W_n\}$, on the other hand, only satisfies one of these. While $\{W_n\}$ is spatially homogeneous, the following example

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demonstrates the fact that it is not temporally homogeneous:

$$\mathrm{P}(W_1 = 1 \mid W_0 = 0) = \tfrac{1}{2} \qquad \text{and} \qquad \mathrm{P}(W_3 = 1 \mid W_2 = 0) = \tfrac{3}{5}.$$

We now generalise the process simulated above. Let $a, b \in \mathbb{N}$ be such that $b \geqslant a$. Suppose that P_1 and P_2 are established simultaneously. The former comprises initially a newly-born pairs of rabbits, and then one month later is bolstered by a further b pairs. The latter, on the other hand, starts with $b - a$ newly-born pairs of rabbits, and then one month later is bolstered by a further a pairs. It is easily checked that the numbers of pairs of rabbits in P_1 and P_2 after n months are given by G_{n+1} and G_n, respectively. Note that the Fibonacci scenario described in Section 1 corresponds to the case $a = b = 1$.

The purpose of the remainder of the current section is to obtain a relatively simple expression for the expected value $\mathrm{E}(W_n)$ of W_n, where the definition of this random variable has been extended to the general case. First, we have

$$W_n = \sum_{k=1}^{n} X_k.$$

where the random variable X_k has mass function given by

$$\mathrm{P}(X_k = 1) = \frac{G_{k+1}}{G_{k+2}} \qquad \text{and} \qquad \mathrm{P}(X_k = -1) = \frac{G_k}{G_{k+2}}.$$

Then, on utilising conditional expectation [2], we obtain

$$\begin{aligned}\mathrm{E}(W_{n+1}) &= \mathrm{E}(W_{n+1} \mid X_{n+1} = 1)\,\mathrm{P}(X_{n+1} = 1) + \mathrm{E}(W_{n+1} \mid X_{n+1} = -1)\,\mathrm{P}(X_{n+1} = -1) \\ &= \frac{G_{n+2}}{G_{n+3}}(\mathrm{E}(W_n) + 1) + \frac{G_{n+1}}{G_{n+3}}(\mathrm{E}(W_n) - 1).\end{aligned}$$

On rearranging, this gives the following recurrence for $n \geqslant 1$:

$$\mathrm{E}(W_{n+1}) = \mathrm{E}(W_n) + \frac{G_n}{G_{n+3}}. \tag{1}$$

Noting that $\mathrm{E}(W_1) = \dfrac{b - a}{a + b}$ and $G_0 = b - a$, it follows from (1) that

$$\mathrm{E}(W_n) = \sum_{k=1}^{n} \frac{G_{k-1}}{G_{k+2}}.$$

This result may also have been obtained by way of

$$\begin{aligned}\mathrm{E}(X_k) &= \mathrm{P}(X_k = 1) - \mathrm{P}(X_k = -1) \\ &= \frac{G_{k+1}}{G_{k+2}} - \frac{G_k}{G_{k+2}} \\ &= \frac{G_{k-1}}{G_{k+2}}\end{aligned}$$

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and

$$\mathrm{E}(W_n) = \mathrm{E}\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} \mathrm{E}(X_k).$$

3. *An asymptotic formula*

We obtain here an asymptotic formula for, and other results associated with, $\mathrm{E}(W_n)$. *Binet's formula* [4, 5] may be written as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2}$$

where α (the *golden ratio*) and β are the solutions of the equation $x^2 = x + 1$, and satisfy $\alpha + \beta = 1$, $\alpha\beta = -1$ and $\alpha - \beta = \sqrt{5}$.

Using (2), we obtain

$$\begin{aligned}\frac{G_n}{G_{n+3}} &= \frac{aF_{n-2} + bF_{n-1}}{aF_{n+1} + bF_{n+2}} \\ &= \frac{a(\alpha^{n-2} - \beta^{n-2}) + b(\alpha^{n-1} - \beta^{n-1})}{a(\alpha^{n+1} - \beta^{n+1}) + b(\alpha^{n+2} - \beta^{n+2})} \\ &= \frac{\alpha^{n-2}(a + b\alpha) - \beta^{n-2}(a + b\beta)}{\alpha^{n+1}(a + b\alpha) - \beta^{n+1}(a + b\beta)}.\end{aligned}$$

Then, after considerable further manipulation, we arrive at

$$\frac{G_n}{G_{n+3}} - \frac{1}{\alpha^3} = \frac{(-1)^{n+1}(a\alpha - b)(\alpha^6 + 1)}{\alpha^3(\alpha^{2n+3}(a + b\alpha) + (-1)^n(a\alpha - b))}.$$

Thus

$$\begin{aligned}\mathrm{E}(W_n) - \frac{n}{\alpha^3} &= \sum_{k=1}^{n} \frac{G_{k-1}}{G_{k+2}} - \frac{n}{\alpha^3} \\ &= \frac{(a\alpha - b)(\alpha^6 + 1)}{\alpha^3} \sum_{k=1}^{n} \frac{(-1)^k}{\alpha^{2n+3}(a + b\alpha) + (-1)^{k-1}(a\alpha - b)}.\end{aligned} \tag{3}$$

By way of the ratio test [6, 7] we know that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha^{2n+3}(a + b\alpha) + (-1)^{k-1}(a\alpha - b)}$$

converges, so we may write

$$\lim_{n \to \infty} \left(\mathrm{E}(W_n) - \frac{n}{\alpha^3}\right) = C(a, b) \tag{4}$$

for some $C(a, b) \in \mathbb{R}$ that is a function of both a and b. In fact, for any $\varepsilon > 0$ we have, by letting $a = F_{j-1}$ and $b = F_j$ for sufficiently large j, $|C(a, b)| < \varepsilon$. To see this, note that $\lim_{j \to \infty} (\alpha F_{j-1} - F_j) = 0$ and

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$\lim_{j \to \infty} \left(F_{j-1} + \alpha F_j\right) = \infty$, from which it follows that we can make the factor outside the sum in (3) as small as we like whilst increasing each of the denominators in the sum. It may also be shown that, for any $a, b \in \mathbb{N}$,

$$\left|C(a, b)\right| < \frac{\alpha^6 + 1}{\alpha^3(\alpha^4 - 1)} < 1. \tag{5}$$

There are several points worthy of note here. From (4) and (5) we see that not only is it true that

$$\mathrm{E}\left(W_n\right) \sim \frac{n}{\alpha^3},$$

but also that $\mathrm{E}\left(W_n\right)$ and n/α^3 remain close to each other in absolute terms. In fact, noting that $\alpha^2 = \alpha + 1$, it is a straightforward matter to show, for the simple random walk given by

$$S_n = \sum_{k=1}^{n} T_k$$

where $\mathrm{P}\left(T_k = 1\right) = 1/\alpha$ and $\mathrm{P}\left(T_k = -1\right) = 1/\alpha^2$ for each $k \in \mathbb{N}$, that

$$\mathrm{E}\left(S_n\right) = \frac{n}{\alpha^3}.$$

Thus, with respect to the mean, $\left(S_n\right)$ provides a good approximation to $\left(W_n\right)$ as n increases without limit (this discussion is continued in the following section). Furthermore, on examining (3), the absolute difference between $\mathrm{E}\left(W_n\right)$ and n/α^3 tends to be small when b/a is close to α, and when a and b are large. This situation arises when $a = F_{n-1}$ and $b = F_n$ for some large $n \in \mathbb{N}$.

Lastly, for a particular pair (a, b), the role $C(a, b)$ plays in (4) is analogous to that played by the Euler-Mascheroni constant γ [5, 8] in the asymptotic relation for the sum of the first n terms of the harmonic series, given by

$$\sum_{k=1}^{n} \frac{1}{k} \sim \log n.$$

It is not known whether either of γ or $C(a, b)$ is irrational (the latter for any choice of $a, b \in \mathbb{N}$).

4. *The variance of W_n*

In the previous section we showed that the expectations of the random variables corresponding to our non-simple random walk and a certain simple random walk remained close in absolute terms as n increased. In order to make further comparisons, we give here expressions for the variance of each of these random variables.

First, it is easy to obtain the result $\mathrm{Var}\left(S_n\right) = 4n/\alpha^3$, so let us go on to consider the calculation of $\mathrm{Var}\left(W_n\right)$. For a particular value of k, we use

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$\left(\sum G_{i_1}G_{i_2}\dots G_{i_n}\right)_k$ to denote the sum comprising all possible products $G_{i_1}G_{i_2}\dots G_{i_n}$ of n terms whose subscripts satisfy both

$$1 \leqslant i_1 \leqslant 2 \leqslant i_2 \leqslant 3 \leqslant i_3 \leqslant \dots \leqslant i_n \leqslant n+1$$

and

$$\sum_{m=1}^{n} i_m = \tfrac{1}{2}n(n+1) + k.$$

It may then be shown, by considering all possible walks consisting of exactly n steps, that

$$\mathrm{P}(W_n = n - 2k) = \begin{cases} \frac{\left(\sum G_{i_1}G_{i_2}\dots G_{i_n}\right)_k}{G_3G_4\dots G_{n+2}} & \text{for } k = 0,1,2,\dots,n \\ 0 & \text{otherwise} \end{cases}.$$

From this it follows that

$$\mathrm{Var}(W_n) = \frac{1}{G_3G_4\dots G_{n+2}} \sum_{k=0}^{n} (n-2k)^2 \left(\sum G_{i_1}G_{i_2}\dots G_{i_n}\right)_k - \left(\sum_{k=1}^{n} \frac{G_{k-1}}{G_{k+2}}\right)^2.$$

It would seem that there is no simple expression for $\mathrm{Var}(W_n)$. However, using computer algebra, we have obtained strong numerical evidence to suggest that

$$\mathrm{Var}(W_n) \sim \frac{4n}{\alpha^3}. \tag{6}$$

which would tell us that $\mathrm{Var}(W_n)$ and $\mathrm{Var}(S_n)$ are relatively close for large n. We may regard (W_n) as a random walk on the integers with 'drift'. This is by virtue of the fact that $\mathrm{E}(W_n)$ increases in a linear fashion with n (at least approximately), while (6) indicates that the standard deviation of W_n is roughly $\sqrt{4n/\alpha^3}$, implying that for large n we would expect W_n to be relatively close to the mean.

5. *An alternative walk*

Finally we consider a variation on the random walk (W_n) which is of interest since it involves a classical result concerning Diophantine representations of Fibonacci numbers. If $n = 2m - 1$ for some $m \in \mathbb{N}$ then the situation is as before, in the sense that the marker moves one unit in the positive direction if a black disc is drawn from the urn, and one unit in the negative direction if a white disc is drawn. However, if $n = 2m$ the marker now moves one unit in the negative direction if a black disc is drawn, and one unit in the positive direction if a white disc appears. Let Y_n be the position of the marker after the nth step. This time we have

$$Y_n = \sum_{k=1}^{n} (-1)^{k+1} X_k,$$

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and hence

$$\mathrm{E}\left(Y_n\right) = \sum_{k=1}^{n} \frac{(-1)^{k+1} G_{k-1}}{G_{k+2}}.$$

Next, for $m \geqslant 2$,

$$\begin{aligned}\mathrm{E}\left(Y_{2m-1}\right) &= \sum_{k=1}^{2m-1} \frac{(-1)^{k+1} G_{k-1}}{G_{k+2}} \\ &= \left(\frac{G_0}{G_3} - \frac{G_1}{G_4}\right) + \left(\frac{G_2}{G_5} - \frac{G_3}{G_6}\right) + \ \ldots + \left(\frac{G_{2m-4}}{G_{2m-1}} - \frac{G_{2m-3}}{G_{2m}}\right) + \frac{G_{2m-2}}{G_{2m+1}} \\ &= \frac{G_0 G_4 - G_1 G_3}{G_3 G_4} + \frac{G_2 G_6 - G_3 G_5}{G_5 G_6} + \ \ldots + \frac{G_{2m-4} G_{2m} - G_{2m-3} G_{2m-1}}{G_{2m-1} G_{2m}} + \frac{G_{2m-2}}{G_{2m+1}}.\end{aligned}$$

Then, using the result (see [9])

$$G_n G_{n+i+j} - G_{n+i} G_{n+j} = (-1)^n \left(G_0 G_{i+j} - G_i G_j\right)$$

for the special case n even, $i = 1$ and $j = 3$ we have

$$\begin{aligned}\mathrm{E}\left(Y_{2m-1}\right) &= \left((b-a)(a+2b) - a(a+b)\right)\left(\frac{1}{G_3 G_4} + \frac{1}{G_5 G_6} + \ \ldots + \frac{1}{G_{2m-1} G_{2m}}\right) + \frac{G_{2m-2}}{G_{2m+1}} \\ &= 2\left(b^2 - ab - a^2\right)\left(\frac{1}{G_3 G_4} + \frac{1}{G_5 G_6} + \ \ldots + \frac{1}{G_{2m-1} G_{2m}}\right) + \frac{G_{2m-2}}{G_{2m+1}},\end{aligned}$$

Then, for $m \geqslant 2$ once more,

$$\begin{aligned}\mathrm{E}\left(Y_{2m}\right) &= \mathrm{E}\left(Y_{2m-1}\right) - \frac{G_{2m-1}}{G_{2m+2}} \\ &= 2\left(b^2 - ab - a^2\right)\left(\frac{1}{G_3 G_4} + \frac{1}{G_5 G_6} + \ \ldots + \frac{1}{G_{2m-1} G_{2m}}\right) + \frac{G_{2m-2}}{G_{2m+1}} - \frac{G_{2m-1}}{G_{2m+2}} \\ &= 2\left(b^2 - ab - a^2\right)\left(\frac{1}{G_3 G_4} + \frac{1}{G_5 G_6} + \ \ldots + \frac{1}{G_{2m+1} G_{2m+2}}\right).\end{aligned}$$

On noting that

$$\sum_{k=1}^{\infty} \frac{1}{G_{2k+1} G_{2k+2}}$$

is a convergent series, and that

$$\lim_{m \to \infty} \frac{G_{2m-1}}{G_{2m+2}} = \frac{1}{\alpha^3},$$

we see that there exists some $D(a, b) \in \mathbb{R}$ such that

$$\lim_{m \to \infty} \mathrm{E}\left(Y_{2m-1}\right) = D(a, b) \qquad \text{and} \qquad \lim_{m \to \infty} \mathrm{E}\left(Y_{2m}\right) = D(a, b) - \frac{1}{\alpha^3}.$$

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An interesting point here is that, for $a, b \in \mathbb{N}$, the expression $|b^2 - ab - a^2|$ is minimised when aF_{j-1} and $b = F_j$ for some $j \in \mathbb{N}$, in which case $|b^2 - ab - a^2| = 1$, leading to a particularly simple form of the series for the expectations. This follows from a result in [10], which asserts that pairs of adjacent Fibonacci numbers, and only these, are to be found among the points with integer coordinates on the hyperbolas $y^2 - yx - x^2 = \pm 1$. Since $G_n = F_{j-1}F_{n-2} - F_jF_{n-1}$ in this case, and $F_{j-1}F_{n-2} + F_jF_{n-1} = F_{(j-1)+(n-1)}$ (again, see [9]), we have $G_n = F_{j+n-2}$ and thus

$$\mathrm{E}(Y_{2m}) = 2\left(\frac{1}{F_{j+1}F_{j+2}} + \frac{1}{F_{j+3}F_{j+4}} + \ldots + \frac{1}{F_{j+2m-1}F_{j+2m}}\right).$$

6. *Acknowledgement*

The author would like to thank an anonymous referee for correcting several typographical errors in the first draft of this article, and for providing comments and suggestions that have improved its clarity.

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doi: 10.1017/mag.2014.8

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https://doi.org/10.1017/mag.2014.8 Published online by Cambridge University Press