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Function space topologies defined by sectional integrals and applications to an extremal problem

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1. *Introduction.* Let $H(t, \boldsymbol{\theta})$ be the hyperplane in R^n ($n \geqslant 2$) which is perpendicular to the unit vector $\boldsymbol{\theta}$, and distant t from the origin; that is $H(t, \boldsymbol{\theta}) = \{\mathbf{x} \in R^n : \mathbf{x} . \boldsymbol{\theta} = t\}$. (Note that $H(t, \boldsymbol{\theta})$ and $H(-t, -\boldsymbol{\theta})$ are the same hyperplane.) If $f(\mathbf{x}) \in \mathscr{L}^1(R^n)$, we will denote the integral of f with respect to $(n-1)$-dimensional Lebesgue measure over $H(t, \boldsymbol{\theta})$ by $F(t, \boldsymbol{\theta})$, termed the projection or sectional integral of f over $H(t, \boldsymbol{\theta})$. By Fubini's theorem, $F(t, \boldsymbol{\theta})$ exists for almost all t for any $\boldsymbol{\theta}$. Throughout this paper we will assume that $f(x)$ has support in X, a compact convex subset of R^n. In Section 2 we examine some of the topologies that may be defined on functions on X in terms of the $F(t, \boldsymbol{\theta})$, and in the remainder of the paper we examine the extremal problem suggested by Croft (4), that of maximizing the integral of f over the set X with the constraint that the $F(t, \boldsymbol{\theta})$ are uniformly bounded above. We examine in particular how the extremal values depend on the convex set X. In the final section the extremal problem is related to a generalization of Bang's plank theorem and the theory of capacities, and several conjectures are proposed.

All functions to be considered will be in the space of integrable functions on X, $\mathscr{L}^1(X)$, functions equal almost everywhere being identified in the usual way. We will be particularly concerned with the following subsets of $\mathscr{L}^1(X)$:

$$\begin{aligned}
\mathscr{L}^\infty &= \{f : \operatorname{ess\,sup} |f| < \infty\},\\
\mathscr{L}^+ &= \{f \in \mathscr{L}^1 : 0 \leqslant f \text{ almost everywhere}\},\\
\mathscr{B} &= \{f \in \mathscr{L}^1 : 0 \leqslant f \leqslant 1 \text{ almost everywhere}\},\\
\mathscr{S} &= \{f : f \text{ is the characteristic function of a measurable set}\},\\
\mathscr{C} &= \{f : f \text{ is continuous on } X\}.
\end{aligned}$$

We will write $\mathscr{L}^\infty(X)$, $\mathscr{C}(X)$, etc. when the set supporting the functions is unclear. $\| \ \|_p$ will denote the usual norm on $L^p(X)$ for $1 \leqslant p \leqslant \infty$. For functions where it is defined, let

$$|||f|||_\infty = \sup_{\boldsymbol{\theta}} (\operatorname*{ess\,sup}_t |F(t, \boldsymbol{\theta})|).$$

From Theorem 1·1 below, it will follow that $|||\ \ |||_\infty$ is a norm on $\mathscr{L}^\infty$. $f \in \mathscr{L}^1(X)$ will be termed a *ridge function* in direction $\boldsymbol{\theta}$ if f may be written as $f(\mathbf{x} . \boldsymbol{\theta})$ for $\mathbf{x} \in X$. Regarding $\mathscr{L}^\infty(X)$ as the continuous dual space to $\mathscr{L}^1(X)$ let τ denote the weak-$*$ topology on $\mathscr{L}^\infty(X)$, and for each $\boldsymbol{\theta}$ let $\tau(\boldsymbol{\theta})$ be the weak-$*$ topology on the essentially bounded functions $F(t, \boldsymbol{\theta})$, regarded as the dual of the functions integrable over the appropriate values of t. Throughout, ν will denote n-dimensional Lebesgue measure.

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We now state some known results that we shall need to refer to.

THEOREM 1.1 (UNIQUENESS THEOREM). *Let $f \in \mathscr{L}^1(R^n)$ be such that $F(t, \boldsymbol{\theta}) = 0$ for almost all t for all $\boldsymbol{\theta}$. Then $f(\mathbf{x}) = 0$ almost everywhere.*

Proof. See Smith, Solmon and Wagner (12), p. 1235, or Besicovitch (2) for different proofs of this.

THEOREM 1·2. *Suppose that $f \in \mathscr{L}^1(X)$ is such that $F(t, \boldsymbol{\theta})$ is constant for all $H(t, \boldsymbol{\theta})$ that intersect X. Then $n = 2$, X is a plane disc of radius R, and taking the centre of the disc as origin*

$$f(\mathbf{x}) = \gamma(R^2 - |\mathbf{x}|^2)^{-\frac{1}{2}} \quad (|\mathbf{x}| < R),$$
$$f(\mathbf{x}) = 0 \quad (|\mathbf{x}| \geqslant R),$$

where γ is some constant.

Proof. See Green (10) or Falconer (9).

THEOREM 1·3. *Let X be a compact convex subset of R^n. Then there exists a function $f \in \mathscr{L}^2(X)$ with projections $F(t, \boldsymbol{\theta})$ if and only if the following conditions hold:*

(i) $F(t, \boldsymbol{\theta}) = 0$ *if* $H(t, \boldsymbol{\theta})$ *misses* X.

(ii) $\int\int |\lambda|^{n-1} |\hat{F}(\lambda, \boldsymbol{\theta})|^2 d\lambda \, d\boldsymbol{\theta} < \infty$ (*where* $\hat{F}(\lambda, \boldsymbol{\theta})$ *is the 1-dimensional Fourier transform of* $F(t, \boldsymbol{\theta})$ *with respect to* t).

(iii) *For each non-negative integer r, $\int F(t, \boldsymbol{\theta}) t^r \, dt$ is a homogeneous polynomial of degree r in $\boldsymbol{\theta}$.*

Proof. See Smith, Solmon and Wagner (12), p. 1262.

2. *Topological results.* In this section we will mainly be concerned with the relationship between convergence of sequences of functions defined on X in the usual topologies ($\mathscr{L}^1, \mathscr{L}^\infty, \tau$, etc.) and their convergence in topologies defined in terms of the sectional integrals. Our first aim is to show that any of the obvious definitions of topologies in terms of sectional integrals are necessarily fairly weak: we will show that in the norm $|||\ |||_\infty$ it is possible to approximate a function in $\mathscr{C} \cap \mathscr{B}$ by functions in $\mathscr{S}$. We proceed using a sequence of lemmas:

LEMMA 2·1. *Let B be the unit ball in R^n, let $\boldsymbol{\theta}_0$ be a unit vector, and let ϵ and δ be positive quantities. Suppose that $f(\mathbf{x}) = k$ if $\mathbf{x} \in B$ and $f(\mathbf{x}) = 0$ otherwise, where $0 \leqslant k \leqslant 1$. Then there exists a function $g(\mathbf{x}) \in \mathscr{S}(B)$ with projections $G(t\ \boldsymbol{\theta})$ such that $|F(t, \boldsymbol{\theta}) - G(t, \boldsymbol{\theta})| < \epsilon$ for all t and $\boldsymbol{\theta}$ such that $|\boldsymbol{\theta} . \boldsymbol{\theta}_0| \leqslant 1 - \delta$.*

Proof. Let $g_r(\mathbf{x} . \boldsymbol{\theta}_0)$ be the ridge function in direction $\boldsymbol{\theta}_0$ defined by

$$g_r(t) = 1 \quad (\text{if } rt - [rt] \leqslant k),$$
$$g_r(t) = 0 \quad (\text{otherwise}),$$

square brackets representing the largest integer less than the argument. Let $G_r(t, \boldsymbol{\theta})$ be the projections of g_r. Then as $r \to \infty$ $|F(t, \boldsymbol{\theta}) - G_r(t, \boldsymbol{\theta})| \to 0$ uniformly for those $(t, \boldsymbol{\theta})$ such that $|\boldsymbol{\theta} . \boldsymbol{\theta}_0| \leqslant 1 - \delta$. The result now follows by taking an r sufficiently large.

We require the following covering lemma of Vitali type:

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LEMMA 2·2. *Let X be a proper compact convex subset of R^n and let r be positive. Then there exists a sequence of closed disjoint balls $\{B_i\}$ all of radius less than r such that* $\nu\left(X\setminus\bigcup_1^\infty B_i\right)=0$.

Proof. This is a special case of the result in De Guzmán (5), p. 25.

THEOREM 2·3. *Let $f(\mathbf{x})\in\mathscr{B}(X)\cap\mathscr{C}(X)$ and let ϵ' be a positive constant. Then there exists $g(\mathbf{x})\in\mathscr{S}(X)$ such that*

(i) $0\leqslant G(t,\boldsymbol{\theta})\leqslant F(t,\boldsymbol{\theta})+\epsilon'$ *for all t and $\boldsymbol{\theta}$*

and (ii) $|G(t,\boldsymbol{\theta})-F(t,\boldsymbol{\theta})|<\epsilon'$ *for almost all t, for each $\boldsymbol{\theta}$.*

Proof. As f is uniformly continuous on X, given $\epsilon>0$ there exists η such that, for $\mathbf{x},\mathbf{y}\in X$, $|\mathbf{x}-\mathbf{y}|<\eta$ implies that $|f(\mathbf{x})-f(\mathbf{y})|<\epsilon$. By Lemma 2·2, there exists a sequence of closed disjoint balls $\{B_i\}_1^\infty$ all with radii less than $\min\{\eta,\epsilon\}$ such that $\nu\left(X\setminus\bigcup_1^\infty B_i\right)=0$. Let $\mathbf{x}_i$ be the centre of B_i for each i, and define the function $f_i(\mathbf{x})$ by

$$\begin{aligned} f_i(\mathbf{x}) &= f(\mathbf{x}_i) \quad (\mathbf{x}\in B_i)\\ &= 0 \qquad\quad (\text{otherwise}).\end{aligned}$$

Let $f_0(\mathbf{x})=\sum_1^\infty f_i(\mathbf{x})$, so that

$$f_0(\mathbf{x})-f(\mathbf{x})<\epsilon \qquad (\text{for all } \mathbf{x}) \tag{2·1}$$

and

$$|f_0(\mathbf{x})-f(\mathbf{x})|<\epsilon \qquad (\text{for almost all } \mathbf{x}). \tag{2·2}$$

Let $\{\boldsymbol{\theta}_i\}_1^\infty$ be a sequence of distinct unit vectors and δ_i a sequence of positive numbers such that for any unit vector $\boldsymbol{\theta}$, $|\boldsymbol{\theta}.\boldsymbol{\theta}_i|>1-\delta_i$ for at most one integer i. By Lemma 2·1, there exists a function $g_i(\mathbf{x})\in\mathscr{S}(B_i)$ such that

$$|F_i(t,\boldsymbol{\theta})-G_i(t,\boldsymbol{\theta})|<2^{-i}\epsilon \tag{2·3}$$

for all t if $|\boldsymbol{\theta}.\boldsymbol{\theta}_i|>1-\delta_i$. Let $g(\mathbf{x})=\sum_1^\infty g_i(\mathbf{x})$. Then $g(\mathbf{x})\in\mathscr{S}(X)$ and

$$\begin{aligned} G(t,\boldsymbol{\theta})-F(t,\boldsymbol{\theta}) &< |G(t,\boldsymbol{\theta})-F_0(t,\boldsymbol{\theta})|+(F_0(t,\boldsymbol{\theta})-F(t,\boldsymbol{\theta}))\\ &< \sum_1^\infty |G_i(t,\boldsymbol{\theta})-F_i(t,\boldsymbol{\theta})|+(F_0(t,\boldsymbol{\theta})-F(t,\boldsymbol{\theta})). \end{aligned} \tag{2·4}$$

But for a given $\boldsymbol{\theta}$, (2·3) gives that $|F_i(t,\boldsymbol{\theta})-G_i(t,\boldsymbol{\theta})|<2^{-i}\epsilon$ for all integers i with at most one exception, in which case $|F_i(t,\boldsymbol{\theta})-G_i(t,\boldsymbol{\theta})|<2\epsilon$ (as the radius of B_i is at most ϵ). Thus using (2·4) and (2·1)

$$G(t,\boldsymbol{\theta})-F(t,\boldsymbol{\theta})<\sum_1^\infty 2^{-i}\epsilon+2\epsilon+\epsilon d<(3+d)\epsilon, \tag{2·5}$$

where d is the diameter of X.

Similarly,

$$|G(t,\boldsymbol{\theta})-F(t,\boldsymbol{\theta})|<\sum_1^\infty |G_i(t,\boldsymbol{\theta})-F_i(t,\boldsymbol{\theta})|+|F_0(t,\boldsymbol{\theta})-F(t,\boldsymbol{\theta})|$$

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so that from (2·2) we get that, for a given $\boldsymbol{\theta}$, for almost all t

$$|G(t,\boldsymbol{\theta})-F(t,\boldsymbol{\theta})| < \sum_1^\infty 2^{-i}\epsilon + 2\epsilon + \epsilon d < (3+d)\epsilon. \tag{2·6}$$

Thus (2·5) and (2·6) together give the result taking $\epsilon' = (3+d)\epsilon$.

COROLLARY 2·4. *Let $f(\mathbf{x})$ be a continuous function in $\mathscr{B}(X)$. Then, given $\epsilon > 0$, there exists $g(\mathbf{x}) \in \mathscr{S}(X)$ such that $|||f-g|||_\infty < \epsilon$.*

Proof. Take the function $g \in \mathscr{S}(X)$ given by Theorem 2·3, so that $|||f-g|||_\infty < \epsilon$ by (ii).

COROLLARY 2·5. *$\|\ \|_\infty$ and $|||\ \ |||_\infty$ are not equivalent norms on $\mathscr{B}(X)$.*

Proof. Consider the function $f(\mathbf{x}) = \frac{1}{2}$ for all $\mathbf{x} \in X$. Then if $g(\mathbf{x}) \in \mathscr{S}(X)$

$$\|f-g\|_\infty = \tfrac{1}{2}\nu(X),$$

but $|||f-g|||_\infty$ may be made arbitrarily small by Corollary 2·4. Thus Theorem 1·1 does not follow from a norm inequality.

We now show that if we consider weak-$*$ topologies rather than norm topologies it becomes possible to describe convergence of functions in terms of convergence of sectional integrals. We proceed by considering sums of ridge functions:

LEMMA 2·6. *The space of finite sums of ridge polynomials on X is dense in the space of continuous functions on X in $\|\ \|_\infty$.*

Proof. Let

$$\mathscr{P} = \left\{\sum_1^N p_i(\mathbf{x}.\boldsymbol{\theta}_i)\colon p_i \text{ is a polynomial and the } \boldsymbol{\theta}_i \text{ are arbitrary and distinct } (\boldsymbol{\theta}_i \neq \pm\boldsymbol{\theta}_j \text{ if } i \neq j)\right\}.$$

We show that $\mathscr{P}$ is an algebra of functions on X, that is $\mathscr{P}$ is a space such that the product of any two members of $\mathscr{P}$ also lies in $\mathscr{P}$. It is enough to show that if $\boldsymbol{\theta}_i$ and $\boldsymbol{\theta}_j$ are distinct unit vectors and p_i and p_j are polynomials, then $p_i(\mathbf{x}.\boldsymbol{\theta}_i)\,p_j(\mathbf{x}.\boldsymbol{\theta}_j)$ lies in $\mathscr{P}$. But this is essentially a 2-dimensional problem, and we have the fact that if $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_N$ are N distinct unit vectors in 2-dimensional space then the homogeneous polynomials of degree r, $(\mathbf{x}.\boldsymbol{\phi}_1)^r, \ldots, (\mathbf{x}.\boldsymbol{\phi}_N)^r$, are linearly independent if $r \geqslant N-1$ by considering the appropriate determinant. Thus if p_i and p_j are of degree at most k, we may choose at most $2k+1$ unit vectors lying in the same 2-dimensional subspace of R^n as $\boldsymbol{\phi}_i$ and $\boldsymbol{\phi}_j$, such that $p_i(\mathbf{x}.\boldsymbol{\phi}_i)\,p_j(\mathbf{x}.\boldsymbol{\phi}_j)$ may be expressed as a sum of ridge polynomials in those directions. Thus $\mathscr{P}$ is an algebra of continuous functions on X, containing the constant functions, and separating the points of X. By the Stone–Weierstrass theorem (see for example Dunford and Schwartz (6), p. 272) we have that $\mathscr{P}$ is dense in the space of continuous functions of X in the uniform norm.

COROLLARY 2·7. *The space of finite sums of ridge polynomials on X is dense in $\mathscr{L}^1(X)$ in $\|\ \|_1$.*

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Proof. If $f \in \mathscr{L}^1$, there exists a continuous g such that $\|f-g\|_1 < \epsilon$, and by Lemma 2·6 a function h that is a finite sum of ridge polynomials with $\|g-h\|_\infty < \epsilon$, and therefore with $\|f-h\|_1 < \epsilon(1+\nu(X))$, as required.

THEOREM 2·8. *Let $f_\tau(\mathbf{x})$ be a sequence in $\mathscr{L}^1(X)$ such that for some constant M we have that $\|f_\tau\|_1 \leqslant M$ for all τ, and let $f(\mathbf{x}) \in \mathscr{L}^1(X)$. Then $f_\tau(\mathbf{x}) \to f(\mathbf{x})$ in τ if and only if $F_\tau(t, \boldsymbol{\theta}) \to F(t, \boldsymbol{\theta})$ in $\tau(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$.*

Proof. First suppose that $f_\tau(\mathbf{x}) \to f(\mathbf{x})$ in τ. If $g(t) \in \mathscr{L}_1(R)$ and $\boldsymbol{\theta}$ is a unit vector, then $g(\mathbf{x}.\boldsymbol{\theta}) \in \mathscr{L}_1(X)$, so that

$$\int f_\tau(\mathbf{x})\, g(\mathbf{x}.\boldsymbol{\theta})\, d\mathbf{x} \to \int f(\mathbf{x})\, g(\mathbf{x}.\boldsymbol{\theta})\, d\mathbf{x},$$

or

$$\int F_\tau(t, \boldsymbol{\theta})\, g(t)\, dt \to \int F(t, \boldsymbol{\theta})\, g(t)\, dt$$

as required.

Conversely, suppose that $F_\tau(t, \boldsymbol{\theta}) \to F(t, \boldsymbol{\theta})$ in $\tau(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$, and that $g(\mathbf{x}) \in \mathscr{L}_1(X)$. By Corollary 2·7 we may find unit vectors $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ (for some N) and ridge polynomials $g_1(\mathbf{x}.\boldsymbol{\theta}_1), \ldots, g_N(\mathbf{x}.\boldsymbol{\theta}_N)$ such that given $\epsilon > 0$

$$\left\| g(\mathbf{x}) - \sum_{i=1}^{N} g_j(\mathbf{x}.\boldsymbol{\theta}_i) \right\|_1 < \epsilon.$$

Thus

$$\int [f_\tau(\mathbf{x}) - f(\mathbf{x})]\, g(\mathbf{x})\, d\mathbf{x} = \sum_i \int [f_\tau(\mathbf{x}) - f(\mathbf{x})]\, g_i(\mathbf{x}.\boldsymbol{\theta}_i)\, d\mathbf{x} + \int [f_\tau(\mathbf{x}) - f(\mathbf{x})]\, [g(\mathbf{x}) - \Sigma g_i(\mathbf{x}.\boldsymbol{\theta}_i)]\, d\mathbf{x}$$

so that

$$\left| \int [f_\tau(\mathbf{x}) - f(\mathbf{x})]\, g(\mathbf{x})\, d\mathbf{x} \right| \leqslant \Sigma \left| \int [F_\tau(t, \boldsymbol{\theta}_i) - F(t, \boldsymbol{\theta}_i)]\, g_i(t)\, dt \right| + \|f_\tau(\mathbf{x}) - f(\mathbf{x})\|_\infty \, \|g(\mathbf{x}) - \Sigma g_i(\mathbf{x}.\boldsymbol{\theta}_i)\|_1.$$

But as $\|f_\tau\|_\infty \leqslant M$ and $F_\tau(t, \boldsymbol{\theta}_i) \to F(t, \boldsymbol{\theta}_i)$ in $\tau(\boldsymbol{\theta}_i)$ for $1 \leqslant i \leqslant N$, it follows that $f_\tau(\mathbf{x}) \to f(\mathbf{x})$ in τ.

Note that it would be possible to obtain results analogous to Theorem 2·8 for functions in $\mathscr{L}^p(X)$ $(1 \leqslant p < \infty)$ and to consider the weak-$*$ topology on $\mathscr{L}^p$ (with dual $\mathscr{L}^q$, where $1/p + 1/q = 1$). However, we shall not require such results in the work that follows.

Finally in this section we remark on the weak-$*$ compactness of certain sets of functions.

LEMMA 2·9. *$B(X)$ is τ-compact.*

Proof. The Banach–Alaoglu theorem states that the unit ball of a topological vector space is compact in the weak-$*$ topology (see Rudin (11), p. 66). $\mathscr{B}(X)$ is the intersection of the unit ball of $\mathscr{L}^\infty(X)$ with the non-negative cone of $\mathscr{L}^\infty(X)$ (which is τ-closed), so that $\mathscr{B}(X)$ is τ-compact.

3. *The extremal problem.* In this section we examine general aspects of the following extremal problem, suggested by Croft (4). If X is a compact convex subset of R^n, what is the maximum possible value of $\int_X f(\mathbf{x})\, d\mathbf{x}$, where $f \in \mathscr{L}^1(X)$ and where $F(t, \boldsymbol{\theta}) \leqslant \Delta$ for almost all t for all $\boldsymbol{\theta}$, with Δ a positive constant?

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Our ultimate aim is to estimate the value of this supremum in terms of the standard measures of the convex set X (width, diameter, etc.). We consider the usual questions associated with such an extremal problem, namely whether the extremal value is attained, and if so whether by a unique function.

The problem will be considered for the classes of function $\mathscr{L}^1$, $\mathscr{L}^+$, $\mathscr{B}$ and $\mathscr{S}$, and we will denote by $\mathscr{L}^1_\Delta$, $\mathscr{L}^+_\Delta$, $\mathscr{B}_\Delta$ and $\mathscr{S}_\Delta$ respectively the subclasses consisting of those functions f such that $F(t,\boldsymbol{\theta}) \leqslant \Delta$ for almost all t for all $\boldsymbol{\theta}$. We will define $M(\mathscr{L}^1_\Delta)$ to be the supremum of $\left\{\int_X f(\mathbf{x})\,d\mathbf{x} : f \in \mathscr{L}^1_\Delta(X)\right\}$, with a similar notation for the other classes of function. Thus our problem is to investigate the values of $M(\mathscr{L}^1_\Delta)$, etc. As $\mathscr{S} \subset \mathscr{B} \subset \mathscr{L}^+ \subset \mathscr{L}^1$ we have immediately that

$$M(\mathscr{S}_\Delta) \leqslant M(\mathscr{B}_\Delta) \leqslant M(\mathscr{L}^+_\Delta) \leqslant M(\mathscr{L}^1_\Delta).$$

We remark that the problem is effectively unaltered if, instead of considering the supremum of $\int_X f(\mathbf{x})\,d\mathbf{x}$ over those functions f with $F(t,\boldsymbol{\theta}) \leqslant \Delta$ for almost all t for all $\boldsymbol{\theta}$, we consider those f with $F(t,\boldsymbol{\theta}) \leqslant \Delta$ for all t and $\boldsymbol{\theta}$. For if $f \in \mathscr{L}^1(X)$ and if $f_\delta(\mathbf{x})$ is (the restriction to X of) the average of f over the ball of centre $\mathbf{x}$ and radius δ, then by the theorem of Lebesgue on the differentiation of the integral (see De Guzmán (5), p. 39) we have that $\lim_{\delta\to 0} f_\delta(\mathbf{x})$ exists and equals $f(\mathbf{x})$ for almost all $\mathbf{x}$. Thus if $f_0(\mathbf{x}) = \overline{\lim_{\delta\to 0}} f_\delta(\mathbf{x})$ then $\int_X f = \int_X f_0$, and also

$$F_0(t,\boldsymbol{\theta}) = \int_{H(t,\boldsymbol{\theta})} \overline{\lim_{\delta\to 0}} f_\delta \leqslant \overline{\lim_{\delta\to 0}} \int_{H(t,\boldsymbol{\theta})} f_\delta \leqslant \Delta,$$

using Fatou's lemma and the fact that (applying Fubini's theorem to f_δ) $F_\delta(t,\boldsymbol{\theta}) \leqslant \Delta$ for all t and $\boldsymbol{\theta}$.

We now deduce some general results in connexion with the extremal problem. We first show that the value $M(\mathscr{B}_\Delta)$ is attained by a function in $\mathscr{B}_\Delta(X)$.

LEMMA 3·1. *$\mathscr{B}_\Delta(X)$ is τ-compact.*

Proof. Using Lemma 2·9, it is enough to show that $\mathscr{B}_\Delta$ is τ-closed in $\mathscr{B}$, and this follows immediately from Theorem 2·8.

COROLLARY 3·2. *$M(\mathscr{B}_\Delta)$ is attained by a function in $\mathscr{B}_\Delta(X)$.*

Proof. $f \mapsto \int_X f$ is trivially a τ-continuous function on $\mathscr{L}^\infty$, so attains its bounds on the τ-compact set $\mathscr{B}_\Delta(X)$.

THEOREM 3·3. *$M(\mathscr{S}_\Delta) = M(\mathscr{B}_\Delta)$.*

Proof. Certainly $M(\mathscr{S}_\Delta) \leqslant M(\mathscr{B}_\Delta)$. Suppose $f \in \mathscr{B}_\Delta(X)$, and again define $f_\delta(\mathbf{x})$ as the average of f over the ball of radius δ and centre $\mathbf{x}$ (if $\mathbf{x} \in X$) and as zero otherwise. We have that $f_\delta \in \mathscr{B}_\Delta(X)$, and by choosing δ sufficiently small, that $\left|\int_X f_\delta - \int_X f\right| < \epsilon$ for

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any given ϵ. As $(1-\epsilon)f_\delta(\mathbf{x})$ is a continuous member of $\mathscr{B}_{(1-\epsilon)\Delta}$, Theorem 2·3 gives that there exists $g(\mathbf{x})\in\mathscr{S}_\Delta(X)$ such that $|G(t,\boldsymbol{\theta})-(1-\epsilon)F_\delta(t,\boldsymbol{\theta})|<\epsilon\Delta$ for almost all t for each $\boldsymbol{\theta}$. Thus

$$\begin{aligned}\left|\int_X g-\int_X f\right| &\leqslant \left|\int_X g-\int_X (1-\epsilon)f_\delta\right|+\left|\int_X (1-\epsilon)f_\delta-\int_X f_\delta\right|+\left|\int_X f_\delta-\int_X f\right|\\ &\leqslant d\epsilon\Delta+\epsilon\left|\int f_\delta\right|+\epsilon\\ &\leqslant \epsilon[d\Delta+\int|f|+1],\end{aligned}$$

where d is the diameter of X. Hence we may find a function $g(\mathbf{x})\in\mathscr{S}_\Delta(X)$ with $\int_X g$ arbitrarily close to $\int_X f$, so that $M(\mathscr{S}_\Delta)=M(\mathscr{B}_\Delta)$.

We end this section with a lemma that will sometimes be useful.

LEMMA 3·4. *Let X be a compact convex subset of R^n. Then $\lim_{\lambda\to 0}\frac{1}{\lambda}M(\mathscr{B}_{\lambda\Delta})$ exists and equals $M(\mathscr{L}^+_\Delta)$.*

Proof. This follows using the fact that if $f\in\mathscr{B}_\Delta$ and $\Delta'\leqslant\Delta$, then $\Delta' f/\Delta\in\mathscr{B}_{\Delta'}$, so that $\frac{1}{\lambda}M(\mathscr{B}_{\lambda\Delta})$ increases as λ decreases; and by considering functions $\lambda\min(f(\mathbf{x}),\,1/\lambda)$ which belong to $\mathscr{B}_{\lambda\Delta}$ if $f(\mathbf{x})$ belongs to $\mathscr{L}^+_\Delta$.

4. *The case where X is the unit ball.* We now examine the special case where X is the n-dimensional unit ball, when we are able to find complete solutions to our problem, as well as getting some useful examples for general X. In 2 dimensions we have from Theorem 1·2 that $M(\mathscr{L}^1_\Delta)=M(\mathscr{L}^+_\Delta)=2\Delta$ ($M(\mathscr{L}^1_\Delta)$ is certainly at most Δw, where w is the width of the set X under consideration), and also that this value of $\int_X f$ is attained uniquely by the function

$$\begin{aligned}f(\mathbf{x})&=\pi^{-1}\Delta(1-|\mathbf{x}|^2)^{-\frac{1}{2}} \quad (|\mathbf{x}|<1),\\ f(\mathbf{x})&=0 \quad (|\mathbf{x}|\geqslant 1).\end{aligned}$$

For $\mathscr{B}_\Delta$ and $\mathscr{S}_\Delta$ we have the following extension of a result of Croft (3).

THEOREM 4·1. *Let X be the plane unit disc, and assume $\Delta<2$. Then*

$$M(\mathscr{B}_\Delta)=M(\mathscr{S}_\Delta)=2\sin^{-1}(\tfrac{1}{2}\Delta)+\Delta(1-\tfrac{1}{4}\Delta^2)^{\frac{1}{2}},$$

and this value of $\int_X f$ is attained uniquely in $\mathscr{B}_\Delta$ by a function not in $\mathscr{S}_\Delta$.

Proof. Croft (3) obtains a bound for the extremal value for $M(\mathscr{S}_\Delta)$ by taking the maximum possible values of $F(t,\boldsymbol{\theta})$ for some fixed $\boldsymbol{\theta}$:

$$\begin{aligned}F(t,\boldsymbol{\theta})&=\Delta \quad (|t|\leqslant b),\\ F(t,\boldsymbol{\theta})&=2(1-t^2)^{\frac{1}{2}} \quad (b<|t|\leqslant 1),\end{aligned}\tag{4·1}$$

where $b=(1-\frac{1}{4}\Delta^2)^{\frac{1}{2}}$, and integrating across the width of the unit disc to get the stated value. An identical argument gives the same bound for $M(\mathscr{B}_\Delta)$. As any $f(\mathbf{x})\in\mathscr{B}_\Delta$ with $\int_X f$ taking this value must have $F(t,\boldsymbol{\theta})$ as in (4·1) for all $\boldsymbol{\theta}$, it follows from Theorem 1·1

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that if such an $f(\mathbf{x})$ exists then it is essentially unique and, further, it is radially symmetric. Clearly $f(\mathbf{x}) = 1$ if $b < |\mathbf{x}| \leqslant 1$, requiring that the restriction of f to $\{\mathbf{x}: |\mathbf{x}| < b\}$, $f_1(\mathbf{x})$ say, has sectional integrals $F_1(t, \boldsymbol{\theta})$ given by

$$\begin{aligned} F_1(t, \boldsymbol{\theta}) &= \Delta - 2[(1-t^2)^{\frac{1}{2}} - (b^2-t^2)^{\frac{1}{2}}] \quad (|t| \leqslant b) \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad (|t| > b). \end{aligned} \tag{4·2}$$

Using the fact that the function

$$\begin{aligned} g(\mathbf{x}) &= \pi^{-1}(u^2 - |\mathbf{x}|^2)^{-\frac{1}{2}} \quad (|\mathbf{x}| < u) \\ &= 0 \qquad\qquad\qquad\qquad (|\mathbf{x}| \geqslant u) \end{aligned}$$

has sectional integrals

$$\begin{aligned} G(t, \boldsymbol{\theta}) &= 1 \quad (|t| < u) \\ &= 0 \quad (|t| \geqslant u), \end{aligned}$$

it follows from an application of Fubini's theorem that the function

$$f_1(\mathbf{x}) = \int_{|\mathbf{x}|}^{b} \pi^{-1}(u^2 - |\mathbf{x}|^2)^{-\frac{1}{2}}\, 2u[(1-u^2)^{-\frac{1}{2}} - (b^2-u^2)^{-\frac{1}{2}}]\, du \tag{4·3}$$

has the sectional integrals given by (4·2). Further, a standard trigonometric substitution allows (4·3) to be evaluated to give that the unique function with sectional integrals (4·1) is

$$\begin{aligned} f(\mathbf{x}) &= 2\pi^{-1} \cos^{-1}[(b^2 - |\mathbf{x}|^2)/(1-|\mathbf{x}|^2)]^{\frac{1}{2}} \quad (|\mathbf{x}| < b) \\ &= 1 \qquad\qquad\qquad\qquad\qquad\qquad\quad (b \leqslant |\mathbf{x}| \leqslant 1). \end{aligned}$$

(Note that, whilst it is possible to show directly that this function has sectional integrals given by (4·1), this necessitates the evaluation of a rather formidable integral.) Thus $f(\mathbf{x})$ has the sectional integrals required for an extremal function in $\mathscr{B}_\Delta$ and $f(\mathbf{x}) \notin \mathscr{S}_\Delta$. Croft (3) shows that the same value of $\int_X f$ may be approximated to by members of $\mathscr{S}_\Delta$, so the result is complete. (Of course, that the extremal exists in $\mathscr{B}_\Delta$ and that

$$M(\mathscr{B}_\Delta) = M(\mathscr{S}_\Delta)$$

also follows from Corollary 3·2 and Theorem 3·3.)

We now turn to the case of the unit ball in R^n for $n \geqslant 3$.

THEOREM 4·2. *Let X be the unit ball in R^n for $n \geqslant 3$. Then $M(\mathscr{L}^1_\Delta) = 2\Delta$ and this value is unattained.*

Proof. Let ϵ be a small positive constant, and let $F(t)$ be an infinitely differentiable 'bump function' such that $0 \leqslant F(t) \leqslant 1$ and

$$\begin{aligned} F(t) &= 1 \quad (|t| \leqslant 1-\epsilon), \\ F(t) &= 0 \quad (|t| \geqslant 1). \end{aligned}$$

Letting $F(t, \boldsymbol{\theta}) = F(t)$ for all $\boldsymbol{\theta}$ we have that $\int F(t, \boldsymbol{\theta})\, t^r\, dt$ is independent of $\boldsymbol{\theta}$ and vanishes if r is odd. As $F(t, \boldsymbol{\theta})$ is infinitely differentiable with respect to t, $F(t, \boldsymbol{\theta})$ has a (1-dimensional) Fourier transform $\hat{F}(\lambda, \boldsymbol{\theta})$ that is rapidly decreasing for large $|\lambda|$ and so fulfils

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condition (ii) of Theorem 1·3. Thus by Theorem 1·3 there exists a function $f(\mathbf{x})$ with support in X, with projections $F(t,\boldsymbol{\theta})$ and with Fourier transform in $\mathscr{L}^2(R^n)$, so that $f(\mathbf{x})\in\mathscr{L}^2(X)\subset\mathscr{L}^1(X)$. Hence we have found a function $f\in\mathscr{L}^1_\Delta(X)$ with $\int f\geqslant 2(1-\epsilon)\Delta$, giving that $M(\mathscr{L}^1_\Delta)=2\Delta$. Theorem 1·2 gives that this value is not attained in $\mathscr{L}^1_\Delta$.

For the case of the unit ball in R^n for $n\geqslant 3$ for the other classes of functions we introduce the following notation: if $f(r,\boldsymbol{\theta})\in\mathscr{L}^1$, where $(r,\boldsymbol{\theta})$ are polar coordinates with respect to some origin, let $f_0(\mathbf{x})$ be the (radial) function defined by

$$f_0(\mathbf{x})=\frac{1}{A_n}\int f(r,\boldsymbol{\theta})\,d\boldsymbol{\theta}\quad(r=|\mathbf{x}|),$$

where the integral is over the n-dimensional unit sphere, of surface measure A_n. Writing $F_0(t)$ as the integral of $f_0(\mathbf{x})$ over the hyperplane $H(t,\boldsymbol{\theta})$ we have that

$$F_0(t)=\frac{1}{A_n}\int F(t,\boldsymbol{\theta})\,d\boldsymbol{\theta} \tag{4·4}$$

$$=A_{n-1}\int_t^\infty f_0(r)\,r(r^2-t^2)^{\frac{1}{2}(n-3)}\,dr \tag{4·5}$$

by considering the integral of a radial function over $H(t,\boldsymbol{\theta})$, and applying Fubini's theorem.

Writing V_n as the measure of the n-dimensional unit ball, we are now in a position to obtain the next result.

THEOREM 4·3. *Let X be the unit ball in R^n $(n\geqslant 3)$, and assume that $\Delta\leqslant V_{n-1}$. Then $M(\mathscr{B}_\Delta)=V_n(1-(1-\Delta/V_{n-1})^{n/(n-1)})$, and this value is uniquely attained by a member of $\mathscr{S}_\Delta\subset\mathscr{B}_\Delta$.*

Proof. Let $f\in\mathscr{B}_\Delta$. Then $f_0(\mathbf{x})$ exists for almost all $\mathbf{x}$, so by (4·4) we have that $F_0(t)\leqslant\Delta$ and that $f_0\in\mathscr{B}_\Delta$. Using Fubini's theorem (4·4) also gives that $\int_X f=\int_X f_0$. Thus:

$$M(\mathscr{B}_\Delta)=\sup\{\textstyle\int f_0\colon f_0\in\mathscr{B}_\Delta \text{ and } f_0 \text{ is radially symmetric}\}.$$

If X is the unit ball and $f_0\in\mathscr{B}_\Delta(X)$, (4·5) holds and, for $n\geqslant 3$, $F_0(t)$ decreases as t increases if $0\leqslant t\leqslant 1$. Thus if $F_0(0)\leqslant\Delta$, we have that $F_0(t)\leqslant\Delta$ for $0\leqslant t\leqslant 1$. Hence

$$\begin{aligned}M(\mathscr{B}_\Delta)&=\sup\{\textstyle\int f_0\colon f_0 \text{ is radial and in } X \text{ and } F_0(0)\leqslant\Delta\}\\&=\sup\left\{A_n\int_0^1 f_0(r)\,r^{n-1}\,dr\colon A_{n-1}\int_0^1 f_0(r)\,r^{n-2}\,dr\leqslant\Delta\right\}.\end{aligned}$$

But the supremum in the last expression is attained if and only if $f_0(\mathbf{x})$ is equal to 0 for almost all $|\mathbf{x}|<k$ and equal to 1 for almost all $|\mathbf{x}|\geqslant k$, where $k=(1-\Delta/V_{n-1})^{1/(n-1)}$. This gives the stated value for $M(\mathscr{B}_\Delta)$, which is therefore attained by a function in $\mathscr{S}_\Delta$. As any f in $\mathscr{B}_\Delta$ for which this extremal is attained must have f_0 of this form, and as $f\leqslant 1$ almost everywhere, the definition of f_0 gives that any such f is necessarily radial, else $f(\mathbf{x})$ would be greater than 1 on some set of positive measure. Thus $f_0(\mathbf{x})$ is the unique function that gives the extremal value.

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Table 1. *Extremal values in the case of the unit ball*

	$n = 2$	$n \geqslant 3$
$M(\mathscr{L}^1_\Delta)$	2Δ, attained	2Δ, not attained
$M(\mathscr{L}^+_\Delta)$	2Δ, attained	$(A_n/A_{n-1})\,\Delta$, not attained
$M(\mathscr{B}_\Delta)$	$2\sin^{-1}(\frac{1}{2}\Delta)+\Delta(1-\frac{1}{4}\Delta^2)^{\frac{1}{2}}$, attained	$V_n(1-(1-\Delta/V_{n-1})^{n/(n-1)})$, attained
$M(\mathscr{S}_\Delta)$	$2\sin^{-1}(\frac{1}{2}\Delta)+\Delta(1-\frac{1}{4}\Delta^2)^{\frac{1}{2}}$, not attained	$V_n(1-(1-\Delta/V_{n-1})^{n/(n-1)})$, attained

THEOREM 4·4. *Let X be the unit ball in R^n ($n \geqslant 3$). Then $M(\mathscr{L}^+_\Delta) = \Delta A_n/A_{n-1}$, and this value is not attained in $\mathscr{L}^+_\Delta$.*

Proof. Proceeding as in the proof of Theorem 4·3, if $f \in \mathscr{L}^+_\Delta$ we define $f_0(\mathbf{x})$ as before. Then $f_0 \in \mathscr{L}^+_\Delta$ and $\int_X f = \int_X f_0$, and (4·2) gives that if $F_0(0) \leqslant \Delta$ then $F_0(t) \leqslant \Delta$ for all t. Thus

$$M(\mathscr{L}^+_\Delta) = \sup\left\{A_n\int_0^1 f_0(r)\,r^{n-1}\,dr\colon A_{n-1}\int_0^1 f_0(r)\,r^{n-2}\,dr \leqslant \Delta\right\}.$$

However, this time we are not restricted to bounded f_0, so $M(\mathscr{L}^+_\Delta)$ is approached by letting $f_0(r)$ approximate to a 'Dirac delta' distribution concentrated at $r = 1$. This gives the stated value of $M(\mathscr{L}^+_\Delta)$ and also that it is not attained in $\mathscr{L}^+_\Delta$.

Alternatively, the extremal value follows by applying Lemma 3·4 to Theorem 4·3 and by using the fact that $nV_n = A_n$.

The extremal values for the case of the unit ball are summarized in Table 1.

5. *Extremal values for general X.* We now obtain estimates for the extremal values of $\int_X f(\mathbf{x})$ with $F(t,\boldsymbol{\theta}) \leqslant \Delta$ for the various classes of functions where X is an arbitrary compact convex set.

THEOREM 5·1. *If X is a compact convex subset of R^n with inradius r then*

$$M(\mathscr{L}^+_\Delta) \leqslant 2\Delta r.$$

Proof. Let $f(\mathbf{x}) \in \mathscr{L}^+_\Delta(X)$. As $\int_X f(\mathbf{x})$ is strictly positive (except in the trivial case), X weighted by f has a centroid (a point $\mathbf{c}$ such that $\int_X f(\mathbf{x})\,(\mathbf{x}-\mathbf{c}) = \mathbf{0}$) which lies inside X, and which we will take to be the origin. For some $\boldsymbol{\theta}$, $h(\boldsymbol{\theta}) \leqslant r$. Taking moments perpendicular to $\boldsymbol{\theta}$ we have that (as the origin is the centroid)

$$\int_{-h(-\boldsymbol{\theta})}^{h(\boldsymbol{\theta})} t\,F(t,\boldsymbol{\theta})\,dt = 0. \tag{5·1}$$

Using $F(t,\boldsymbol{\theta}) \leqslant \Delta$ for all t, and $h(\boldsymbol{\theta}) \leqslant r$ for this $\boldsymbol{\theta}$,

$$\int_0^{h(\boldsymbol{\theta})} t\,F(t,\boldsymbol{\theta})\,dt \leqslant \tfrac{1}{2}\Delta r^2$$

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so that from (5·1) we also have

$$\int_0^{h(-\theta)} t\,F(t,\theta)\,dt \leqslant \tfrac{1}{2}\Delta r^2.$$

But it is easy to see that

$$\sup\left\{\int_0^\infty F(t)\,dt\colon 0 \leqslant F(t) \leqslant \Delta \quad \text{and} \quad \int_0^\infty t\,F(t)\,dt \leqslant M\right\} = (2M\Delta)^{\frac{1}{2}}$$

with the supremum being attained if and only if $F(t)$ is the function equal to Δ for almost all $t \leqslant (2M/\Delta)^{\frac{1}{2}}$ and equal to 0 for almost all $t \geqslant (2M/\Delta)^{\frac{1}{2}}$. Thus

$$\begin{aligned}\int_X f(\mathbf{x}) &= \int_{-h(-\theta)}^{h(\theta)} F(t,\theta)\,dt \\ &\leqslant 2\Delta r,\end{aligned}$$

giving that $M(\mathscr{L}_\Delta^+) \leqslant 2\Delta r$.

Theorem 5·1 enables us to obtain the exact value of $M(\mathscr{L}_\Delta^+)$ in 2 and 3 dimensions:

THEOREM 5·2. *Let X be a compact convex subset of R^n with inradius r. Then if $n = 2$ or $n = 3$, $M(\mathscr{L}_\Delta^+) = 2\Delta r$. If $n = 2$ this value is attained for all X (not necessarily by a unique function), and if $n = 3$ it may or may not be attained.*

Proof. If S is an insphere of X, the results for the unit ball give that $M(\mathscr{L}_\Delta^+(S)) = 2\Delta r$ if $n = 2$ or $n = 3$, so that combining this with Theorem 5·1 gives that $M(\mathscr{L}_\Delta^+(X)) = 2\Delta r$. If $n = 2$, this value is attained by a function supported by an insphere of X, which will not be a unique function if X does not have a unique insphere.

In 3 dimensions, the unit ball has already provided us with a case where the extremal value is not attained. We now give a construction to show that in the case of a circular cylinder there is a function that gives the extremal value. Let X be the cylinder defined in cylindrical polar coordinates (r,θ,z) by $|r| \leqslant 1$, $-1 \leqslant z \leqslant 3$. Let μ_z be the measure given by the uniform mass distribution on the surface of the sphere of radius 1 and centre $(0,0,z)$, with total mass 2. Note that, if $g(\mathbf{x}.\boldsymbol{\theta})$ is a ridge function, $\int g\,d\mu_z = \int g(t)\,dt$, the second integration being over those t for which the plane $H(t,\boldsymbol{\theta})$ cuts the sphere. Let $\mu^{(i)} = 2^{-i}\Sigma\mu_z$, the sum being over $z = k \times 2^{1-i}$ $(k = 1, 2, \ldots, 2^i)$. Then if $g(\mathbf{x})$ is any continuous function on X we have that $\int g\,d\mu^{(i)} \to \int fg\,d\nu$ as $i \to \infty$, where

$$f(r,\theta,z) = m/4\pi(1-r^2)^{\frac{1}{2}}$$

with $m = 0$ unless $r^2+z^2 < 1$ or $r^2+(z-2)^2 < 1$, in which case $m = 1$, or $0 \leqslant z \leqslant 2$, $r^2+z^2 > 1$ and $r^2+(z-2)^2 > 1$, in which case $m = 2$. But for any non-negative ridge function $g(\mathbf{x}.\boldsymbol{\theta})$, $\int g\,d\mu^{(i)} \leqslant \int g(t)\,dt$, so that $\int fg\,d\nu \leqslant \int g(t)\,dt$, implying that $F(t,\boldsymbol{\theta}) \leqslant 1$ for almost all t for all $\boldsymbol{\theta}$. Also $f \in \mathscr{L}^+(X)$ and $\int f\,d\nu = 2$, so that f is an extremal function.

We now investigate the value of $M(\mathscr{L}_\Delta^1)$. Except in the case of X a centro-symmetric set it seems hard to obtain an exact value, but we can get the following estimates:

THEOREM 5·3. *Let X be a convex subset of R^n, and for each $\mathbf{x}$ let $h_{\mathbf{x}}(\boldsymbol{\theta})$ be the support function of X relative to $\mathbf{x}$ as origin. Let $w(\boldsymbol{\theta})$ be the width function of X and let r be the inradius of X. Then if $\tilde{h}_{\mathbf{x}}(\boldsymbol{\theta}) = \min\{h_{\mathbf{x}}(\boldsymbol{\theta}), h_{\mathbf{x}}(-\boldsymbol{\theta})\}$,*

$$2\Delta r \leqslant M(\mathscr{L}_\Delta^1) \leqslant \sup_{\mathbf{x}\in X}\inf_{\boldsymbol{\theta}}\left[\frac{\Delta}{2}\,\frac{w(\boldsymbol{\theta})^2}{(w(\boldsymbol{\theta})-\tilde{h}_{\mathbf{x}}(\boldsymbol{\theta}))}\right].$$

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Proof. The left-hand inequality follows from the fact that X contains a ball of radius r, together with the result for the unit ball (see Theorem 4·2). To obtain the right-hand inequality we proceed by taking $\mathbf{c}$ to be the centroid of X weighted by the function $f(\mathbf{x})$, as in Theorem 5·1. Then for any $\boldsymbol{\theta}$,

$$\int_X f \leqslant \sup\left\{\int_{-h_{\mathbf{c}}(-\boldsymbol{\theta})}^{h_{\mathbf{c}}(\boldsymbol{\theta})} F(t)\,dt\colon F(t) \leqslant \Delta \quad \text{and} \quad \int_{-h_{\mathbf{c}}(-\boldsymbol{\theta})}^{h_{\mathbf{c}}(\boldsymbol{\theta})} t\,F(t)\,dt = 0\right\}.$$

It is easy to see that the above expression is maximized by the 'function' that equals Δ for $-h_{\mathbf{c}}(-\boldsymbol{\theta}) \leqslant t \leqslant h_{\mathbf{c}}(\boldsymbol{\theta})$ and is a negative delta distribution at either $-h_{\mathbf{c}}(-\boldsymbol{\theta})$ or $h_{\mathbf{c}}(\boldsymbol{\theta})$, giving that

$$\int_X f \leqslant \frac{\Delta}{2}\,\frac{w(\boldsymbol{\theta})^2}{(w(\boldsymbol{\theta}) - \tilde{h}_{\mathbf{c}}(\boldsymbol{\theta}))}. \tag{5·2}$$

Thus, allowing $\mathbf{c}$ and $\boldsymbol{\theta}$ to vary over all possible values, we get the result.

Note in particular that (5·2) gives that $M(\mathscr{L}^1_\Delta) \leqslant \Delta w$ (where w is the width of X) with equality if and only if X is centro-symmetric. Note also that the value given by the right-hand inequality cannot be attained by any function unless X is centro-symmetric.

We now apply Theorem 5·3 to a particular type of convex set, namely the regular n-simplex. This is of interest as the set in which the ratio of width/inradius is maximal.

COROLLARY 5·4. *Let T be the regular simplex in R^n of width w and inradius r. Then*

$$2\Delta r \leqslant M(\mathscr{L}^1_\Delta(T)) \leqslant \frac{\Delta}{2}\,\frac{(n+1)^2 r}{n}.$$

The right-hand inequality is better than Δw if $n \leqslant 11$.

Proof. If $f \in \mathscr{L}^1_\Delta(T)$, then by averaging f over all possible symmetries of T we may obtain a new function in $\mathscr{L}^1_\Delta$ and with the same integral over T, such that the new function has the symmetries of T. Thus in estimating extremal values we need only consider such functions in the proof of Theorem 5·3, and in particular we need only consider functions whose centroids coincide with the centroid of the set T. Thus taking $\mathbf{x}$ as the centroid of T and $\boldsymbol{\theta}$ as an outward unit vector perpendicular to a face of the simplex T, we have that $w(\boldsymbol{\theta}) = (n+1)r$ and $h_{\mathbf{x}}(\boldsymbol{\theta}) = r$, which gives the stated inequality using (5·2).

The right-hand inequality is only of use if the value given is less than the obvious estimate, Δw. But for the regular n-simplex we have that

$$\begin{aligned} r &= w/(2\sqrt{n}) && (n \text{ odd}) \\ &= w\sqrt{(n+2)/2(n+1)} && (n \text{ even}) \end{aligned}$$

(see Eggleston (8), p. 112).

Thus

$$\frac{w\Delta}{\sqrt{n}} \leqslant M(\mathscr{L}^1_\Delta(T)) \leqslant w\Delta\,\frac{(n+1)^2}{4n^{\frac{3}{2}}} \quad (n \text{ odd}),$$

$$w\Delta\,\frac{\sqrt{(n+2)}}{n+1} \leqslant M(\mathscr{L}^1_\Delta(T)) \leqslant w\Delta\,\frac{(n+1)\sqrt{(n+2)}}{4n} \quad (n \text{ even})$$

and this gives a better upper estimate for $M(\mathscr{L}^1_\Delta(T))$ if $n \leqslant 11$.

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Table 2. *Extremal values for a general convex set*

	$n = 2$	$n = 3$	$n \geqslant 4$
$M(\mathscr{L}^1_\Delta)$		$2\Delta r \leqslant M(\mathscr{L}^1_\Delta) \leqslant \Delta w$	
$M(\mathscr{L}^+_\Delta)$	$2\Delta r$, attained, not necessarily uniquely	$2\Delta r$, not necessarily attained	$(A_n/A_{n-1})\Delta r \leqslant M(\mathscr{L}^+_\Delta) \leqslant 2\Delta r$
$M(\mathscr{B}_\Delta)$		Attained	
$M(\mathscr{S}_\Delta)$		$M(\mathscr{S}_\Delta) = M(\mathscr{B}_\Delta)$	

We summarize the extremal values for a general convex set in Table 2.

6. *Relation to other problems.* The well-known 'Plank Theorem' proved by Bang (1) states that, if X is a compact convex subset of R^n covered by a finite collection of planks, P_i $(1 \leqslant i \leqslant N)$ (a plank is the closed region between two parallel hyperplanes), then $\sum_1^N w_i \geqslant w$ where w is the width of X and w_i is the width of P_i. One possible generalization of this result would be to ask, if X is covered by a collection of planks P_i such that each point of X belongs to at least k planks, then how small can $\sum_1^N w_i$ be? If for example we work in 2 dimensions and take $k = 2$ (that is we require a double covering of X) then, as we shall see below, if X is centro-symmetric, $\sum_1^N w_i \geqslant 2w$, where w is the width of X. However if X is the equilateral triangle, by taking 3 planks each of width $w/2$ placed next to the sides of X, we see that $\sum_1^3 w_i$ is as little as $\frac{3}{2}w$. This question may be asked for any k, but the limiting case as $k \to \infty$ is of particular interest. Effectively we then ask how small can $\sum_1^N \int g_i(t)$ be where $g_i(\mathbf{x}.\boldsymbol{\theta}_i)\,(1 \leqslant i \leqslant N)$ are a collection of non-negative ridge functions such that $\sum_1^N g_i(\mathbf{x}.\boldsymbol{\theta}_i) \geqslant 1$ for all $\mathbf{x} \in X$, and where N and the $\boldsymbol{\theta}_i$ are arbitrary. Whilst we do not answer this question in general, the following theorem relates these problems to our extremal problem:

THEOREM 6·1. *Let X be a convex subset of R^n and let $g_i(\mathbf{x}.\boldsymbol{\theta}_i)$ $(1 \leqslant i \leqslant N)$ be non-negative ridge functions defined on X for any N and $\boldsymbol{\theta}_i$. Then*

(i) *if $\sum_i g_i(\mathbf{x}.\boldsymbol{\theta}_i) \geqslant 1$ for all $\mathbf{x} \in X$, $\sum_i \int g_i(t)\,dt \geqslant M(\mathscr{L}^+_1)$,*

(ii) *if $\sum_i g_i(\mathbf{x}.\boldsymbol{\theta}_i) = 1$ for all $\mathbf{x} \in X$, $\sum_i \int g_i(t)\,dt \geqslant M(\mathscr{L}^1_1)$.*

Proof. If $f(\mathbf{x}) \in \mathscr{L}^1_1$ or $\mathscr{L}^+_1$ we have that

$$\int f(\mathbf{x}) \sum_i g_i(\mathbf{x}.\boldsymbol{\theta}_i)\,d\mathbf{x} = \sum_i \int g_i(t)\,F(t, \boldsymbol{\theta}_i)\,dt \leqslant \sum_i \int g_i(t)\,dt,$$

and (i) and (ii) both follow from this inequality.

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Combining the above with Theorem 5·2 we get from (i) that in 2 or 3 dimensions if $g_i(\mathbf{x}.\boldsymbol{\theta}_i)$ are non-negative ridge functions with $\Sigma g_i(\mathbf{x}.\boldsymbol{\theta}_i) \geqslant 1$ for $\mathbf{x} \in X$, then

$$\Sigma \int g_i(t)\,dt \geqslant 2r$$

(where r is the inradius of X). Thus for sets X for which $2r = w$ (including all centrosymmetric X) the basic Plank Theorem follows, as well as the generalized result.

We may also deduce Corollary 5·4, the extremal problem for the regular n-simplex T, from Theorem 6·1. For let $\boldsymbol{\theta}_i$ $(1 \leqslant i \leqslant n+1)$ be outward unit vectors perpendicular to the 'faces' of T. Letting

$$g_i = \frac{1}{n(n+1)}\left(\frac{t}{r}+n\right) \quad (-nr \leqslant t \leqslant r),$$

we have that $\Sigma g_i(\mathbf{x}.\boldsymbol{\theta}_i) = 1$ if $\mathbf{x} \in X$. Thus Theorem 6·1 gives that $M(\mathscr{L}_1^1) \leqslant \dfrac{r(n+1)^2}{2n}$, as before.

We now consider the relationship between the extremal problem and potential theory, and in particular the theory of capacities of sets. (For general details of this subject see, for example, Du Plessis(7).) Here we shall only be considering the capacity kernel $1/|\mathbf{x}-\mathbf{y}|$, where $\mathbf{x}$ and $\mathbf{y}$ are points in R^n (in the case of $n = 3$ this is the Newtonian kernel that occurs in electrostatic theory). The potential $K_\mu(\mathbf{x})$ at a point $\mathbf{x}$ due to a charge distribution μ is given by

$$K_\mu(\mathbf{x}) = \int_{R^n} \frac{d\mu(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}$$

and the capacity of a compact set X in R^n, $C(X)$, may be defined by

$$C(X) = \sup\{\mu(X)\colon \mu \text{ is a non-negative Borel measure on } X \text{ with } K_\mu(\mathbf{x}) \leqslant 1 \text{ for all } \mathbf{x} \text{ in } R^n\}.$$

We get the following inequality between $C(X)$ and $M(\mathscr{L}_1^+(X))$ for any compact set X.

THEOREM 6·2. *Let X be a compact subset of R^n. Then $M(\mathscr{L}_1^+(X)) \leqslant \left(\dfrac{A_n}{2A_{n-1}}\right) C(X)$, where A_n is the surface measure of the n-dimensional ball.*

Proof. Take $f \in \mathscr{L}_1^+(X)$. Then identifying f with a measure on X in the usual way, we have that

$$K_f(\mathbf{x}) = \int \frac{f(\mathbf{y})\,d\nu(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}.$$

But this is precisely the integral of $\left\{\int_H f\right\}$ over all hyperplanes H that contain the point $\mathbf{x}$, multiplied by a factor of $1/A_{n-1}$. As $f \in \mathscr{L}_1^+$, $K_f(\mathbf{x}) \leqslant \left(\dfrac{A_n}{2A_{n-1}}\right)$ for all points $\mathbf{x}$ in R^n, so that from the definition of $C(X)$ we get that $M(\mathscr{L}_1^+(X)) \leqslant \left(\dfrac{A_n}{2A_{n-1}}\right) C(X)$.

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In 2 or 3 dimensions, the maximum principle holds for the kernel $1/|\mathbf{x}-\mathbf{y}|$, that is if μ is a non-negative mass distribution, then $K_\mu(\mathbf{x}) \leqslant M$ for $\mathbf{x} \in$ support (μ) implies that $K_\mu(\mathbf{x}) \leqslant M$ for all $\mathbf{x}$ in R^n. Thus in R^2 or R^3 we may define $C(X)$ by

$$C(X) = \sup\{\mu(X): \mu \text{ is a non-negative Borel measure on } X \text{ such that } K_\mu(\mathbf{x}) \leqslant 1 \text{ for all } \mathbf{x} \in \text{support}(\mu)\}$$

and this gives a stronger form of the inequality in Theorem 6·2.

Capacity theory also gives that, for a compact set X, there exists an equilibrium measure $\bar{\mu}$, that is a measure such that $K_{\bar{\mu}}(\mathbf{x}) \leqslant 1$ for all $\mathbf{x}$, and such that $C(X) = \bar{\mu}(X)$. For kernels of the form $1/|\mathbf{x}-\mathbf{y}|$ it may be shown that the equilibrium measure is unique, and in particular that in 3 or more dimensions the support of the equilibrium measure is contained in the outside boundary of X. Although the equilibrium measures do not correspond to extremals in $M(\mathscr{L}_1^+)$ (as $K_f(\mathbf{x}) \leqslant \frac{1}{2}A_n/A_{n-1}$ for all $\mathbf{x}$ does not imply that $\int_H f \leqslant 1$ for all hyperplanes H), it may be possible to adapt ideas from capacity theory for use with our problem. For example it seems plausible that in the case of $n \geqslant 4$ (or even $n \geqslant 3$) any function $f \in \mathscr{L}_\Delta^+$ that gives a value of $\int f$ near the maximum is likely to be concentrated near the boundary of X.

We end by listing some conjectures which, if proved, would help to answer some of the outstanding questions.

CONJECTURE 6·3. (i) *For $n \geqslant 4$ (or possibly even for $n \geqslant 3$) $M(\mathscr{B}_\Delta)$ is attained by a function in S_Δ, and any such function takes the value* 1 *'near' the boundary of X and* 0 *elsewhere (at least for Δ sufficiently small).*

(ii) *For $n \geqslant 4$ (or possibly even for $n \geqslant 3$) $M(\mathscr{L}_\Delta^+)$ is never attained.*

Remarks. (ii) follows from (i) using the idea of Lemma 3·4, since the support of any extremal for $M(\mathscr{L}_\Delta^+)$ would then have to be contained in the boundary of X, a set of n-dimensional measure zero. These results seem plausible since they hold in certain special cases (for example in the case of the sphere), and also from comparison with the equilibrium distribution in capacity theory, as mentioned above.

CONJECTURE 6·4. *If (for any n) f is an extremal function for $M(\mathscr{L}_\Delta^+)$, then f is the τ-limit as $\Delta \to 0$ of functions $(1/\Delta)f_\Delta$, where f_Δ is an extremal function for $M(\mathscr{B}_\Delta)$.*

CONJECTURE 6·5. (i) *Let S be an insphere of X, and let $f \in \mathscr{L}^1(X)$. Suppose that $F(t, \boldsymbol{\theta}) \geqslant 0$ for all hyperplanes $H(t, \boldsymbol{\theta})$ that cut S. Then $\int f \geqslant 0$.*

(ii) $M(\mathscr{L}_\Delta^1) = 2\Delta r$.

Remarks. (ii) follows from (i) by subtracting near-extremal functions for $\mathscr{L}_\Delta^1(X)$ from near-extremal functions for $\mathscr{L}_\Delta^1(S)$. Whilst it would be useful to prove (i), its validity seems doubtful, though counterexamples seem hard to construct.

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