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Artinian modules over commutative rings

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0. *Introduction*

In [**5**], I provided a method whereby the study of an Artinian module A over a commutative ring R (throughout the paper, R will denote a commutative ring with identity) can, for some purposes at least, be reduced to the study of an Artinian module A' over a complete (Noetherian) local ring; in the latter situation, Matlis' duality [**1**] (alternatively, see [**6**], ch. 5) is available, and this means that the investigation can often be converted into a dual one about a finitely generated module over a complete (Noetherian) local ring.

However, that method is quite complicated, because it involves localization at the outset, and the module A' is, in general, different from the A with which one starts. The purpose of the present paper is to show, in a direct manner, that A (assumed henceforth to be non-zero) itself has a natural structure as a faithful (Artinian) module over a complete semi-local commutative Noetherian ring R' which is naturally determined by A, in such a way that a subset of A is an R-submodule if and only if it is an R'-submodule.

This result can be thought of as an Artinian analogue of the elementary fact that, if N is a Noetherian module over our commutative ring R, then N has a natural structure as a module over $R/(0:N)$, and the latter ring is Noetherian. For a long time, it had been a source of irritation to me that I did not know of any method which would reduce the study of an Artinian module over a commutative ring to algebra over a commutative Noetherian ring: [**5**] provided one method, and the present paper provides a simpler and more direct method.

The ring R' is obtained by completion of R with respect to a suitable topology and then factoring out an appropriate annihilator, as follows. There exist only finitely many maximal ideals M of R for which $\mathrm{Soc}(A)$, the socle of A, has a submodule isomorphic to R/M: let the distinct such maximal ideals be $M_1, \ldots, M_s$. Set $J := \bigcap_{i=1}^{s} M_i$; each element of A is annihilated by a power of J, and so A has a natural structure as a module over

$$\hat{R}^{(J)} := \lim_{\leftarrow n} R/J^n,$$

the J-adic completion of R. The ring R' referred to above is $\hat{R}^{(J)}/(0:_{\hat{R}^{(J)}} A)$: we shall show that this is a complete semi-local commutative Noetherian ring. (Incidentally, when we refer to a commutative semi-local Noetherian ring as 'complete', we shall mean that it is complete with respect to the topology defined by its Jacobson radical.)

There are roughly three stages in the proof presented below. Section 1 concerns an analogue of Matlis' duality which is valid over an Artinian ring; this could be

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approached by use of the fact that an Artinian ring is isomorphic to a direct product of finitely many Artinian local rings and appeal to the standard version of Matlis' duality, but, in fact, the necessary facts are very easy to obtain directly by use of length arguments.

We use this duality in Section 2, where the main result is proved in the special case in which $\mathrm{Soc}(A) \cong \bigoplus_{i=1}^{s} R/M_i$. The method here involves what at first sight appears to be a rather different completion of R, described as follows. For each $n \in \mathbb{N}_0$ (we use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers), set

$$I_n := (0 :_R (0 :_A J^n)).$$

Then it is easy to check that $I_n \supseteq I_{n+1}$ and $I_n I_k \subseteq I_{n+k}$ for all $n, k \in \mathbb{N}_0$, so that the family of ideals $(I_n)_{n \in \mathbb{N}_0}$ is a multiplicative filtration of R in the sense of [**4**, p. 408]. We shall define

$$\hat{R} := \lim_{\leftarrow n} R/I_n.$$

To prove the main result in this special case, we shall use an associated graded ring in conjunction with [**4**, ch. 9, theorem 13, p. 409] and the result [**7**, p. 260, corollary 4] that if a finitely generated ideal J'' of a commutative ring R'' (with identity) is such that R''/J'' is Noetherian and R'' is (Hausdorff) complete with respect to its J''-adic topology, then R'' is itself a Noetherian ring.

The final section involves reduction of the general case to that for the special case considered in Section 2: a use of the Artin–Rees Lemma is involved in the proof. Although the arguments in the final section are reminiscent of arguments in [**5**, section 3], it should be borne in mind that the approach in Section 2 below is rather different from work at the comparable stage in [**5**], and the main result of this paper gives a more direct method of study of an Artinian R-module than that provided by [**5**].

Throughout the paper, R will denote a (non-trivial) commutative ring with identity. Also, we shall use $\mathscr{C}(R)$ to denote the category of all R-modules and R-homomorphisms.

1. *Duality over Artinian rings*

In this section, we develop an analogue of 'Matlis' duality' which is valid over a commutative Artinian ring; while one could approach this by use of the fact that a commutative Artinian ring is isomorphic to a direct product of finitely many Artinian local rings and appeal to the standard version of Matlis' duality, the method chosen for the presentation below is a direct and simple one based on length arguments.

Notation and remarks 1·1. Suppose that R is Artinian, so that it has only finitely many distinct maximal ideals, $M_1, \ldots, M_s$, say. Set $S := \bigoplus_{i=1}^{s} R/M_i$ and $E := E(S)$, the injective envelope of S. We shall use D to denote the additive, exact, R-linear [**3**, p. 35] functor $\mathrm{Hom}_R(\ , E)$ from $\mathscr{C}(R)$ to itself.

(i) Recall that an R-module G is Artinian if and only if it is Noetherian, and that this is the case if and only if G is finitely generated.

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(ii) Observe that S is semi-simple: we shall make considerable tacit use of standard properties of semi-simple modules, such as those developed in [**6**, pp. 58–61]. Note in particular that, for $i,j \in \mathbb{N}$ with $1 \leqslant i,j \leqslant n$, the homomorphism

$$\mathrm{Hom}_R(R/M_i, S) \to \mathrm{Hom}_R(R/M_i, E)$$

induced by the inclusion map is an isomorphism, and that, if $i \neq j$, then

$$\mathrm{Hom}_R(R/M_i, R/M_j) = 0.$$

(iii) It follows easily from (ii), by induction on length, that, for every finitely generated R-module G, its 'dual' $D(G)$ is also finitely generated and $l(G) = l(D(G))$ (we use 'l' to denote lengths of modules). In particular, $E \cong D(R)$ is finitely generated.

LEMMA 1·2. *Assume that R is Artinian and use the notation of* 1·1. *For each R-module G, let*

$$\mu_G : G \to DD(G) = \mathrm{Hom}_R(\mathrm{Hom}_R(G, E), E)$$

be the natural R-homomorphism for which $(\mu_G(x))(f) = f(x)$ for all $x \in G$ and $f \in \mathrm{Hom}_R(G, E)$.

Then, as G varies through $\mathscr{C}(R)$, the μ_G constitute a morphism from the identity functor to the functor DD. Furthermore, μ_G is an isomorphism whenever G is finitely generated.

Proof. All the claims apart from that in the last sentence are completely straightforward to check. To prove the final claim, one only has to observe that E is an injective cogenerator [**6**, p. 46] for R, so that μ_G is injective for each R-module G, and then note that, when G is finitely generated, it is a consequence of 1·1 (iii) that G and $DD(G)$ have the same finite length.

COROLLARY 1·3. *Assume that R is Artinian and use the notation of* 1·1. *Then, for each $f \in \mathrm{Hom}_R(E, E)$, there is a unique $r_f \in R$ such that $f(x) = r_f x$ for all $x \in E$.*

Proof. By 1·2, the map $\mu_R : R \to \mathrm{Hom}_R(\mathrm{Hom}_R(R, E), E)$ is an isomorphism. The claim therefore follows from the fact that the composition

$$\phi : R \xrightarrow{\mu_R} \mathrm{Hom}_R(\mathrm{Hom}_R(R, E), E) \xrightarrow{\cong} \mathrm{Hom}_R(E, E),$$

in which the second isomorphism is the obvious natural one, is such that $\phi(r)$, for $r \in R$, is the endomorphism of E given by multiplication by r.

Our next result is an extension of [**5**, corollary 2.3] to an Artinian ring which is not necessarily local. The (brief) proof provided in [**5**], when modified in the obvious way, works in the more general situation, but we include the argument here for the convenience of the reader.

COROLLARY 1·4. *Assume that R is Artinian and use the notation of* 1·1. *Suppose that G is a faithful R-submodule of E. Then $G = E$.*

Proof. The canonical exact sequence $0 \to G \xrightarrow{\psi} E \to E/G \to 0$ induces an exact sequence

$$0 \longrightarrow D(E/G) \longrightarrow D(E) \xrightarrow{D(\psi)} D(G) \longrightarrow 0.$$

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By 1·3, each element of $D(E) = \mathrm{Hom}_R(E, E)$ is given by multiplication by an element of R, and since G is a faithful R-module, $D(\psi)$ is injective. Therefore $D(E/G) = 0$, so that, by 1·1 and 1·2, we have $E/G \cong DD(E/G) = 0$.

COROLLARY 1·5. *Assume that R is Artinian and use the notation of* 1·1. *Suppose that I_1, I_2 are ideals of R such that $I_1 \subseteq I_2$ and $(0 :_E I_1) = (0 :_E I_2)$. Then $I_1 = I_2$.*

Proof. This is an easy consequence of the results about duality in 1·2: apply the functor D to the canonical exact sequence

$$0 \to I_2/I_1 \to R/I_1 \to R/I_2 \to 0.$$

2. *A special case of the main result*

We begin by setting the scene for the remainder of the paper. The reader should note that we are no longer working under the hypothesis that the commutative ring R is Artinian.

Notation 2·1. Throughout the remainder of the paper, A will denote a non-zero Artinian R-module, and the following notation will be used.

Since $\mathrm{Soc}(A)$ is Artinian, there exist only finitely many maximal ideals M of R for which $\mathrm{Soc}(A)$ has a submodule isomorphic to R/M: let the distinct such maximal ideals be $M_1, \ldots, M_s$. Set $J := \bigcap_{i=1}^{s} M_i$ and

$$\hat{R}^{(J)} := \lim_{\leftarrow n} R/J^n,$$

the J-adic completion of R.

LEMMA 2·2. *Each non-zero element $x \in A$ is annihilated by some power of J. Hence A has a natural structure as a module over $\hat{R}^{(J)}$ in such a way that a subset of A is an R-submodule if and only if it is an $\hat{R}^{(J)}$-submodule.*

Proof. Note that $Rx \cong R/(0 : x)$ as R-modules, so that $R/(0 : x)$ is an Artinian ring with the property that each of its maximal ideals has the form $M_i/(0 : x)$ for an i with $1 \leqslant i \leqslant s$; hence $(J + (0 : x))/(0 : x)$ is contained in the (necessarily nilpotent) Jacobson radical of $R/(0 : x)$.

Let $g \in A$ and $\hat{r} = (r_n + J^n)_{n \in \mathbb{N}_0} \in \hat{R}^{(J)}$. Then the sequence $(r_n g)_{n \in \mathbb{N}_0}$ is ultimately constant, and it is easy to check that A has an $\hat{R}^{(J)}$-module structure for which $\hat{r}g$ is the ultimate constant value of the above sequence. This structure has the specified property.

LEMMA 2·3. *Let the situation be as in* 2·1, *so that A is a non-zero Artinian R-module. Then A is an essential extension of* $\mathrm{Soc}(A)$.

Proof. Let $0 \neq g \in A$. By 2·2, there exist $h \in \mathbb{N}$ and (not necessarily distinct) maximal ideals $M'_1, \ldots, M'_h$ of R such that $M'_1 \ldots M'_h g = 0$. Suppose that h is chosen as small as possible subject to these conditions. Then

$$0 \neq M'_1 \ldots M'_{h-1} g \subseteq \mathrm{Soc}(A).$$

Further notation 2·4. In addition to the notation introduced in 2·1, the following additional assumption and notation will be in force throughout the remainder of this section.

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We shall assume that $\mathrm{Soc}(A) \cong \bigoplus_{i=1}^{s} R/M_i$.

For each $n \in \mathbb{N}_0$, set $A_n = (0 :_A J^n)$ and

$$I_n := (0 :_R A_n) = (0 :_R (0 :_A J^n)).$$

In fact, $I_1 = J$, since I_1 must annihilate R/M_i for $i = 1, \ldots, s$.

Observe that $I_n \supseteq I_{n+1}$ for each $n \in \mathbb{N}_0$: we define

$$\hat{R} := \lim_{\leftarrow n} R/I_n.$$

For each $n \in \mathbb{N}_0$, let $\pi_n : R \to R/I_n$ denote the natural map, and let

$$\hat{I}_n := \{(\pi_j(r_j))_{j \in \mathbb{N}_0} \in \hat{R} : \pi_n(r_n) = 0\},$$

an ideal of $\hat{R}$. In addition, let $\hat{J} := \hat{I}_1$, the Jacobson radical of $\hat{R}$.

Remarks 2·5. Let the situation be as in 2·1 and 2·4.

(i) For each $n \in \mathbb{N}_0$, we have $J^n \subseteq I_n$, so that $(0 :_A I_n) \subseteq (0 :_A J^n)$; however $(0 :_A I_n) \supseteq (0 :_A J^n)$ by definition, and so

$$(0 :_A I_n) = (0 :_A J^n) = A_n.$$

(ii) Let $n, k \in \mathbb{N}_0$. Then $J^n I_k \subseteq I_{n+k}$ because

$$J^n I_k (0 :_A J^{n+k}) \subseteq I_k (0 :_A J^k) = 0.$$

Hence $J^{n+k} \subseteq J^n I_k \subseteq I_{n+k}$, and so it follows from (i) that

$$(0 :_A I_{n+k}) = (0 :_A J^n I_k) = (0 :_A J^{n+k}) = A_{n+k}.$$

(iii) Let $n, k \in \mathbb{N}_0$. It follows from (ii) that

$$I_n I_k (0 :_A J^{n+k}) = I_n I_k (0 :_A J^n I_k) \subseteq I_n (0 :_A J^n) = 0.$$

Hence $I_n I_k \subseteq I_{n+k}$. Therefore $(I_n)_{n \in \mathbb{N}_0}$ is a multiplicative filtration of R (in the sense of [**4**, p. 408]), so that $(\hat{I}_n)_{n \in \mathbb{N}_0}$ is a multiplicative filtration of $\hat{R}$ with respect to which $\hat{R}$ is complete.

(iv) It also follows from (ii) that, for $n, k, l \in \mathbb{N}_0$, we have

$$(0 :_{A_{n+k+l}} I_{n+k}) = (0 :_{A_{n+k+l}} J^n I_k) = (0 :_{A_{n+k+l}} J^{n+k}) = A_{n+k}$$

simply because $J^{n+k+l} \subseteq J^{n+k}$. Consequently, when A_{n+k+l} is regarded as an R/I_{n+k+l}-module in the natural way,

$$\begin{aligned}(0 :_{A_{n+k+l}} I_{n+k}/I_{n+k+l}) &= (0 :_{A_{n+k+l}} (J^n I_k + I_{n+k+l})/I_{n+k+l}) \\ &= (0 :_{A_{n+k+l}} (J^{n+k} + I_{n+k+l})/I_{n+k+l}).\end{aligned}$$

(v) Next, note that ($I_0 = R$ and), for all $n \in \mathbb{N}$, the ring R/I_n is Artinian: to see this, note that A_n has a natural structure as a faithful Artinian R/I_n-module, and, since A_n is annihilated by a power of a product of maximal ideals of R/I_n, it has finite length over this ring; therefore the elementary result about Noetherian modules mentioned in the third paragraph of the Introduction shows that R/I_n is Noetherian; and since $J^n \subseteq I_n$, it follows that R/I_n is Artinian.

(vi) Finally, we point out that A has a natural structure as an $\hat{R}$-module: for $g \in A$, we have seen in 2·2 that there exists $n \in \mathbb{N}$ such that $J^n g = 0$, so that, by (ii),

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$I_n g = 0$; hence, for $\hat{r} = (\pi_n(r_n))_{n\in\mathbb{N}_0} \in \hat{R}$, we have $r_m g = r_n g$ for all $m \geqslant n$, and we can obtain a structure of $\hat{R}$-module on A for which $\hat{r}g := r_n g$.

It should be noted that, with this $\hat{R}$-module structure, a subset of A is an R-submodule if and only if it is an $\hat{R}$-submodule, so that A is again an Artinian $\hat{R}$-module.

Furthermore, A is faithful as $\hat{R}$-module, since if

$$\hat{r} = (\pi_n(r_n))_{n\in\mathbb{N}_0} \in (0 :_{\hat{R}} A),$$

then the above definition of the action of $\hat{r}$ on A means that $r_n \in (0 :_R A_n)$ for all $n \in \mathbb{N}_0$, and so $\hat{r} = 0$.

LEMMA 2·6. *Let the situation be as in* 2·1 *and* 2·4.

Then, for all $n, k \in \mathbb{N}_0$,

$$I_{n+k}/I_{n+k+1} = (J^n I_k + I_{n+k+1})/I_{n+k+1} = (J^{n+k} + I_{n+k+1})/I_{n+k+1},$$

and

$$\hat{I}_{n+k}/\hat{I}_{n+k+1} = (\hat{J}^n \hat{I}_k + \hat{I}_{n+k+1})/\hat{I}_{n+k+1} = (\hat{J}^{n+k} + \hat{I}_{n+k+1})/\hat{I}_{n+k+1}.$$

Proof. Set $S := \operatorname{Soc}(A)$ and $E := E(S)$, the injective envelope of the R-module S. By 2·3, we can, and do, regard A as a submodule of E. Let $n \in \mathbb{N}$, and note that the maximal ideals of R/I_n are precisely $M_1/I_n, \ldots, M_s/I_n$. Set $E_n := (0 :_E I_n)$. Now it follows easily from the natural equivalence of functors

$$\operatorname{Hom}_R(\quad \otimes_{R/I_n} R/I_n, E) \to \operatorname{Hom}_{R/I_n}(\quad, \operatorname{Hom}_R(R/I_n, E))$$

of [3, ch. 8, theorem 4, p. 165] that there is an isomorphism of R/I_n-modules

$$E_n \cong E_{R/I_n}\left(\bigoplus_{i=1}^{s} (R/I_n)/(M_i/I_n)\right).$$

Since A_n is a faithful R/I_n-submodule of E_n, it follows from 1·4 and 2·5(v) that $A_n = E_n$. Note that this is true for all $n \in \mathbb{N}$.

Now let $n, k \in \mathbb{N}_0$. We have

$$I_{n+k}/I_{n+k+1} \supseteq (J^n I_k + I_{n+k+1})/I_{n+k+1} \supseteq (J^{n+k} + I_{n+k+1})/I_{n+k+1}$$

by 2·5(ii), while 2·5(iv) shows that

$$\begin{aligned}(0 :_{E_{n+k+1}} I_{n+k}/I_{n+k+1}) &= (0 :_{E_{n+k+1}} (J^n I_k + I_{n+k+1})/I_{n+k+1}) \\ &= (0 :_{E_{n+k+1}} (J^{n+k} + I_{n+k+1})/I_{n+k+1})\end{aligned}$$

in view of the preceding paragraph in this proof. It therefore follows from 1·5 that

$$I_{n+k}/I_{n+k+1} = (J^n I_k + I_{n+k+1})/I_{n+k+1} = (J^{n+k} + I_{n+k+1})/I_{n+k+1},$$

and the final claim follows from the observation that, under the natural isomorphism $R/I_{n+k+1} \to \hat{R}/\hat{I}_{n+k+1}$, the image of $(J^n I_k + I_{n+k+1})/I_{n+k+1}$ is $(\hat{J}^n \hat{I}_k + \hat{I}_{n+k+1})/\hat{I}_{n+k+1}$.

THEOREM 2·7. *Let the situation be as in* 2·1 *and* 2·4. (*In particular, recall that* $\operatorname{Soc}(A) \cong \bigoplus_{i=1}^{s} R/M_i$.) *Then*

(i) $\hat{J}^n = \hat{I}_n$ *for all* $n \in \mathbb{N}$, *so that* $\hat{R}$ *is complete with respect to the topology defined by its Jacobson radical*;

(ii) $\hat{J}$ *is a finitely generated ideal of* $\hat{R}$; *and*

(iii) $\hat{R}$ *is Noetherian.*

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Proof. (i), (ii) Let $n \in \mathbb{N}$. Consider the ideal

$$G(\hat{I}_n) := (\hat{I}_n/\hat{I}_{n+1}) \oplus (\hat{I}_{n+1}/\hat{I}_{n+2}) \oplus (\hat{I}_{n+2}/\hat{I}_{n+3}) \oplus \ldots \oplus (\hat{I}_{n+k}/\hat{I}_{n+k+1}) \oplus \ldots$$

of the graded ring

$$G(\hat{R}) := (\hat{R}/\hat{I}_1) \oplus (\hat{I}_1/\hat{I}_2) \oplus (\hat{I}_2/\hat{I}_3) \oplus \ldots \oplus (\hat{I}_k/\hat{I}_{k+1}) \oplus \ldots$$

associated with the filtration $(\hat{I}_n)_{n \in \mathbb{N}_0}$ of $\hat{R}$. By 2·6,

$$\hat{I}_{n+k}/\hat{I}_{n+k+1} = (\hat{J}^n \hat{I}_k + \hat{I}_{n+k+1})/\hat{I}_{n+k+1}$$

for all $k \in \mathbb{N}_0$, and so, since $\hat{R}/\hat{I}_{n+1}$ is an Artinian ring by 2·1 (v), we deduce that there exist finitely many elements of $\hat{J}^n$ whose natural images in $(\hat{I}_n/\hat{I}_{n+1})$ generate $G(\hat{I}_n)$ as an ideal of $G(\hat{R})$. It therefore follows from [**4**, ch. 9, theorem 13, p. 409] that $\hat{J}^n = \hat{I}_n$, and this is a finitely generated ideal of $\hat{R}$.

(iii) In view of parts (i) and (ii), this now follows from [**7**, p. 260, corollary 4] since $\hat{R}/\hat{J} = \hat{R}/\hat{I}_1$ is an Artinian ring by 2·5 (v).

COROLLARY 2·8. *Let the situation be as in* 2·1 *and* 2·4. (*In particular, recall that* $\operatorname{Soc}(A) \cong \bigoplus_{i=1}^{s} R/M_i$.)

Since $J^n \subseteq I_n$ *for all* $n \in \mathbb{N}_0$ *by* 2·5 (i), *there is a natural ring homomorphism*

$$\theta : \hat{R}^{(J)} = \varprojlim_n R/J^n \longrightarrow \hat{R} = \varprojlim_n R/I_n.$$

Then θ *is surjective and* $\operatorname{Ker}\theta = (0 :_{\hat{R}^{(J)}} A)$. *Hence*

$$\hat{R}^{(J)}/(0 :_{\hat{R}^{(J)}} A) \cong \hat{R},$$

and so is a complete semi-local Noetherian ring with exactly s *maximal ideals.*

Proof. By 2·6, we have $I_n = J^n + I_{n+1}$ for all $n \in \mathbb{N}$, and it is straightforward to use this in an inductive argument to see that θ is surjective. The remaining claims follow easily from the facts that A is a faithful $\hat{R}$-module (by 2·5 (vi)) and that if we regard the $\hat{R}$-module A as an $\hat{R}^{(J)}$-module by means of θ, then we obtain precisely the $\hat{R}^{(J)}$-module structure given in 2·2.

3. *The general case*

Our strategy in this section is to use a generalization of [**5**, (3.4)] in order to reduce the proof of our main theorem in the general case to a situation where we can apply the results of Section 2.

LEMMA 3·1. *Let the situation be as in* 2·1. *Then there exist* $t \in \mathbb{N}$ *and submodules* $A_1, \ldots, A_t$ *of* A *such that* $0 = \bigcap_{j=1}^{t} A_j$ *and, for each* $j = 1, \ldots, t$, *the Artinian* R*-module* A/A_j *satisfies*

$$\operatorname{Soc}(A/A_j) \cong \bigoplus_{i=1}^{s} R/M_i.$$

Proof. Recall from 2·1 that $M_1, \ldots, M_s$ are distinct maximal ideals of R. By their definition, there exist $t_1, \ldots, t_s \in \mathbb{N}$ and simple R-modules S_{ij} $(1 \leqslant j \leqslant t_i,\ 1 \leqslant i \leqslant s)$ such that

$$\operatorname{Soc}(A) = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{t_i} S_{ij}$$

and $S_{ij} \cong R/M_i$ for all $j = 1, \ldots, t_i$ and $i = 1, \ldots, s$.

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For each $j = 1, \ldots, t_i$ and $i = 1, \ldots, s$, let x_{ij} be a generator for the simple R-module S_{ij}. Let $t := \max\{t_1, \ldots, t_s\}$. For each $i = 1, \ldots, s$ and $j = 1, \ldots, t$, let $j(i) := \min\{j, t_i\}$. For each $j = 1, \ldots, t$, let $\mathscr{T}_j$ be the set of all submodules of A which contain none of $x_{1,j(1)}, \ldots, x_{s,j(s)}$ but do contain all the other x_{ik}: by Zorn's Lemma, $\mathscr{T}_j$ has a maximal member A_j with respect to inclusion. We shall show that $A_1, \ldots, A_t$ have the desired properties.

Set $B := \bigcap_{i=1}^{t} A_i$. We suppose that $B \neq 0$ and look for a contradiction. Then $\mathrm{Soc}(B)$ is a non-zero submodule of the (semi-simple) R-module $\mathrm{Soc}(A)$, and so there exist a $k \in \mathbb{N}$ with $1 \leqslant k \leqslant s$ and $a_1, \ldots, a_{t_k} \in R$ with $a_j \notin M_k$ for some j satisfying $1 \leqslant j \leqslant t_k$ such that

$$a_1 x_{k1} + \ldots + a_{t_k} x_{kt_k} \in (0 :_B M_k).$$

But $B \subseteq A_j$ and $x_{k1}, \ldots, x_{k,j-1}, x_{k,j+1}, \ldots, x_{kt_k} \in A_j$ (note that $j(k) = j$). It therefore follows that $x_{kj} \in A_j$, that is $x_{k,j(k)} \in A_j$. This contradiction shows that $B = \bigcap_{i=1}^{t} A_i = 0$.

Next, consider an integer j such that $1 \leqslant j \leqslant t$. Our next aim is to show that $\mathrm{Soc}(A/A_j) \cong \bigoplus_{i=1}^{s} R/M_i$. We first note that

$$x_{1,j(1)} + A_j, \ldots, x_{s,j(s)} + A_j$$

are non-zero elements of $\mathrm{Soc}(A/A_j)$ whose annihilators are $M_1, \ldots, M_s$ respectively. It is therefore sufficient for us to show that these elements generate $\mathrm{Soc}(A/A_j)$, and this is what we shall do. Suppose, on the contrary, that they generate a proper submodule of $\mathrm{Soc}(A/A_j)$. Then, on use of standard properties of semi-simple modules, we deduce that there is an integer k with $1 \leqslant k \leqslant s$ and a $y \in A$ such that $x_{k,j(k)} + A_j, y + A_j$ are elements of $(0 :_{A/A_j} M_k)$ which are linearly independent over R/M_k.

Next, we observe that none of $x_{1,j(1)}, \ldots, x_{s,j(s)}$ belongs to $Ry + A_j$, for if i were an integer between 1 and s such that $x_{i,j(i)} - ry \in A_j$ for some $r \in R$, then we would have

$$x_{i,j(i)} + A_j = r(y + A_j),$$

consideration of annihilators would lead to the conclusion that $i = k$, and we would have a contradiction to the linear independence mentioned in the last paragraph.

Thus $Ry + A_j \in \mathscr{T}_j$ and $Ry + A_j \supset A_j$, contrary to the fact that A_j is a maximal member of $\mathscr{T}_j$. With this contradiction, the proof is complete.

We are now in a position to prove the main result of this paper in full generality.

THEOREM 3·2. *Let A be a non-zero Artinian module over the commutative ring R. There exist only finitely many maximal ideals M of R for which $\mathrm{Soc}(A)$ has a submodule isomorphic to R/M: let the distinct such maximal ideals be $M_1, \ldots, M_s$. Set $J := \bigcap_{i=1}^{s} M_i$ and*

$$\hat{R}^{(J)} := \lim_{\leftarrow n} R/J^n,$$

the J-adic completion of R. Then A has a natural structure as a module over $\hat{R}^{(J)}$.

(i) *The ring $\hat{R}^{(J)}/(0 :_{\hat{R}^{(J)}} A)$ is a semi-local commutative Noetherian ring which is complete in the topology defined by its Jacobson radical.*

(ii) *The module A is, in a natural way, a faithful Artinian module over*

$$\hat{R}^{(J)}/(0 :_{\hat{R}^{(J)}} A);$$

moreover, a subset of A is an R-submodule if and only if it is an $\hat{R}^{(J)}/(0 :_{\hat{R}^{(J)}} A)$-submodule.

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Proof. The claims in (ii) follow from Lemma 2·2, and so we have only to prove (i). Set $R' = \hat{R}^{(J)}/(0:_{\hat{R}^{(J)}} A)$, and let J' denote the Jacobson radical of R'. Note that R' has exactly s maximal ideals.

By 3·1, there exist finitely many submodules $A_1, \ldots, A_t$ of A such that $0 = \bigcap_{j=1}^{t} A_j$ and, for each $j = 1, \ldots, t$, the Artinian R-module A/A_j satisfies

$$\operatorname{Soc}(A/A_j) \cong \bigoplus_{i=1}^{s} R/M_i.$$

For $i = 1, \ldots, t$, let

$$R_i = \hat{R}^{(J)}/(A_i:_{\hat{R}^{(J)}} A),$$

and note that, by 2·8, this is a complete semi-local Noetherian ring with exactly s maximal ideals. Thus, when we view such an R_i as an R'-module via the natural surjective ring homomorphism $R' \to R_i$, we see that R_i is Noetherian. Since

$$\bigcap_{i=1}^{t} (A_i:_{\hat{R}^{(J)}} A) = (0:_{\hat{R}^{(J)}} A),$$

R' is isomorphic, as R'-module, to a submodule of $\bigoplus_{i=1}^{t} R_i$, and so R' is a Noetherian ring. Since each R_i $(1 \leqslant i \leqslant t)$ is J'-adically complete, so too is $\bigoplus_{i=1}^{t} R_i$, and so we can deduce from the Artin–Rees Lemma that R' is J'-adically complete, as claimed.

Remark 3·3. Perhaps it should be pointed out that Theorem 3·2 allows Matlis' duality [**1**] and the commutative algebra of finitely generated modules over complete Noetherian local rings to be used in the study of Artinian modules over general commutative rings, simply because, by [**2**, theorem 8.15], a complete semi-local commutative Noetherian ring is isomorphic to a direct product of complete Noetherian local rings.

REFERENCES

[1] E. Matlis. Injective modules over Noetherian rings. *Pacific J. Math.* **8** (1958), 511–528.
[2] H. Matsumura. *Commutative Ring Theory* (Cambridge University Press, 1986).
[3] D. G. Northcott. *An Introduction to Homological Algebra* (Cambridge University Press, 1960).
[4] D. G. Northcott. *Lessons on Rings, Modules and Multiplicities* (Cambridge University Press, 1968).
[5] R. Y. Sharp. A method for the study of Artinian modules, with an application to asymptotic behavior. In *Commutative Algebra – Proceedings of a Microprogram held June* 15–*July* 2, 1987. Mathematical Sciences Research Institute Publications no. 15 (Springer-Verlag, 1989), pp. 443–465.
[6] D. W. Sharpe and P. Vámos. *Injective Modules* (Cambridge University Press, 1972).
[7] O. Zariski and P. Samuel. *Commutative Algebra*, vol. II. Graduate Texts in Math. no. 29 (Springer-Verlag, 1975).

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