Ergod. Th. & Dynam. Sys., (2025), **45**, 2601–2635 © The Author(s), 2025. Published by Cambridge University Press.
doi:10.1017/etds.2025.6

Transitive mappings on the Cantor fan

IZTOK BANIČ †‡§, GORAN ERCEG ¶, JUDY KENNEDY ‖,
CHRIS MOURON* and VAN NALL**

† *Faculty of Natural Sciences and Mathematics, University of Maribor, Maribor 2000, Slovenia*
‡ *Institute of Mathematics, Physics and Mechanics, Ljubljana 1000, Slovenia*
§ *Andrej Marušič Institute, University of Primorska, Koper 6000, Slovenia*
(e-mail: iztok.banic@um.si)
¶ *Faculty of Science, University of Split, Split 21000, Croatia*
(e-mail: goran.erceg@pmfst.hr)
‖ *Department of Mathematics, Lamar University, Beaumont 77710, Texas, USA*
(e-mail: kennedy9905@gmail.com)
* *Rhodes College, Memphis 38112, Tennessee, USA*
(e-mail: mouronc@rhodes.edu)
** *Department of Mathematics, University of Richmond, Richmond 23173, Virginia, USA*
(e-mail: vnall@richmond.edu)

(*Received* 9 *May* 2023 *and accepted in revised form* 12 *January* 2025)

Abstract. Many continua that admit a transitive homeomorphism may be found in the literature. The circle is probably the simplest non-degenerate continuum that admits such a homeomorphism. However, most of the known examples of such continua have a complicated topological structure. For example, they are indecomposable (such as the pseudo-arc or the Knaster bucket-handle continuum), or they are not indecomposable but have some other complicated topological structure, such as a dense set of ramification points (such as the Sierpiński carpet) or a dense set of end-points (such as the Lelek fan). In this paper, we continue our mission of finding continua with simpler topological structures that admit a transitive homeomorphism. We construct a transitive homeomorphism on the Cantor fan. In our approach, we use two different techniques, each of them giving two constructions of a transitive homeomorphism on the Cantor fan: one technique using quotient spaces of products of compact metric spaces and Cantor sets, and one using Mahavier products of closed relations on compact metric spaces. We also demonstrate how our technique using Mahavier products of closed relations may be used to construct a transitive function f on a Cantor fan X such that $\varprojlim(X, f)$ is a Lelek fan.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

Key words: closed relations, Mahavier products, transitive dynamical systems, Cantor fans, Lelek fans
2020 Mathematics Subject Classification: 37B02, 37B45 (Primary); 54C60, 54F15, 54F17 (Secondary)

1. *Introduction*

In topological dynamical systems theory, an interesting behavior of a dynamical system (X, f) is often inherited from topological properties of X and from the properties of the mapping f. One of the commonly studied properties in the theory of topological dynamical systems is transitivity of dynamical systems (X, f). In this paper, we are interested in transitive topological dynamical systems (X, f), where X is a continuum and f is a homeomorphism. We say that a continuum X admits a transitive homeomorphism if there is a homeomorphism f on X such that (X, f) is transitive. In the literature, there are many examples of continua that admit a transitive homeomorphism (such as the circle, the pseudo-arc, the Sierpiński carpet, or the Lelek fan). It is often the case that when constructing a non-trivial continuum X that admits a transitive homeomorphism, the continuum X should have a complicated topological structure. Our main focus in this paper is to present various techniques that allow us to construct examples of simple continua that admit transitive homeomorphisms. Then, we demonstrate how they can be applied to show that the Cantor fan admits a transitive homeomorphism.

We present two different techniques for constructing a transitive dynamical system (X, f) on a compact metric space (or a continuum) X.

- *Our technique using quotient spaces.* For a given compact metric space X and for a given family $\mathcal{F}$ of continuous functions on X, we construct a transitive dynamical system (Y, f) (depending on X and $\mathcal{F}$) by defining Y as a quotient space of the topological product $X \times C$, where C is a Cantor set obtained as an infinite product of a finite set. The transitive mapping f on the space Y is then defined using the family $\mathcal{F}$ as well as the shift map on C. We use this technique to present two different proofs that there is a transitive homeomorphism on the Cantor fan.
- *Our technique using closed relations on compact metric spaces.* When constructing a dynamical system with closed relations (X, F), two standard topological dynamical systems (X_F^+, σ_F^+) and (X_F, σ_F) are constructed at the same time. This is a new way of constructing topological dynamical systems with interesting properties. Therefore, it is only natural to study the properties of (X, F) that imply interesting topological or dynamical properties of the systems (X_F^+, σ_F^+) and (X_F, σ_F), where σ_F is a homeomorphism while σ_F^+ is usually not. Here, we show that the shift homeomorphism on X_F is transitive if and only if the shift map on X_F^+ is transitive. We also study closed relations on X that are unions of graphs of continuous functions. Then, we apply these results to obtain two transitive homeomorphisms on the Cantor fan.

  At the end, we also study σ-transitivity of dynamical systems with closed relations. One of the results we prove is that σ-transitivity of (X, F) is equivalent to the transitivity of the dynamical system (X_F^+, σ_F^+). We conclude the paper by giving an illustrative example showing that there is a transitive function f on a Cantor fan X such that $\varprojlim(X, f)$ is a Lelek fan.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

In the second technique, the homeomorphism is uniquely defined just by defining the relation F, while in the first technique, the homeomorphism is uniquely defined by defining two things: the family $\mathcal{F}$ of continuous functions on X and the shift map on the Cantor set.

As already mentioned, to construct an interesting dynamical system (X, φ), where X is a continuum and φ a homeomorphism, the continuum X often needs to have some complicated topological structure (see [3–8, 14, 22–25, 27, 28], where examples of such continua are obtained). In this paper, we are interested in the transitivity of such dynamical systems (X, φ). To date, many transitive dynamical systems on continua have been constructed. For example, on a circle, any irrational rotation is a transitive homeomorphism. In [16], Kennedy, and in [21], Minc and Transue constructed independently transitive homeomorphisms on the pseudo-arc. Minc and Transue's construction is based on inverse limits of unit intervals. In [11], Činč and Oprocha constructed a complete space of transitive homeomorphisms with various additional dynamical properties on the pseudo-arc. This generalizes the results from [16, 21]. In [12], Handel constructed transitive homeomorphisms on the pseudo-circle. In [20], Martínez-de-la-Vega, Martínez-Montejano, and Mouron constructed a transitive mapping (actually mixing) on Ważewski's universal dendrite, which, after taking the inverse limit, produces a mixing homeomorphism on a hereditarily decomposable tree-like continuum by using the shift, and in [8], Boroński, and Oprocha constructed a transitive homeomorphism on the Sierpiński carpet. In [3], Banič, Erceg, and Kennedy present a transitive homeomorphism on the Lelek fan. Their homeomorphism has non-zero entropy. In [27], motivated by [3], Oprocha constructed a transitive homeomorphism on the Lelek fan; the entropy of his homeomorphism is zero.

In this paper, we develop various techniques that are applied to study transitive mappings on the Cantor fan. We use our techniques to obtain four different elementary proofs of the following theorem.

THEOREM 1.1. *There is a transitive homeomorphism on the Cantor fan.*

Using closed relations on compact metric spaces, we also prove the following theorem.

THEOREM 1.2. *There is a transitive function f on a Cantor fan X such that $\varprojlim(X, f)$ is a Lelek fan. In addition, the shift map on $\varprojlim(X, f)$ is a transitive homeomorphism.*

We believe that our techniques can be applied to produce other homeomorphisms and mappings to study dynamical properties of other compact metric spaces.

We proceed as follows. In §2, the basic definitions and results that are needed later in the paper are presented. In §3, we construct various dynamical systems using quotient spaces of topological products $X \times C$, where X is a compact metric space and C is a Cantor set. Here, we give our first two (of four) proofs of Theorem 1.1. In §4, we study dynamical systems with closed relations. Using this, we give two additional proofs of Theorem 1.1. In §5, we study σ-transitivity of dynamical systems with closed relations. Then we prove Theorem 1.2.

2. *Definitions and notation*

The following definitions, notation, and well-known results are needed in the paper.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

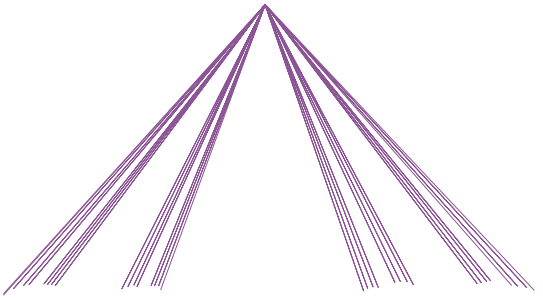


FIGURE 1. The Cantor fan.

Definition 2.1. Let X and Y be metric spaces, and let $f : X \to Y$ be a function. We use $\Gamma(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$ to denote *the graph of the function f*.

Definition 2.2. Let X be a metric space, $x \in X$, and $\varepsilon > 0$. We use $B(x, \varepsilon)$ to denote the open ball centered at x with radius ε.

Definition 2.3. We use $\mathbb{N}$ to denote the set of positive integers and $\mathbb{Z}$ to denote the set of integers.

Definition 2.4. Let X be a non-empty compact metric space and let $F \subseteq X \times X$ be a relation on X. If F is closed in $X \times X$, then we say that F is *a closed relation on X*.

Definition 2.5. *A continuum* is a non-empty compact connected metric space. *A subcontinuum* is a subspace of a continuum, which is itself a continuum.

Definition 2.6. Let X be a continuum. We say that X is *a Cantor fan* if X is homeomorphic to the continuum $\bigcup_{c \in C} A_c$, where $C \subseteq [0, 1]$ is a Cantor set and for each $c \in C$, A_c is the convex segment in the plane from $(0, 0)$ to $(c, -1)$.

Definition 2.7. Let X be a Cantor fan and let Y be a subcontinuum of X. A point $x \in Y$ is called an *end-point of the continuum Y* if for every arc A in Y that contains x, x is an end-point of A. The set of all end-points of Y will be denoted by $E(Y)$.

Definition 2.8. Let X be a Cantor fan and let Y be a subcontinuum of X. We say that Y is *a Lelek fan* if $\mathrm{Cl}(E(Y)) = Y$.

The Lelek fan was constructed by Lelek in [18]. An interesting property of the Lelek fan X is the fact that the set of its end-points is a dense one-dimensional set in X. It is also unique, that is, it is the only non-degenerate smooth fan with a dense set of end-points. This was proved independently by Bula and Oversteegen in [9] and by Charatonik in [10]. They proved that any two non-degenerate subcontinua of the Cantor fan with a dense set of endpoints are homeomorphic. See [26] for more information about continua, fans, and their properties.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

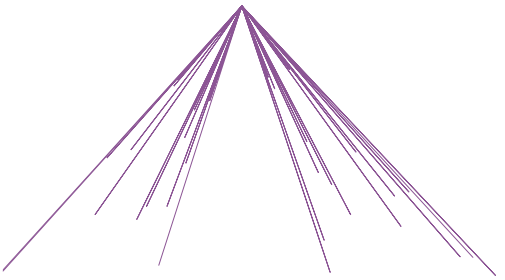


FIGURE 2. The Lelek fan.

Definition 2.9. Let X be a non-empty compact metric space and let F be a closed relation on X. Then, we call

$$X_F^m = \left\{(x_1, x_2, \ldots, x_{m+1}) \in \prod_{i=1}^{m+1} X \;\middle|\; \text{for each } i \in \{1, 2, \ldots, m\}, (x_i, x_{i+1}) \in F\right\}$$

for each positive integer m, *the mth Mahavier product of* F, we call

$$X_F^+ = \left\{(x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X \;\middle|\; \text{for each positive integer } i, (x_i, x_{i+1}) \in F\right\}$$

the Mahavier product of F, and we call

$$X_F = \left\{(\ldots, x_{-1}, x_0; x_1, x_2, \ldots) \in \prod_{i=-\infty}^{\infty} X \;\middle|\; \text{for each integer } i, (x_i, x_{i+1}) \in F\right\}$$

the two-sided Mahavier product of F.

Definition 2.10. Let X be a non-empty compact metric space and let F be a closed relation on X. The function $\sigma_F^+ : X_F^+ \to X_F^+$, defined by

$$\sigma_F^+(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_3, x_4, \ldots)$$

for each $(x_1, x_2, x_3, x_4, \ldots) \in X_F^+$, is called *the shift map on* X_F^+. The function $\sigma_F : X_F \to X_F$, defined by

$$\sigma_F(\ldots, x_{-3}, x_{-2}, x_{-1}, x_0; x_1, x_2, x_3, \ldots) = (\ldots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1; x_2, x_3, \ldots)$$

for each $(\ldots, x_{-3}, x_{-2}, x_{-1}, x_0; x_1, x_2, x_3, \ldots) \in X_F$, is called *the shift map on* X_F.

Observation 2.1. Note that σ_F is always a homeomorphism, while σ_F^+ may not be a homeomorphism.

Definition 2.11. Let (X, f) be a dynamical system. We say that (X, f) is:

(1) *transitive* if for all non-empty open sets U and V in X, there is a non-negative integer n such that $f^n(U) \cap V \neq \emptyset$;

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

(2) *dense orbit transitive* if there is a point $x \in X$ such that its trajectory $\mathcal{O}_f^{\oplus}(x) = \{x, f(x), f^2(x), f^3(x), \ldots\}$ is dense in X. We call such a point x *a transitive point in* (X, f).

Definition 2.12. Let (X, f) be a dynamical system. We say that the mapping f is *transitive* if (X, f) is transitive.

Observation 2.2. It is a well-known fact that if X has no isolated points, then (X, f) is transitive if and only if (X, f) is dense orbit transitive. Also, if f is a homeomorphism, then (X, f) is transitive if and only if (X, f^{-1}) is transitive. See [1, 17] for more information about transitive dynamical systems.

Definition 2.13. Suppose X is a compact metric space. If $f : X \to X$ is a continuous function, the *inverse limit space* generated by f is the subspace

$$\varprojlim(X, f) = \left\{(x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X \;\middle|\; \text{for each positive integer } i,\ x_i = f(x_{i+1})\right\}$$

of the topological product $\prod_{i=1}^{\infty} X$. The function $\sigma : \varprojlim(X, f) \to \varprojlim(X, f)$, defined by

$$\sigma(x_1, x_2, x_3, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$$

for each $(x_1, x_2, x_3, \ldots) \in \varprojlim(X, f)$, is called *the shift map on* $\varprojlim(X, f)$.

The following theorem is a well-known result proved in [13], and it also easily follows from the results from [19].

THEOREM 2.3. *Let (X, f) be a dynamical system. If (X, f) is transitive and $\sigma : \varprojlim(X, f) \to \varprojlim(X, f)$ is the shift map on $\varprojlim(X, f)$, then also $(\varprojlim(X, f), \sigma)$ is a transitive dynamical system.*

Observation 2.4. For any dynamical system (X, f), the shift map σ on $\varprojlim(X, f)$ is a homeomorphism from $\varprojlim(X, f)$ to $\varprojlim(X, f)$.

3. *Transitive homeomorphisms on quotient spaces*

In this section, we give our first two proofs of Theorem 1.1. We start the section with some well-known results that are needed later. To make the paper more reader friendly, we give their proofs (since the proofs are elementary and short).

Definition 3.1. Let X be a compact metric space and let $\sim$ be an equivalence relation on X. We use:

(1) $[x]$ to denote the equivalence class $\{y \in X \mid y \sim x\}$ of the element x for any $x \in X$;
(2) $X/_\sim$ to denote the quotient space $\{[x] \mid x \in X\}$;
(3) q to denote the natural quotient mapping $X \to X/_\sim$, which is defined by $q(x) = [x]$ for any $x \in X$;
(4) and for any $U \subseteq X/_\sim$, we define

$$U \text{ is open in } X/_\sim \iff q^{-1}(U) \text{ is open in } X.$$

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

Observation 3.1. Let X be a compact metric space and let $\sim$ be an equivalence relation on X. Then, the following hold.

(1) The quotient map q is a continuous surjection.
(2) If the quotient space $X/\sim$ is a Hausdorff space, then it is a metrizable compactum. Therefore, in this case, $X/\sim$ is a continuum, if it is connected.

Definition 3.2. Let X be a compact metric space, let $\sim$ be an equivalence relation on X, and let $f : X \to X$ be a function such that for all $x, y \in X$,

$$x \sim y \Longleftrightarrow f(x) \sim f(y).$$

Then, we let $f^\star : X/\sim \to X/\sim$ be defined by $f^\star([x]) = [f(x)]$ for any $x \in X$.

The following theorem is a well-known result. Since its proof is short, we give it for the completeness of the paper.

THEOREM 3.2. *Let X be a compact metric space, let $\sim$ be an equivalence relation on X, and let $f : X \to X$ be a function such that for all $x, y \in X$,*

$$x \sim y \Longleftrightarrow f(x) \sim f(y).$$

Then, the following hold.

(1) *$f^\star$ is a well-defined function from $X/\sim$ to $X/\sim$.*
(2) *If f is continuous, then $f^\star$ is continuous.*
(3) *If f is a homeomorphism, then $f^\star$ is a homeomorphism.*
(4) *If f is transitive, then $f^\star$ is transitive.*

Proof. To see that $f^\star$ is a well-defined function from $X/\sim$ to $X/\sim$, let $x, y \in X$ be such that $x \sim y$. It follows that

$$f^\star([x]) = [f(x)] = [f(y)] = f^\star([y]).$$

Next, suppose that f is continuous. To show that $f^\star$ is continuous, let U be an open set in $X/\sim$. We show that $(f^\star)^{-1}(U)$ is open in $X/\sim$ by showing that $q^{-1}((f^\star)^{-1}(U))$ is open in X. Since $f^\star \circ q = q \circ f$, it follows that

$$q^{-1}((f^\star)^{-1}(U)) = (f^\star \circ q)^{-1}(U) = (q \circ f)^{-1}(U).$$

Since f and q are continuous, it follows that $q^{-1}((f^\star)^{-1}(U))$ is open in X. Therefore, $f^\star$ is continuous. Now, suppose that f is a homeomorphism. To see that $f^\star$ is bijective, let $g : X/\sim \to X/\sim$ be defined by $g([x]) = [f^{-1}(x)]$ for any $x \in X$. Note that g is a well-defined function (since $g = (f^{-1})^\star$). It follows that for each $x \in X$,

$$\begin{aligned} f^\star(g([x])) &= f^\star([f^{-1}(x)]) = [f(f^{-1}(x))] = [x], \\ g(f^\star([x])) &= g([f(x)]) = [f^{-1}(f(x))] = [x]. \end{aligned}$$

Therefore, $f^\star$ is bijective and $(f^\star)^{-1} = g$. The proof that g is continuous is similar to the proof that $f^\star$ is continuous. We leave the details to the reader. Since $f^\star$ and $(f^\star)^{-1}$ are both continuous, it follows that $f^\star$ is a homeomorphism. Finally, suppose that f is transitive. Let U and V be any non-empty open sets in $X/\sim$. We prove that there

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

is a non-negative integer n such that $(f^\star)^n(U) \cap V \neq \emptyset$. Since the quotient map is a continuous surjection, it follows that $q^{-1}(U)$ and $q^{-1}(V)$ are non-empty open subsets of X. Let n be a non-negative integer such that $f^n(q^{-1}(U)) \cap q^{-1}(V) \neq \emptyset$ and let $x \in f^n(q^{-1}(U)) \cap q^{-1}(V)$. We show that $[x] \in (f^\star)^n(U) \cap V$. It follows from $x \in q^{-1}(V)$ that $q(x) \in q(q^{-1}(V))$ and, therefore, $[x] \in V$. Also, it follows from $x \in f^n(q^{-1}(U))$ that $q(x) \in q(f^n(q^{-1}(U)))$ and, therefore, $[x] \in q(f^n(q^{-1}(U)))$. To show that also $[x] \in (f^\star)^n(U)$, we prove that $q(f^n(q^{-1}(U))) = (f^\star)^n(U)$:

$$\begin{aligned}(f^\star)^n(U) &= \{(f^\star)^n([t]) \mid [t] \in U\} = \{[f^n(t)] \mid q(t) \in U\} \\ &= \{q(f^n(t)) \mid q(t) \in U\} = q(f^n(\{t \mid q(t) \in U\})) = q(f^n(q^{-1}(U))).\end{aligned}$$

This completes the proof. □

Let X be a set and let n be a positive integer. For each positive integer i and for each point $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \prod_{k=1}^{\infty} X$ (or for each $i \in \{1, 2, 3, \ldots, n\}$ and for each point $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n) \in \prod_{k=1}^{n} X$ or for each integer i and for each point $\mathbf{x} = (\ldots, x_{-2}, x_{-1}, x_0; x_1, x_2, \ldots) \in \prod_{k=-\infty}^{\infty} X$), we use $\mathbf{x}(i)$ to denote the ith coordinate x_i of the point $\mathbf{x}$. For all positive integers i and j (or for all $i, j \in \{1, 2, 3, \ldots, n\}$ or for all integers i and j) such that $i \leq j$, we use $[i, j]$ to denote the set $\{i, i+1, i+2, \ldots, j\}$ and $\pi_{[i,j]}$ to denote the projection $\pi_{[i,j]} : \prod_{k=1}^{\infty} X \to \prod_{k=i}^{j} X$ (or the projection $\pi_{[i,j]} : \prod_{k=1}^{n} X \to \prod_{k=i}^{j} X$ or the projection $\pi_{[i,j]} : \prod_{k=-\infty}^{\infty} X \to \prod_{k=i}^{j} X$) that is defined by $\pi_{[i,j]}(x_1, x_2, x_3, \ldots) = (x_i, x_{i+1}, x_{i+2}, \ldots, x_j)$ for any $(x_1, x_2, x_3, \ldots) \in \prod_{k=1}^{\infty} X$ (or $\pi_{[i,j]}(x_1, x_2, \ldots, x_n) = (x_i, x_{i+1}, \ldots, x_j)$ for any $(x_1, x_2, \ldots, x_n) \in \prod_{k=1}^{n} X$ or $\pi_{[i,j]}(\ldots, x_{-2}, x_{-1}, x_0; x_1, x_2, \ldots) = (x_i, x_{i+1}, x_{i+2}, \ldots, x_j)$ for any $(\ldots, x_{-2}, x_{-1}, x_0; x_1, x_2, \ldots) \in \prod_{k=-\infty}^{\infty} X$). For $i = j$, we use π_i to denote the projection $\pi_{[i,i]}$.

For a non-empty compact metric space X, we use $p_1 : X \times X \to X$ and $p_2 : X \times X \to X$ to denote *the standard projections* defined by $p_1(s, t) = s$ and $p_2(s, t) = t$ for all $(s, t) \in X \times X$.

Definition 3.3. For each positive integer n, we use the following.

(1) C_n to denote the topological product $C_n = \prod_{k=1}^{\infty} \{1, 2, 3, \ldots, n\}$, where $\{1, 2, 3, \ldots, n\}$ is equipped with the discrete topology. We also use σ_n to denote the shift map $\sigma_n : C_n \to C_n$, defined by

$$\sigma_n(\mathbf{x}) = \sigma_n(\mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3), \ldots) = (\mathbf{x}(2), \mathbf{x}(3), \mathbf{x}(4), \ldots)$$

for any $\mathbf{x} \in C_n$.

(2) D_n to denote the topological product $D_n = \prod_{k=-\infty}^{\infty} \{1, 2, 3, \ldots, n\}$, where the set $\{1, 2, 3, \ldots, n\}$ is equipped with the discrete topology. We also use τ_n to denote the shift map $\tau_n : D_n \to D_n$, defined by

$$\tau_n(\mathbf{x}) = \tau_n(\ldots, \mathbf{x}(-1), \mathbf{x}(0); \mathbf{x}(1), \mathbf{x}(2), \ldots) = (\ldots, \mathbf{x}(-1), \mathbf{x}(0), \mathbf{x}(1); \mathbf{x}(2), \ldots)$$

for any $\mathbf{x} \in D_n$.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

Observation 3.3. Note that for each positive integer $n \geq 2$, C_n and D_n (equipped with the product topologies) are Cantor sets and that the shift maps σ_n and τ_n are transitive. Also, note that for each positive integer n, τ_n is a homeomorphism while σ_n is not.

Definition 3.4. Let X be a compact metric space, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any functions. Then, we use:

(1) $(f_1, \ldots, f_n)_{C_n}$ to denote the mapping from $C_n \times X$ to $C_n \times X$, which is defined by

$$(f_1, \ldots, f_n)_{C_n}(\mathbf{x}, t) = (\sigma_n(\mathbf{x}), f_{\mathbf{x}(1)}(t))$$

for any $(\mathbf{x}, t) \in C_n \times X$. We call $(f_1, \ldots, f_n)_{C_n}$ *a sorting function with respect to the first projection of* C_n;

(2) $(f_1, \ldots, f_n)_{D_n}$ to denote the mapping from $D_n \times X$ to $D_n \times X$, which is defined by

$$(f_1, \ldots, f_n)_{D_n}(\mathbf{x}, t) = (\tau_n(\mathbf{x}), f_{\mathbf{x}(1)}(t))$$

for any $(\mathbf{x}, t) \in D_n \times X$. We call $(f_1, \ldots, f_n)_{D_n}$ *a sorting function with respect to the first projection of* D_n.

In Example 3.4, we demonstrate how the defined sorting functions work.

Example 3.4. Let $n = 2$, let $X = [0, 1]$, and let $f_1, f_2 : [0, 1] \to [0, 1]$ be defined by $f_1(t) = t$ and $f_2(t) = t^2$ for any $t \in [0, 1]$. Then, for $\mathbf{x} = (1, 2, 1, 2, 1, \ldots)$, $\mathbf{x}(1) = 1$ and it follows that for each $t \in [0, 1]$,

$$\begin{aligned}(f_1, f_2)_{C_2}(\mathbf{x}, t) &= (\sigma_2(\mathbf{x}), f_{\mathbf{x}(1)}(t)) \\ &= (\sigma_2(1, 2, 1, 2, 1, \ldots), f_1(t)) = ((2, 1, 2, 1, \ldots), t),\end{aligned}$$

and for $\mathbf{x} = (\ldots, 2, 1; 2, 1, \ldots)$, $\mathbf{x}(1) = 2$ and it follows that for each $t \in [0, 1]$,

$$\begin{aligned}(f_1, f_2)_{D_2}(\mathbf{x}, t) &= (\tau_2(\mathbf{x}), f_{\mathbf{x}(1)}(t)) \\ &= (\tau_2(\ldots, 2, 1; 2, 1, \ldots), f_2(t)) = ((\ldots, 2, 1, 2; 1, \ldots), t^2).\end{aligned}$$

PROPOSITION 3.5. *Let X be a compact metric space, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any functions. Then, the following hold.*

(1) *If $f_1, f_2, f_3, \ldots, f_n$ are continuous, then $(f_1, \ldots, f_n)_{C_n}$ and $(f_1, \ldots, f_n)_{D_n}$ are continuous.*

(2) *If $f_1, f_2, f_3, \ldots, f_n$ are surjective, then $(f_1, \ldots, f_n)_{C_n}$ and $(f_1, \ldots, f_n)_{D_n}$ are surjective.*

(3) *If $f_1, f_2, f_3, \ldots, f_n$ are homeomorphisms, then $(f_1, \ldots, f_n)_{D_n}$ is a homeomorphism.*

Proof. Suppose that the functions $f_1, f_2, f_3, \ldots, f_n$ are continuous. First, we prove that $(f_1, \ldots, f_n)_{C_n}$ is continuous. Let W be an open set in X and let V be an open set in C_n such that $\operatorname{diam}(V) < 1$ and such that for some $k \in \{1, 2, 3, \ldots, n\}$, $\pi_1(V) = \{k\}$ (this can be done since $\{1, 2, 3, \ldots, n\}$ is equipped with the discrete topology). Fix such an element k. Also, let $U = V \times W$. To see that $(f_1, \ldots, f_n)_{C_n}$ is continuous, we prove that

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

$(f_1, \ldots, f_n)_{C_n}^{-1}(U)$ is open in $C_n \times X$. Observe that

$$(f_1, \ldots, f_n)_{C_n}^{-1}(U) = (f_1, \ldots, f_n)_{C_n}^{-1}(V \times W) = \sigma_n^{-1}(V) \times f_k^{-1}(W).$$

Since the functions σ_n and f_k are continuous, it follows that $(f_1, \ldots, f_n)_{C_n}^{-1}(U)$ is open in $C_n \times X$.

The proof that $(f_1, \ldots, f_n)_{D_n}$ is continuous is analogous to the proof above—we leave it for the reader.

Now, suppose that the functions $f_1, f_2, f_3, \ldots, f_n$ are surjective. To show that $(f_1, \ldots, f_n)_{C_n}$ is surjective, let $(\mathbf{x}, t) \in C_n \times X$ be any point. Since σ_n is surjective, there is a point $\mathbf{t} \in C_n$ such that $\sigma_n(\mathbf{t}) = \mathbf{x}$. Choose such a point $\mathbf{t}$. Then, for each $x \in f_{\mathbf{t}(1)}^{-1}(t)$,

$$(f_1, \ldots, f_n)_{C_n}(\mathbf{t}, x) = (\sigma_n(\mathbf{t}), f_{\mathbf{t}(1)}(x)) = (\mathbf{x}, t);$$

therefore, $(f_1, \ldots, f_n)_{C_n}$ is surjective since $f_{\mathbf{t}(1)}^{-1}(t) \neq \emptyset$.

The proof that $(f_1, \ldots, f_n)_{D_n}$ is surjective is analogous to the proof above that $(f_1, \ldots, f_n)_{C_n}$ is surjective—we leave it for the reader.

Finally, suppose that $f_1, f_2, f_3, \ldots, f_n$ are homeomorphisms. It follows from item (1) that $(f_1, \ldots, f_n)_{D_n}$ is a continuous function. To see that $(f_1, \ldots, f_n)_{D_n}$ is a bijection, let $h = (f_1, \ldots, f_n)_{D_n}$ and let $g : D_n \times X \to D_n \times X$ be defined by

$$g(\mathbf{x}, t) = (\tau_n^{-1}(\mathbf{x}), f_{\tau_n^{-1}(\mathbf{x})(1)}^{-1}(t))$$

for any $(\mathbf{x}, t) \in D_n \times X$. Observe that for each $(\mathbf{x}, t) \in D_n \times X$,

$$\begin{aligned} g(h(\mathbf{x}, t)) &= g(\tau_n(\mathbf{x}), f_{\mathbf{x}(1)}(t)) \\ &= (\tau_n^{-1}(\tau_n(\mathbf{x})), f_{\tau_n^{-1}(\tau_n(\mathbf{x}))(1)}^{-1}(f_{\mathbf{x}(1)}(t))) = (\mathbf{x}, t), \\ h(g(\mathbf{x}, t)) &= h(\tau_n^{-1}(\mathbf{x}), f_{\tau_n^{-1}(\mathbf{x})(1)}^{-1}(t)) \\ &= (\tau_n(\tau_n^{-1}(\mathbf{x})), f_{\tau_n^{-1}(\mathbf{x})(1)}(f_{\tau_n^{-1}(\mathbf{x})(1)}^{-1}(t))) = (\mathbf{x}, t). \end{aligned}$$

It follows that $(f_1, \ldots, f_n)_{D_n}$ is a bijection. Since $(f_1, \ldots, f_n)_{D_n}$ is a continuous bijection from a compact metric space onto itself, it is a homeomorphism. □

In the following subsections, we examine separately the quotient spaces $C_n \times X$ and $D_n \times X$. We will use the notion of different types of concatenation that is defined in the following definition.

Definition 3.5. Let X be a compact metric space, let n and m be positive integers, and let $\mathbf{x} \in \prod_{k=1}^{n} X$, $\mathbf{y} \in \prod_{k=1}^{m} X$, $\mathbf{z} \in \prod_{i=1}^{\infty} X$, and $\mathbf{w} \in \prod_{i=-\infty}^{0} X$. Then, we define:

(1) *the concatenation* $\mathbf{x} \oplus \mathbf{y} \in \prod_{k=1}^{m+n} X$ of $\mathbf{x}$ and $\mathbf{y}$ as follows:

$$\mathbf{x} \oplus \mathbf{y} = (\mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3), \ldots, \mathbf{x}(n), \mathbf{y}(1), \mathbf{y}(2), \mathbf{y}(3), \ldots, \mathbf{y}(m));$$

(2) *the concatenation* $\mathbf{x} \oplus \mathbf{z} \in \prod_{k=1}^{\infty} X$ of $\mathbf{x}$ and $\mathbf{z}$ as follows:

$$\mathbf{x} \oplus \mathbf{z} = (\mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3), \ldots, \mathbf{x}(n), \mathbf{z}(1), \mathbf{z}(2), \mathbf{z}(3), \ldots);$$

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

(3) *the concatenation* $\mathbf{w} \oplus \mathbf{x} \in \prod_{k=-\infty}^{0} X$ of $\mathbf{w}$ and $\mathbf{x}$ as follows:

$$\mathbf{w} \oplus \mathbf{x} = (\ldots, \mathbf{w}(-3), \mathbf{w}(-2), \mathbf{w}(-1), \mathbf{w}(0), \mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3), \ldots, \mathbf{x}(n));$$

(4) *the concatenation* $\mathbf{w} \oplus \mathbf{z} \in \prod_{k=-\infty}^{\infty} X$ of $\mathbf{w}$ and $\mathbf{z}$ as follows:

$$\mathbf{w} \oplus \mathbf{z} = (\ldots, \mathbf{w}(-3), \mathbf{w}(-2), \mathbf{w}(-1), \mathbf{w}(0); \mathbf{z}(1), \mathbf{z}(2), \mathbf{z}(3), \ldots).$$

3.1. *The quotient spaces of* $C_n \times X$.

Definition 3.6. Let X be a compact metric space, let n be a positive integer, let $\mathbf{x} \in \prod_{k=1}^{n} X$ and $\mathbf{y} \in \prod_{k=1}^{\infty} X$, and let $\varepsilon > 0$. We say that $|\mathbf{x} - \mathbf{y}|_n < \varepsilon$, if:

(1) for each $k \in \{1, 2, 3, \ldots, n\}$, $\mathbf{x}(k) = \mathbf{y}(k)$ and

(2) $\sum_{k=n}^{\infty} \operatorname{diam}(X)/2^k < \varepsilon$.

Definition 3.7. Let X be a compact metric space, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any functions. Also, let i and j be integers such that $i \leq j$ and let $\mathbf{x} \in \prod_{k=i}^{j} \{1, 2, 3, \ldots, n\}$. Then, we use $f_{\mathbf{x}}$ to denote the function $f_{\mathbf{x}} : X \to [0, 1]$, defined by

$$f_{\mathbf{x}}(t) = (f_{\mathbf{x}(j)} \circ f_{\mathbf{x}(j-1)} \circ f_{\mathbf{x}(j-2)} \circ \cdots \circ f_{\mathbf{x}(i+2)} \circ f_{\mathbf{x}(i+1)} \circ f_{\mathbf{x}(i)})(t)$$

for any $t \in X$.

Definition 3.8. Let (X, d) be a compact metric space, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any functions. Also, let U be a dense open set in X such that for each $j \in \{1, 2, 3, \ldots, n\}$, $f_j(U) \subseteq U$. We say that *the n-tuple* $(f_1, f_2, f_3, \ldots, f_n)$ *has property* $\mathcal{L}(U)$ if for each $\varepsilon > 0$, for each $x \in U$, and for each $z \in X$, there are non-negative integers $k_1, k_2, k_3, \ldots, k_n$ such that

$$d((f_1^{k_1} \circ f_2^{k_2} \circ f_3^{k_3} \circ \cdots \circ f_n^{k_n})(x), z) < \varepsilon.$$

We say that *the n-tuple* $(f_1, f_2, f_3, \ldots, f_n)$ *has property* $\mathcal{L}$ if there is an open set U in X such that the n-tuple $(f_1, f_2, f_3, \ldots, f_n)$ has property $\mathcal{L}(U)$.

THEOREM 3.6. *Let X be a compact metric space that does not have any isolated points, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any homeomorphisms. If the n-tuple $(f_1, f_2, f_3, \ldots, f_n)$ has property $\mathcal{L}$, then $(f_1, \ldots, f_n)_{C_n}$ is a transitive continuous surjection.*

Proof. Let U be an open set in X such that the n-tuple $(f_1, f_2, f_3, \ldots, f_n)$ has property $\mathcal{L}(U)$. To prove that $(f_1, \ldots, f_n)_{C_n}$ is a transitive continuous surjection, let $h = (f_1, \ldots, f_n)_{C_n}$. It follows from Proposition 3.5 that h is a continuous surjection. We show that h is transitive by constructing a transitive point in $(C_n \times X, h)$. Let $\{(\mathbf{x}_i, t_i) \mid i \text{ is a positive integer}\}$ be a countable dense subset of $C_n \times X$. Also, for each positive integer i, let $\varepsilon_i > 0$ be such that $\lim_{i \to \infty} \varepsilon_i = 0$. For each positive integer i, let m_i be a positive integer and let $\hat{\mathbf{x}}_i \in \prod_{k=1}^{m_i} \{1, 2, 3, \ldots, n\}$ be such that $|\hat{\mathbf{x}}_i - \mathbf{x}_i|_{m_i} < \varepsilon_i$. Note that for each positive integer i, such elements m_i and $\hat{\mathbf{x}}_i$ do exist since $\{(\mathbf{x}_i, t_i) \mid i \text{ is a positive integer}\}$ is a dense subset of $C_n \times X$ and since $C_n \times X$ is equipped

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

with the product topology. For each $x \in U$, for each $z \in X$, and for each $\varepsilon > 0$, let $k_1(x, z, \varepsilon), k_2(x, z, \varepsilon), k_3(x, z, \varepsilon), \ldots, k_n(x, z, \varepsilon)$ be positive integers such that

$$d((f_1^{k_1(x,z,\varepsilon)} \circ f_2^{k_2(x,z,\varepsilon)} \circ f_3^{k_3(x,z,\varepsilon)} \circ \cdots \circ f_n^{k_n(x,z,\varepsilon)})(x), z) < \varepsilon,$$

and let

$$Y(x, z, \varepsilon) = (\underbrace{n, n, \ldots, n}_{k_n(x,z,\varepsilon)}, \underbrace{n-1, n-1, \ldots, n-1}_{k_{n-1}(x,z,\varepsilon)}, \underbrace{n-2, n-2, \ldots, n-2}_{k_{n-2}(x,z,\varepsilon)}, \ldots, \underbrace{1, 1, \ldots, 1}_{k_1(x,z,\varepsilon)}).$$

Now, we inductively define points $\mathbf{w}_i$ in C_n (based on i being even or odd) as follows. Let $x_0 \in U$ and let $\hat{\mathbf{w}}_1 = Y(x_0, t_1, \varepsilon_1)$, let $\hat{\mathbf{w}}_2 = \hat{\mathbf{w}}_1 \oplus \hat{\mathbf{x}}_1$, and let $\hat{\mathbf{w}}_3 = \hat{\mathbf{w}}_2 \oplus Y(f_{\hat{\mathbf{w}}_2}(x_0), t_2, \varepsilon_2)$. Note that since $f_{\hat{\mathbf{w}}_2}(U) \subseteq U$, it follows that $f_{\hat{\mathbf{w}}_2}(x_0) \in U$. Let i be any positive integer and suppose that $\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, \hat{\mathbf{w}}_3, \ldots, \hat{\mathbf{w}}_{2i-1}$ have already been defined. Then, we define $\hat{\mathbf{w}}_{2i}$ and $\hat{\mathbf{w}}_{2i+1}$ as follows: $\hat{\mathbf{w}}_{2i} = \hat{\mathbf{w}}_{2i-1} \oplus \hat{\mathbf{x}}_i$ and $\hat{\mathbf{w}}_{2i+1} = \hat{\mathbf{w}}_{2i} \oplus Y(f_{\hat{\mathbf{w}}_{2i}}(x_0), t_{i+1}, \varepsilon_{i+1})$. Next, for each positive integer i, let $\mathbf{w}_i = \hat{\mathbf{w}}_i \oplus (1, 1, 1, \ldots)$. Note that the sequence $(\mathbf{w}_i)$ is convergent in C_n and let $\mathbf{w} = \lim_{i\to\infty} \mathbf{w}_i$. Finally, we show that $(\mathbf{w}, x_0)$ is a transitive point in $(C_n \times X, h)$. Let V be a non-empty open set in C_n, let W be a non-empty open set in X, and let $U_0 = V \times W$. We show that there is a positive integer k such that $h^k(\mathbf{w}, x_0) \in U_0$. Let k_0 be a positive integer such that $B((\mathbf{x}_{k_0}, t_{k_0}), \varepsilon_{k_0}) \subseteq U_0$. Also, let i and k be any positive integers such that $\pi_{[1,i]}(\sigma_n^k(\mathbf{w})) = \hat{\mathbf{x}}_{k_0}$. Note that k is the length of $\hat{\mathbf{w}}_{2k_0-1}$. Then, $h^k(\mathbf{w}, x_0) \in U_0$ and we are done. □

Observation 3.7. Note that $(f_1, \ldots, f_n)_{C_n}$ from Theorem 3.6 is not a homeomorphism. This follows from the fact that σ_n is not a homeomorphism.

Definition 3.9. Let m and n be positive integers, and let $\mathbf{y} \in C_n$. We use $\mathbf{y}[m]$ to denote $\mathbf{y}[m] = (\mathbf{y}(m), \mathbf{y}(m-1), \mathbf{y}(m-2), \ldots, \mathbf{y}(1))$.

Definition 3.10. Let X be a compact metric space, let m and n be positive integers, let $\mathbf{y} \in C_n$, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any homeomorphisms. We use $f_{\mathbf{y}[m]}^{-1}$ to denote the function $f_{\mathbf{y}[m]}^{-1} : X \to X$ defined by

$$f_{\mathbf{y}[m]}^{-1}(t) = (f_{\mathbf{y}(m)}^{-1} \circ f_{\mathbf{y}(m-1)}^{-1} \circ f_{\mathbf{y}(m-2)}^{-1} \circ \cdots \circ f_{\mathbf{y}(1)}^{-1})(t)$$

for each $t \in X$.

Observation 3.8. Let X be a compact metric space, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any homeomorphisms. Note that for each point $\mathbf{x} \in \varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$, there are uniquely determined points $\mathbf{a}, \mathbf{b} \in C_n$ and $t \in X$ such that

$$\mathbf{x} = ((\mathbf{a}, t), (\mathbf{b}[1] \oplus \mathbf{a}, f_{\mathbf{b}[1]}^{-1}(t)), (\mathbf{b}[2] \oplus \mathbf{a}, f_{\mathbf{b}[2]}^{-1}(t)), (\mathbf{b}[3] \oplus \mathbf{a}, f_{\mathbf{b}[3]}^{-1}(t)), \ldots).$$

THEOREM 3.9. *Let X be a compact metric space, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any homeomorphisms. Then, $\varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$ is homeomorphic to $C_n \times X$.*

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

Proof. Note that $C_n \times X$ is homeomorphic to $(C_n \times C_n) \times X$. We show that the inverse limit $\varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$ is homeomorphic to $(C_n \times C_n) \times X$. For each $\mathbf{x} \in \varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$, let $\mathbf{a}_\mathbf{x}, \mathbf{b}_\mathbf{x} \in C_n$ and $t_\mathbf{x} \in X$ be such that

$$\mathbf{x} = ((\mathbf{a}_\mathbf{x}, t_\mathbf{x}), (\mathbf{b}_\mathbf{x}[1] \oplus \mathbf{a}_\mathbf{x}, f^{-1}_{\mathbf{b}_\mathbf{x}[1]}(t_\mathbf{x})), (\mathbf{b}_\mathbf{x}[2] \oplus \mathbf{a}_\mathbf{x}, f^{-1}_{\mathbf{b}_\mathbf{x}[2]}(t_\mathbf{x})), \ldots).$$

Let $T : \varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n}) \to (C_n \times C_n) \times X$ be defined by $T(\mathbf{x}) = ((\mathbf{a}_\mathbf{x}, \mathbf{b}_\mathbf{x}), t_\mathbf{x})$ for each $\mathbf{x} \in \varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$. Clearly, T is a homeomorphism. □

Definition 3.11. Let X be a compact metric space, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any homeomorphisms. We use $\sigma_{(f_1,\ldots,f_n)_{C_n}}$ to denote the shift homeomorphism on $\varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$.

THEOREM 3.10. *Let X be a compact metric space, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any homeomorphisms. If the n-tuple $(f_1, f_2, f_3, \ldots, f_n)$ has property $\mathcal{L}$, then the shift homeomorphism $\sigma_{(f_1,\ldots,f_n)_{C_n}}$ is transitive.*

Proof. By Theorem 3.6, the function $(f_1, \ldots, f_n)_{C_n}$ is transitive. It follows from Theorem 2.3 that $\sigma_{(f_1,\ldots,f_n)_{C_n}}$ is transitive. □

Definition 3.12. Let X be a compact metric space, let n be a positive integer, let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any homeomorphisms, and let $\sim$ be an equivalence relation on $(C_n \times C_n) \times X$. Then, we define a relation $\sim_{(f_1,\ldots,f_n)_{C_n}}$ on $\varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$ by

$$\mathbf{x} \sim_{(f_1,\ldots,f_n)_{C_n}} \mathbf{y} \iff \mathbf{x} = \mathbf{y} \text{ or } ((\mathbf{a}_\mathbf{x}, \mathbf{b}_\mathbf{x}), t_\mathbf{x}) \sim ((\mathbf{a}_\mathbf{y}, \mathbf{b}_\mathbf{y}), t_\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$.

Observation 3.11. Observe that the defined relation $\sim_{(f_1,\ldots,f_n)_{C_n}}$ is an equivalence relation on $\varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$.

THEOREM 3.12. *Let X be a compact metric space, let n be a positive integer, let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any homeomorphisms, and let $\sim$ be an equivalence relation on $(C_n \times C_n) \times X$. Then, the inverse limit $\varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})/_{\sim_{(f_1,\ldots,f_n)_{C_n}}}$ is homeomorphic to $(C_n \times C_n) \times X/\sim$.*

Proof. Let $\varphi : \varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})/_{\sim_{(f_1,\ldots,f_n)_{C_n}}} \to (C_n \times C_n) \times X/\sim$ be defined by $\varphi([\mathbf{x}]) = [T(\mathbf{x})]$ for any $\mathbf{x} \in \varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$, where T is the homeomorphism from the proof of Theorem 3.9. Then, φ is a well-defined function, which is a homeomorphism. □

THEOREM 3.13. *Let X be a compact metric space, let $\sim$ be an equivalence relation on X, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any homeomorphisms. If the n-tuple $(f_1, f_2, f_3, \ldots, f_n)$ has property $\mathcal{L}$, then there is a transitive homeomorphism on the quotient space*

$$\varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})/_{\sim_{(f_1,\ldots,f_n)_{C_n}}}.$$

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

Proof. By Theorem 3.10, $\sigma_{(f_1,\dots,f_n)_{C_n}}$ is a transitive homeomorphism on the inverse limit $\varprojlim(C_n \times X, (f_1, \dots, f_n)_{C_n})$. Therefore, by Theorem 3.2, the function $\sigma^{\star}_{(f_1,\dots,f_n)_{C_n}}$ is a transitive homeomorphism on the quotient space $\varprojlim(C_n \times X, (f_1, \dots, f_n)_{C_n})/\sim_{(f_1,\dots,f_n)_{C_n}}$. □

Next, we construct, using Theorems 3.12 and 3.13, a transitive homeomorphism on the Cantor fan. First, we prove Lemmas 3.14 and 3.15.

LEMMA 3.14. *Let $\varepsilon > 0$, let $x \in (0, \frac{2}{3}]$, and let $z \in [0, \frac{2}{3}]$. Then, there are positive integers k and n such that*

$$|(\tfrac{1}{2})^{k/2^n} \cdot x^{1/2^n} - z| < \varepsilon.$$

Proof. Note that the set $\{k/2^n \mid k \text{ and } n \text{ are positive integers}\}$ is dense in $[0, \infty)$. Therefore, $\{(\frac{1}{2})^{k/2^n} \mid k \text{ and } n \text{ are positive integers}\}$ is dense in $[0, 1]$. For each positive integer n, we choose a positive integer k_n as follows.

(1) If $z > 0$, then for each positive integer n, let k_n be such a positive integer that for each positive integer k, $|(\frac{1}{2})^{k_n/2^n} - z| \le |(\frac{1}{2})^{k/2^n} - z|$.
(2) If $z = 0$, then for each positive integer n, let k_n be a positive integer such that $(\frac{1}{2})^{k_n/2^n} \le 1/2^n$.

Then, $\lim_{n\to\infty}(\frac{1}{2})^{k_n/2^n} = z$. Since $\lim_{n\to\infty} x^{1/2^n} = 1$, it follows that

$$\lim_{n\to\infty}((\tfrac{1}{2})^{k_n/2^n} \cdot x^{1/2^n}) = \lim_{n\to\infty}(\tfrac{1}{2})^{k_n/2^n} \cdot \lim_{n\to\infty} x^{1/2^n} = z \cdot 1 = z$$

and hence there is a positive integer n such that $|(\frac{1}{2})^{k_n/2^n} \cdot x^{1/2^n} - z| < \varepsilon$. Choose any such n and let $k = k_n$. □

Definition 3.13. We use f_1, f_2, and f_3 to denote the homeomorphisms $f_1, f_2, f_3 : [0, 1] \to [0, 1]$ that are defined by

$$f_1(t) = \sqrt{t}, \quad f_2(t) = \begin{cases} \frac{1}{2}t; & t \le \frac{2}{3} \\ 2t - 1; & t \ge \frac{2}{3} \end{cases}, \quad \text{and} \quad f_3(t) = f_1^{-1}(t)$$

for any $t \in [0, 1]$.

LEMMA 3.15. *Let $\varepsilon > 0$, let $x \in (0, 1)$, and let $z \in [0, 1]$. Then, there are positive integers k, m, and n such that*

$$|f_1^{m+n}(f_2^k(f_3^m(x))) - z| < \varepsilon.$$

Proof. We consider the following possible cases.

(1) $z = 1$. Let $m = k = 1$ and let $y = f_1^m(f_2^k(f_3^m(x)))$. Then, $y \in (0, 1)$. Also, let n be a positive integer such that $|f_1^n(y) - 1| < \varepsilon$. Then,

$$|f_1^{m+n}(f_2^k(f_3^m(x))) - z| = |f_1^n(y) - z| = |f_1^n(y) - 1| < \varepsilon.$$

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

(2) $z < 1$. Let m be a positive integer such that both $f_3^m(x) < \frac{2}{3}$ and $f_3^m(z) < \frac{2}{3}$. Since the map f_1^m is uniformly continuous, there is a $\delta > 0$ such that for all $p, q \in [0, 1]$,

$$|p - q| < \delta \Longrightarrow |f_1^m(p) - f_1^m(q)| < \varepsilon.$$

Choose and fix such a $\delta > 0$. By Lemma 3.14, there are positive integers k and n such that

$$|(\tfrac{1}{2})^{k/2^n} \cdot (f_3^m(x))^{1/2^n} - f_3^m(z)| < \delta.$$

Choose and fix such positive integers k and n. Note that $f_1^n(f_2^k(f_3^m(x))) = (\frac{1}{2})^{k/2^n} \cdot (f_3^m(x))^{1/2^n}$ and that $z = f_1^m(f_3^m(z))$. Therefore,

$$|f_1^{m+n}(f_2^k(f_3^m(x))) - z| = |f_1^m((\tfrac{1}{2})^{k/2^n} \cdot (f_3^m(x))^{1/2^n}) - f_1^m(f_3^m(z))| < \varepsilon. \qquad \square$$

Observation 3.16. Note that it follows from Lemma 3.15 that the 3-tuple (f_1, f_2, f_3) has property $\mathcal{L}$.

Definition 3.14. Let n be a positive integer. If $n \geq 2$, then we use $\sim$ to denote the equivalence relation on $(C_n \times C_n) \times [0, 1]$, defined by

$$((\mathbf{x}_1, \mathbf{x}_2), t) \sim ((\mathbf{y}_1, \mathbf{y}_2), s) \Longleftrightarrow ((\mathbf{x}_1, \mathbf{x}_2), t) = ((\mathbf{y}_1, \mathbf{y}_2), s) \quad \text{or} \quad s = t = 1$$

for all $((\mathbf{x}_1, \mathbf{x}_2), t), ((\mathbf{y}_1, \mathbf{y}_2), s) \in (C_n \times C_n) \times [0, 1]$.

Observation 3.17. Observe that for each positive integer $n \geq 2$, $((C_n \times C_n) \times [0, 1])/\sim$ is a Cantor fan.

Here is our first proof of Theorem 1.1.

Proof. (Our first proof of Theorem 1.1) It follows from Theorem 3.12 that $\varprojlim(C_3 \times X, (f_1, f_2, f_3)_{C_3})/\sim_{(f_1,f_2,f_3)_{C_3}}$ is homeomorphic to $((C_3 \times C_3) \times [0, 1])/\sim$ and it is, therefore, a Cantor fan. Since f_1, f_2, and f_3 have property $\mathcal{L}$, it follows from Theorem 3.13 that there is a transitive homeomorphism on $\varprojlim(C_3 \times X, (f_1, f_2, f_3)_{C_3})/\sim_{(f_1,f_2,f_3)_{C_3}}$. $\square$

3.2. *The quotient spaces of $D_n \times X$.*

THEOREM 3.18. *Let X be a compact metric space that does not have any isolated points, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any homeomorphisms. If the n-tuple $(f_1, f_2, f_3, \ldots, f_n)$ has property $\mathcal{L}$, then $(f_1, \ldots, f_n)_{D_n}$ is a transitive homeomorphism.*

Proof. By Theorem 3.10, the shift homeomorphism $\sigma_{(f_1,\ldots,f_n)_{C_n}}$ on the inverse limit $\varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$ is transitive. Therefore, $\sigma^{-1}_{(f_1,\ldots,f_n)_{C_n}}$ is transitive. We show that $(f_1, \ldots, f_n)_{D_n}$ is transitive by showing that $(f_1, \ldots, f_n)_{D_n}$ and $\sigma^{-1}_{(f_1,\ldots,f_n)_{C_n}}$ are topological conjugates. Let the homeomorphism T from $\varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$ to $(C_n \times C_n) \times X$ be defined as in the proof of Theorem 3.9 by $T(\mathbf{x}) = ((\mathbf{a}_\mathbf{x}, \mathbf{b}_\mathbf{x}), t_\mathbf{x})$ for each

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

$\mathbf{x} \in \varprojlim(C_n \times X, (f_1, \ldots, f_n)_{C_n})$. Also, let $S : (C_n \times C_n) \times X \to D_n \times X$ be defined by

$$S((\mathbf{a}, \mathbf{b}), t) = ((\ldots, \mathbf{b}(3), \mathbf{b}(2), \mathbf{b}(1); \mathbf{a}(1), \mathbf{a}(2), \mathbf{a}(3), \ldots), t)$$

for each $((\mathbf{a}, \mathbf{b}), t)$. Clearly, S is a homeomorphism. One can easily check that

$$(f_1, \ldots, f_n)_{D_n}(\mathbf{x}, t) = (S \circ T \circ \sigma^{-1}_{(f_1,\ldots,f_n)_{C_n}} \circ T^{-1} \circ S^{-1})(\mathbf{x}, t)$$

for each $(\mathbf{x}, t) \in D_n \times X$. □

THEOREM 3.19. *Let X be a compact metric space, let n be a positive integer, and let $f_1, f_2, f_3, \ldots, f_n : X \to X$ be any homeomorphisms. If $f_1, f_2, f_3, \ldots, f_n$ have property $\mathcal{L}$, then there is a transitive homeomorphism on $D_n \times X/\sim$ for any equivalence relation $\sim$ on $D_n \times X$ for which*

$$(\mathbf{x}_1, t_1) \sim (\mathbf{x}_2, t_2) \iff (f_1, \ldots, f_n)_{D_n}(\mathbf{x}_1, t_1) \sim (f_1, \ldots, f_n)_{D_n}(\mathbf{x}_2, t_2)$$

for all $(\mathbf{x}_1, t_1), (\mathbf{x}_2, t_2) \in D_n \times X$.

Proof. Let $\sim$ be an equivalence relation on $D_n \times X$, for which

$$(\mathbf{x}_1, t_1) \sim (\mathbf{x}_2, t_2) \iff (f_1, \ldots, f_n)_{D_n}(\mathbf{x}_1, t_1) \sim (f_1, \ldots, f_n)_{D_n}(\mathbf{x}_2, t_2)$$

for all $(\mathbf{x}_1, t_1), (\mathbf{x}_2, t_2) \in D_n \times X$. By Theorem 3.18, $h = (f_1, \ldots, f_n)_{D_n}$ is a transitive homeomorphism on $D_n \times X$. By Theorem 3.2, $h^\star$ is a transitive homeomorphism on $D_n \times X/\sim$. □

Definition 3.15. We use $\sim$ to denote the equivalence relation on $D_3 \times [0, 1]$, defined by

$$(\mathbf{x}, t) \sim (\mathbf{y}, s) \iff (\mathbf{x}, t) = (\mathbf{y}, s) \quad \text{or} \quad s = t = 1$$

for all $(\mathbf{x}, t), (\mathbf{y}, s) \in D_3 \times [0, 1]$.

Observation 3.20. Observe that $(D_3 \times [0, 1])/\sim$ is a Cantor fan.

We conclude this section by giving our second proof of Theorem 1.1.

Proof. (Our second proof of Theorem 1.1) Let f_1, f_2, and f_3 be the homeomorphisms from Definition 3.13. Recall that f_1, f_2, and f_3 have property $\mathcal{L}$. Also, note that

$$(\mathbf{x}_1, t_1) \sim (\mathbf{x}_2, t_2) \iff (f_1, f_2, f_3)_{D_3}(\mathbf{x}_1, t_1) \sim (f_1, f_2, f_3)_{D_3}(\mathbf{x}_2, t_2)$$

for all $(\mathbf{x}_1, t_1), (\mathbf{x}_2, t_2) \in D_3 \times [0, 1]$. Therefore, by Theorem 3.19, there is a transitive homeomorphism on $(D_3 \times [0, 1])/\sim$. □

4. *Transitive homeomorphisms on Mahavier products of closed relations*

Here, we study relationships between X_F and X_F^+. In §4.1, we show that the shift homeomorphism on X_F is transitive if and only if the shift map on X_F^+ is transitive. In §4.2, we study closed relations on X that are unions of graphs of continuous functions. Then, we apply our results to obtain transitive homeomorphisms on the Cantor fan.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

4.1. *Two-sided Mahavier products.* Theorems 4.1 and 4.5 are the main results of §4.1, where a relation between X_F and X_F^+ is established.

THEOREM 4.1. *Let X be a compact metric space and let F be a closed relation on X. Then, $\varprojlim(X_F^+, \sigma_F^+)$ is homeomorphic to the two-sided Mahavier product X_F. Also, the inverse of the shift map σ_F on X_F is topologically conjugate to the shift map on $\varprojlim(X_F^+, \sigma_F^+)$.*

Proof. Note that for any $\mathbf{x} \in \varprojlim(X_F^+, \sigma_F^+)$, there are unique points $\mathbf{a} \in X_F^+$ and $\mathbf{b} \in X_{F^{-1}}^+$ such that $\mathbf{x} = (\mathbf{a}, (\mathbf{b}(1)) \oplus \mathbf{a}, (\mathbf{b}(2), \mathbf{b}(1)) \oplus \mathbf{a}, (\mathbf{b}(3), \mathbf{b}(2), \mathbf{b}(1)) \oplus \mathbf{a}, \ldots)$. Note that in this case, $(\mathbf{b}(1), \mathbf{a}(1)) \in F$. So, for each $\mathbf{x} \in \varprojlim(X_F^+, \sigma_F^+)$, let $\mathbf{a}_\mathbf{x} \in X_F^+$ and $\mathbf{b}_\mathbf{x} \in X_{F^{-1}}^+$ be such that $\mathbf{x} = (\mathbf{a}_\mathbf{x}, (\mathbf{b}_\mathbf{x}(1)) \oplus \mathbf{a}_\mathbf{x}, (\mathbf{b}_\mathbf{x}(2), \mathbf{b}_\mathbf{x}(1)) \oplus \mathbf{a}_\mathbf{x}, (\mathbf{b}_\mathbf{x}(3), \mathbf{b}_\mathbf{x}(2), \mathbf{b}_\mathbf{x}(1)) \oplus \mathbf{a}_\mathbf{x}, \ldots)$. Let $\varphi : \varprojlim(X_F^+, \sigma_F^+) \to X_F$ be defined by

$$\varphi(\mathbf{x}) = (\ldots, \mathbf{b}_\mathbf{x}(3), \mathbf{b}_\mathbf{x}(2), \mathbf{b}_\mathbf{x}(1); \mathbf{a}_\mathbf{x}(1), \mathbf{a}_\mathbf{x}(2), \mathbf{a}_\mathbf{x}(3), \ldots)$$

for any $\mathbf{x} \in \varprojlim(X_F^+, \sigma_F^+)$. Then, φ is a homeomorphism. Next, let σ be the shift map on $\varprojlim(X_F^+, \sigma_F^+)$. Then, one can easily check that $\sigma_F^{-1} = \varphi \circ \sigma \circ \varphi^{-1}$. □

Next, we give an example of a dynamical system (X, f) such that $(\varprojlim(X, f), \sigma)$ is transitive while (X, f) is not ($\sigma : \varprojlim(X, f) \to \varprojlim(X, f)$ is the shift map on $\varprojlim(X, f)$).

Example 4.2. Let $f : [0, 1] \to [0, 1]$ be defined by $f(t) = 0$ for any $t \in [0, 1]$. Then, $\varprojlim([0, 1], f) = \{(0, 0, 0, \ldots)\}$ and it follows that $(\varprojlim([0, 1], f), \sigma)$ is transitive. Since f is not surjective, $([0, 1], f)$ is not a transitive dynamical system.

However, the following holds.

PROPOSITION 4.3. *Let X be a compact metric space, let $f : X \to X$ be a continuous surjection, and let $\sigma : \varprojlim(X, f) \to \varprojlim(X, f)$ be the shift map on $\varprojlim(X, f)$. The following statements are equivalent.*

(1) *The dynamical system (X, f) is transitive.*
(2) *The dynamical system $(\varprojlim(X, f), \sigma)$ is transitive.*

Proof. The implication from statement (1) to statement (2) follows from Theorem 2.3. To prove the implication from statement (2) to statement (1), suppose that $(\varprojlim(X, f), \sigma)$ is transitive. By Observation 2.2, also $(\varprojlim(X, f), \sigma^{-1})$ is transitive since σ is a homeomorphism. Let U and V be any non-empty open sets in X, and let $W_U = U \times \prod_{k=2}^{\infty} X$ and $W_V = V \times \prod_{k=2}^{\infty} X$. Also, let $\mathbf{x} \in W_U \cap \varprojlim(X, f)$. Since $(\varprojlim(X, f), \sigma^{-1})$ is transitive, there is a non-negative integer n such that $(\sigma^{-1})^n(\mathbf{x}) \in W_V \cap \varprojlim(X, f)$. It follows that for such an n, $f^n(\mathbf{x}(1)) \in V$; therefore, $f^n(\mathbf{x}(1)) \in f^n(U) \cap V$ and $f^n(U) \cap V \neq \emptyset$ follows. □

Example 4.4. Let $X = [0, 1]$ and let $F = \{(t, 1) \mid t \in [0, 1]\}$. Then, $X_F^+ = \{(t, 1, 1, 1, \ldots) \mid t \in [0, 1]\}$ and $X_F = \{(\ldots, 1, 1, 1; 1, 1, 1, \ldots)\}$. Note that in this case, σ_F is transitive but σ_F^+ is not since it is not surjective.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

THEOREM 4.5. *Let X be a compact metric space and let F be a closed relation on X. The following statements are equivalent.*

(1) *The map σ_F^+ is transitive.*
(2) *The homeomorphism σ_F is transitive and σ_F^+ is surjective.*

Proof. Let σ be the shift map on $\varprojlim(X_F^+, \sigma_F^+)$. First, suppose that σ_F^+ is transitive. It follows that σ_F^+ is surjective. By Theorem 2.3, σ is also transitive. By Theorem 4.1, σ is topologically conjugate to σ_F^{-1}; therefore, σ_F^{-1} is transitive. It follows that σ_F is transitive.

Now, suppose that σ_F is transitive and that σ_F^+ is surjective. Since σ_F^{-1} is transitive, it follows from Theorem 4.1 that σ is transitive. Since σ_F^+ is surjective, it follows from Proposition 4.3 that σ_F^+ is transitive. □

4.2. *Unions of graphs of continuous functions.* Theorems 4.8 and 4.13 are the main results of this sections. After proving them, we apply them to obtain transitive homeomorphisms on the Cantor fan. First, we prove the following lemma.

LEMMA 4.6. *Let X be a compact metric space and let $\{f_\alpha \mid \alpha \in \Lambda\}$ be a collection of continuous functions from X to X such that $F = \bigcup_{\alpha\in\Lambda} \Gamma(f_\alpha)$ is closed in $X \times X$. Then, for each open set U in $\prod_{k=1}^{\infty} X$ and for each $\mathbf{x} \in U \cap X_F^+$,*

$$\mathbf{x}(1) \in \operatorname{Int}(\pi_1(U \cap X_F^+)).$$

Proof. Let U be an open set in $\prod_{k=1}^{\infty} X$ such that $U \cap X_F^+ \neq \emptyset$ and let $\mathbf{x} \in U \cap X_F^+$. Also, let n be a positive integer and let $U_1, U_2, U_3, \ldots, U_n$ be open sets in X such that

$$\mathbf{x} \in \left(U_1 \times U_2 \times U_3 \times \cdots \times U_{n-1} \times U_n \times \prod_{k=n+1}^{\infty} X\right) \cap X_F^+ \subseteq U \cap X_F^+.$$

Next, let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \in \Lambda$ be such that $\mathbf{x}(k+1) = f_{\alpha_k}(\mathbf{x}(k))$ for each integer $k \in \{1, 2, 3, \ldots, n-1\}$ and let $V_n = U_n$. Since $f_{\alpha_{n-1}}$ is continuous at the point $\mathbf{x}(n-1)$, there is an open set V_{n-1} in X such that $\mathbf{x}(n-1) \in V_{n-1} \subseteq U_{n-1}$ and $f_{\alpha_{n-1}}(V_{n-1}) \subseteq V_n$. Since $f_{\alpha_{n-2}}$ is continuous at the point $\mathbf{x}(n-2)$, there is an open set V_{n-2} in X such that $\mathbf{x}(n-2) \in V_{n-2} \subseteq U_{n-2}$ and $f_{\alpha_{n-2}}(V_{n-2}) \subseteq V_{n-1}$. Continuing inductively, we construct open sets $V_1, V_2, V_3, \ldots, V_n$ in X such that for each $k \in \{1, 2, 3, \ldots, n-1\}$, $\mathbf{x}(k) \in V_k \subseteq U_k$ and $f_{\alpha_k}(V_k) \subseteq V_{k+1}$.

Finally, we show that $\mathbf{x}(1) \in \operatorname{Int}(\pi_1(U \cap X_F^+))$. Since $\mathbf{x}(1) \in V_1$ and V_1 is open in X, it suffices to see that $V_1 \subseteq \pi_1(U \cap X_F^+)$. Let $x \in V_1$. To see that $x \in \pi_1(U \cap X_F^+)$, let $y_1 = x$, for each $k \in \{2, 3, \ldots, n\}$, let $y_k = f_{\alpha_{k-1}}(x_{k-1})$, for each $k > n$, let $y_k = f_{\alpha_1}(x_{k-1})$, and let $\mathbf{y} = (y_1, y_2, y_3, \ldots)$. Then, $\mathbf{y} \in U \cap X_F^+$ and, therefore, $\mathbf{y}(1) \in \pi_1(U \cap X_F^+)$. It follows that $x \in \pi_1(U \cap X_F^+)$. □

The following observation follows directly from the proof of Lemma 4.6.

Observation 4.7. Let X be a compact metric space, let $\{f_\alpha \mid \alpha \in \Lambda\}$ be a collection of continuous functions from X to X such that $F = \bigcup_{\alpha\in\Lambda} \Gamma(f_\alpha)$ is closed in $X \times X$, let U be an open set in $\prod_{k=1}^{\infty} X$, let $\mathbf{x} \in U \cap X_F^+$, and for each positive integer k, let $\alpha_k \in \Lambda$ be

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

such that $\mathbf{x}(k+1) = f_{\alpha_k}(\mathbf{x}(k))$. Then, there are a positive integer m and open sets $U_1, U_2, U_3, \ldots, U_m$ in X such that:

(1) $\mathbf{x} \in (U_1 \times U_2 \times U_3 \times \cdots \times U_{m-1} \times U_m \times \prod_{k=m+1}^{\infty} X) \cap X_F^+ \subseteq U \cap X_F^+$ and
(2) for each $k \in \{1, 2, 3, \ldots, m-1\}$, $f_{\alpha_k}(U_k) \subseteq U_{k+1}$.

Definition 4.1. Let X be a compact metric space, let F be a closed relation on X, and let $x \in X$. Then, we define

$$\mathcal{U}_F^{\oplus}(x) = \{y \in X \mid \text{ there are } n \in \mathbb{N} \text{ and } \mathbf{x} \in X_F^n \text{ such that } \mathbf{x}(1) = x, \mathbf{x}(n) = y\}$$

and we call it the forward impression of x by F.

THEOREM 4.8. *Let X be a compact metric space, let F be a closed relation on X, let $\{f_\alpha \mid \alpha \in A\}$ be a non-empty collection of continuous functions from X to X such that $F^{-1} = \bigcup_{\alpha \in A} \Gamma(f_\alpha)$, and let $\{g_\beta \mid \beta \in B\}$ be a non-empty collection of continuous functions from X to X such that $F = \bigcup_{\beta \in B} \Gamma(g_\beta)$. If there is a dense set D in X such that for each $s \in D$, $\mathrm{Cl}(\mathcal{U}_F^{\oplus}(s)) = X$, then (X_F^+, σ_F^+) is transitive.*

Proof. Let U and V be open sets in $\prod_{k=1}^{\infty} X$ such that $U \cap X_F^+ \neq \emptyset$ and $V \cap X_F^+ \neq \emptyset$, let $\mathbf{x} \in U \cap X_F^+$, and let for each positive integer k:

(1) $\alpha_k \in A$ be such that $\mathbf{x}(k) = f_{\alpha_k}(\mathbf{x}(k+1))$ and
(2) $\beta_k \in B$ be such that $\mathbf{x}(k+1) = g_{\beta_k}(\mathbf{x}(k))$.

Also, let m be a positive integer and let $U_1, U_2, U_3, \ldots, U_m$ be non-empty open sets in X such that:

(1) $\mathbf{x} \in (U_1 \times U_2 \times U_3 \times \cdots \times U_{m-1} \times U_m \times \prod_{k=m+1}^{\infty} X) \cap X_F^+ \subseteq U \cap X_F^+$ and
(2) for each $k \in \{1, 2, 3, \ldots, m-1\}$, $g_{\beta_k}(U_k) \subseteq U_{k+1}$.

Such an integer and such sets do exist by Observation 4.7. By Lemma 4.6, $\mathbf{x}(1) \in \mathrm{Int}(\pi_1(U \cap X_F^+))$. Let $V_1 = U_1 \cap \mathrm{Int}(\pi_1(U \cap X_F^+))$. Then, V_1 is open in X, $V_1 \subseteq U_1$, and $\mathbf{x}(1) \in V_1$. Next, let V_2 be an open set in X such that $V_2 \subseteq U_2$, $\mathbf{x}(2) \in V_2$, and $f_{\alpha_1}(V_2) \subseteq V_1$. Since $f_{\alpha_1} : X \to X$ is continuous, it follows that such a set V_2 does exist. Continuing inductively, we construct open sets $V_1, V_2, V_3, \ldots, V_m$ in X such that for each $k \in \{1, 2, 3, \ldots, m-1, m\}$, $V_k \subseteq U_k$, $\mathbf{x}(k) \in V_k$, and for each $k \in \{1, 2, 3, \ldots, m-1\}$, $f_{\alpha_k}(V_{k+1}) \subseteq V_k$.

Next, let $d_m \in V_m \cap D$, $d_{m-1} = f_{\alpha_{m-1}}(d_m)$, $d_{m-2} = f_{\alpha_{m-2}}(d_{-1}), \ldots, d_2 = f_{\alpha_2}(d_3)$, and $d_1 = f_{\alpha_1}(d_2)$. Also, let $\mathbf{d} \in X_F^+$ be such a point that $\mathbf{d}(1) = g_{\beta_m}(d_m)$ and let

$$\mathbf{x}' = (d_1, d_2, d_3, \ldots, d_m) \oplus \mathbf{d}.$$

Note that $\mathbf{x}' \in (U_1 \times U_2 \times U_3 \times \cdots \times U_{m-1} \times U_m \times \prod_{k=m+1}^{\infty} X) \cap X_F^+$ is such a point that $\mathbf{x}'(m) \in U_m \cap D$. Also, note that $\mathbf{x}' \in U \cap X_F^+$. Since for each $s \in D$, $\mathrm{Cl}(\mathcal{U}_F^{\oplus}(s)) = X$, it follows that $\mathrm{Cl}(\mathcal{U}_F^{\oplus}(\mathbf{x}'(m))) = X$.

It follows from Lemma 4.6 that $\mathrm{Int}(\pi_1(V \cap X_F^+)) \neq \emptyset$. Therefore, it follows from $\mathrm{Cl}(\mathcal{U}_F^{\oplus}(\mathbf{x}'(m))) = X$ that there are a positive integer ℓ and a point $\mathbf{y} \in X_F^{\ell}$ such that $\mathbf{y}(1) = \mathbf{x}'(m)$ and $\mathbf{y}(\ell) \in \mathrm{Int}(\pi_1(V \cap X_F^+))$. Finally, let $\mathbf{z} \in V \cap X_F^+$ be such that $\mathbf{z}(1) = \mathbf{y}(\ell)$ and let

$$\mathbf{w} = (\mathbf{x}'(1), \mathbf{x}'(2), \mathbf{x}'(3), \ldots, \mathbf{x}'(m), \mathbf{y}(2), \mathbf{y}(3), \mathbf{y}(4), \ldots, \mathbf{y}(\ell), \mathbf{z}(2), \mathbf{z}(3), \mathbf{z}(4), \ldots).$$

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

Then, $\mathbf{w} \in U \cap X_F^+$ and $(\sigma_F^+)^{m+\ell-2}(\mathbf{w}) = \mathbf{z} \in V \cap X_F^+$. Therefore, $(\sigma_F^+)^{m+\ell-2}(U \cap X_F^+) \cap (V \cap X_F^+) \neq \emptyset$ and it follows that (X_F^+, σ_F^+) is transitive. □

Example 4.9. Let F be the union of the graphs of functions f_1, f_2, $f_3 : [0, 1] \to [0, 1]$, defined by $f_1(x) = 1$, $f_2(x) = x^2$, and $f_3(x) = x^{1/3}$ for each $x \in [0, 1]$. Then, $\sigma_F^+ : X_F^+ \to X_F^+$ is not transitive since the open set $U = [0, .01) \times (.99, 1] \times [0, 1] \times [0, 1] \times [0, 1] \times \cdots$ has the property that $\sigma_F^+(U) = (1, 1, 1, \ldots)$. Note that each $\mathcal{U}_F^{\oplus}(x)$ is dense in $[0, 1]$ for each $x \in (0, 1)$, and while F is the union of the graphs of continuous functions defined on $[0, 1]$, F^{-1} is not the union of the graphs of continuous functions defined on $[0, 1]$.

The following well-known result, which is presented in Lemma 4.10, is used later in the paper.

LEMMA 4.10. *For each $x \in (0, 1)$, the set $\{x^{2^m/3^n} \mid m$ and n are positive integers$\}$ is dense in $[0, 1]$.*

Proof. An irrational rotation on a circle produces a dense orbit of any point on a circle. This translates to the fact that for any irrational number γ, the set

$$\{m - n \cdot \gamma \mid m \text{ and } n \text{ are positive integers}\}$$

is dense in $\mathbb{R}$. Therefore, the set

$$\left\{ m - n \cdot \frac{\ln 3}{\ln 2} \;\middle|\; m \text{ and } n \text{ are positive integers} \right\}$$

is dense in $\mathbb{R}$, since $\ln 3/\ln 2$ is irrational. Thus, the set

$$\{m \cdot \ln 2 - n \cdot \ln 3 \mid m \text{ and } n \text{ are positive integers}\}$$

is dense in $\mathbb{R}$. Therefore, the set

$$\{e^{m \cdot \ln 2 - n \cdot \ln 3} \mid m \text{ and } n \text{ are positive integers}\}$$

is dense in $[0, \infty)$ and it follows that the set

$$\left\{ \frac{2^m}{3^n} \;\middle|\; m \text{ and } n \text{ are positive integers} \right\}$$

is dense in $[0, \infty)$. Let $x \in (0, 1)$ and let $a, b \in (0, 1)$ be such that $a < b$. Then, $\ln a < \ln b < 0$ and $\ln x < 0$. So $0 < \ln b/\ln x < \ln a/\ln x$. Let m and n be positive integers such that

$$\frac{\ln b}{\ln x} < \frac{2^m}{3^n} < \frac{\ln a}{\ln x}.$$

Such integers m and n do exist since $\{2^m/3^n \mid m$ and n are positive integers$\}$ is dense in $[0, \infty)$. Then, $\ln a < (2^m/3^n) \cdot \ln x < \ln b$ and, therefore, $\ln a < \ln x^{2^m/3^n} < \ln b$. It follows that $a < x^{2^m/3^n} < b$. □

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

In the following example, we show that, in general, there are closed relations F on compact metric spaces X such that (X_F^+, σ_F^+) is not transitive even though there is a dense set D in X such that for each $s \in D$, $\mathrm{Cl}(\mathcal{U}_F^{\oplus}(s)) = X$.

Example 4.11. Let

$$F = \{(x, y) \in [0, 1] \times [0, 1] \mid y = x^2 \text{ or } y = x^{\frac{1}{3}}\} \cup \{(0, 1), (1, 0)\}.$$

Then, F is a closed relation on $[0, 1]$ such that:

(1) for each $x \in (0, 1)$, $\mathcal{U}_F^{\oplus}(x)$ is dense in $[0, 1]$ (by Lemma 4.10);

(2) for each $x \in (0, 1)$, $\mathcal{U}_F^{\oplus}(x) \cap \{0, 1\} = \emptyset$.

Let $U = ((0, 1) \times \prod_{k=2}^{\infty}[0, 1]) \cap X_F^+$ and $V = ([0, (1/100)) \times ((99/100), 1] \times \prod_{k=3}^{\infty}[0, 1]) \cap X_F^+$. Then, $U \neq \emptyset$, $V \neq \emptyset$, and $\pi_1(V) = \{0\}$. Therefore, for each non-negative integer n, $(\sigma_F^+)^n(U) \cap V = \emptyset$ and it follows that the dynamical system $([0, 1]_F^+, \sigma_F^+)$ is not transitive.

In the following example, we demonstrate how Theorem 4.8 may be used to obtain a well-known result.

Example 4.12. Let $f : [0, 1] \to [0, 1]$ be the tent-map defined by $f(x) = 2x$, if $x \leq \frac{1}{2}$, and $f(x) = 2 - 2x$, if $x \geq \frac{1}{2}$. Also, let $F = \Gamma(f)^{-1}$. It is a well-known fact that the inverse limit of closed intervals $[0, 1]$ with a tent map as a bonding function is a Knaster continuum. Since $[0, 1]_F^+$ is the inverse limit of closed intervals with a tent map as a bonding function, it is the Knaster continuum. It is also a well-known fact that the shift homeomorphism on $[0, 1]_F^+$ is transitive. Note that this follows easily also from Theorem 4.8 since:

(1) F^{-1} is the graph of f;

(2) F is the union of graphs of two continuous functions; and

(3) there is a dense subset of $[0, 1]$ with a dense forward impression.

Definition 4.2. Let X be a compact metric space (with metric d) and let F be a closed relation on X. We use D_F to denote the metric on X_F and D_F^+ to denote the metric on X_F^+:

$$D_F(\mathbf{x}, \mathbf{y}) = \sum_{k=-\infty}^{\infty} \frac{d(\mathbf{x}(k), \mathbf{y}(k))}{2^{|k|}} \quad \text{and} \quad D_F^+(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{\infty} \frac{d(\mathbf{x}(k), \mathbf{y}(k))}{2^k}$$

for all $\mathbf{x}, \mathbf{y} \in X_F$ and $\mathbf{x}, \mathbf{y} \in X_F^+$.

THEOREM 4.13. *Let X be a compact metric space, let n be a positive integer, and for each $k \in \{1, 2, 3, \ldots, n\}$, let $f_k : X \to X$ be a homeomorphism. Let $F = \bigcup_{k=1}^{n} \Gamma(f_k)$ and assume that for each $x \in X$ and for all $i, j \in \{1, 2, 3, \ldots, n\}$,*

$$f_i(x) = f_j(x) \quad \textit{and} \quad i \neq j \Longrightarrow F(x) = \{x\}.$$

Then, X_F is homeomorphic to X_F^+.

Proof. Without any loss of generality, we assume that the diameter of X is less than or equal to 1. Let $A = \{a \in X \mid \text{for each } k \in \{1, 2, 3, \ldots, n\}, f_k(a) = a\}$ and let

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

$A^\star = \{(\ldots, a, a; a, a, a, \ldots) \mid a \in A\}$. Note that A is a closed subset of X and that $A^\star$ is a closed subset of X_F. Also, note that for each $\mathbf{x} \in X_F \setminus A^\star$, there is a unique function $m_{\mathbf{x}} : \mathbb{Z} \to \{1, 2, 3, \ldots, n\}$ such that for each $k \in \mathbb{Z}$, $\mathbf{x}(k+1) = f_{m_{\mathbf{x}}(k)}(\mathbf{x}(k))$. For each $\mathbf{x} \in X_F$, we define $x_1 = f_{m_{\mathbf{x}}(0)}(\mathbf{x}(0))$, $x_2 = f_{m_{\mathbf{x}}(-1)}(x_1)$, $x_3 = f_{m_{\mathbf{x}}(1)}(x_2)$, $x_4 = f_{m_{\mathbf{x}}(-2)}(x_3)$, and for each positive integer k, we define

$$x_{k+1} = \begin{cases} f_{m_{\mathbf{x}}(\frac{k}{2})}(x_k); & k \text{ is even}, \\ f_{m_{\mathbf{x}}(-(k+1)/2)}(x_k); & k \text{ is odd}. \end{cases}$$

Then, we define the function $T : X_f \to X_F^+$ by

$$T(\mathbf{x}) = \begin{cases} (x_1, x_2, x_3, \ldots); & \mathbf{x} \in X_F \setminus A^\star, \\ (\mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3), \ldots); & \mathbf{x} \in A^\star. \end{cases}$$

Let $\mathbf{x} \in X_F$ be any point. We show that T is continuous at the point $\mathbf{x}$. Let $\varepsilon > 0$. We consider the following two possible cases.

(1) $\mathbf{x} \in X_F \setminus A^\star$. Let m be a positive integer such that $\sum_{k=m}^{\infty} 1/2^k < \varepsilon/3$. Note that for each $i \in \{-m, -m+1, -m+2, \ldots, m-2, m-1, m\}$ and for each $k \in \{1, 2, 3, \ldots, n\} \setminus \{m_{\mathbf{x}}(i)\}$, $f_k(\mathbf{x}(i)) \neq \mathbf{x}(i+1)$. Let

$$U = \left(\prod_{k=-\infty}^{-m-1} X \right) \times \left(\prod_{k=-m}^{m} U_k \right) \times \left(\prod_{k=m+1}^{\infty} X \right),$$

where for each $k \in \{-m, -m+1, -m+2, \ldots, m-2, m-1, m\}$, U_k is open in X, such that $U \cap A^\star = \emptyset$ and such that for each $\mathbf{y} \in U \cap X_F$, $\mathbf{y}(i+1) = f_{m_{\mathbf{x}}(i)}(\mathbf{y}(i))$ for each $i \in \{-m, -m+1, -m+2, \ldots, m-2, m-1\}$. Let $T(\mathbf{x}) = (x_1, x_2, x_3, \ldots)$. Then, $x_1 = f_{m_{\mathbf{x}}(0)}(\mathbf{x}(0))$ and for each positive integer k,

$$x_{k+1} = \begin{cases} f_{m_{\mathbf{x}}(\frac{k}{2})}(x_k); & k \text{ is even}, \\ f_{m_{\mathbf{x}}(-(k+1)/2)}(x_k); & k \text{ is odd}. \end{cases}$$

Also, for each $\mathbf{y} \in U \cap X_F$, if $T(\mathbf{y}) = (y_1, y_2, y_3, \ldots)$, then $y_1 = f_{m_{\mathbf{x}}(0)}(\mathbf{y}(0))$ and for each positive integer $k \in \{1, 2, 3, \ldots, 2m+2\}$,

$$y_{k+1} = \begin{cases} f_{m_{\mathbf{x}}(\frac{k}{2})}(y_k); & k \text{ is even}, \\ f_{m_{\mathbf{x}}(-(k+1)/2)}(y_k); & k \text{ is odd}. \end{cases}$$

Let $\delta_1 > 0$ be such that for each $\mathbf{y} \in U \cap X_F$,

$$d(\mathbf{x}(0), \mathbf{y}(0)) < \delta_1 \Longrightarrow \sum_{k=1}^{2m+2} \frac{d(x_k, y_k)}{2^k} < \frac{\varepsilon}{3},$$

where $T(\mathbf{y}) = (y_1, y_2, y_3, \ldots)$. Such a δ_1 does exist since all the functions f_1, f_2, $f_3, \ldots, f_m$ are continuous. Also, let $\delta > 0$ be such that for all $\mathbf{y} \in X_F$,

$$D_F(\mathbf{x}, \mathbf{y}) < \delta \Longrightarrow \mathbf{y} \in U \cap X_F \quad \text{and} \quad d(\mathbf{x}(0), \mathbf{y}(0)) < \delta_1.$$

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

Therefore, for each $\mathbf{y} \in X_F$ such that $D_F(\mathbf{x}, \mathbf{y}) < \delta$,

$$D_F^+(T(\mathbf{x}), T(\mathbf{y})) \leq \sum_{k=1}^{2m+2} \frac{d(x_k, y_k)}{2^k} + \sum_{k=2m+3}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon,$$

where $T(\mathbf{y}) = (y_1, y_2, y_3, \ldots)$.

(2) $\mathbf{x} \in A^\star$. Let m be a positive integer such that $\sum_{k=m}^{\infty} 1/2^k < \varepsilon/3$ and let $a \in A$ be such that $\mathbf{x} = (\ldots, a, a; a, a, a, \ldots)$. Since all the functions $f_1, f_2, f_3, \ldots, f_m$ are continuous and since for each $k \in \{1, 2, 3, \ldots, m\}$, $f_k(a) = a$, there is a $\delta_1 > 0$ such that for each $\mathbf{y} \in X_F^+$,

$$d(\mathbf{y}(0), a) < \delta_1 \Longrightarrow \sum_{k=1}^{m-1} \frac{d(\mathbf{y}(k), a)}{2^k} < \frac{\varepsilon}{3}.$$

Choose and fix such a δ_1. Also, let $\delta > 0$ be such that for each $\mathbf{y} \in X_F$,

$$D_F(\mathbf{x}, \mathbf{y}) < \delta \Longrightarrow d(\mathbf{y}(0), a) < \delta_1.$$

Therefore, for each $\mathbf{y} \in X_F$ such that $D_F(\mathbf{x}, \mathbf{y}) < \delta$,

$$D_F^+(T(\mathbf{x}), T(\mathbf{y})) = D_F^+((a, a, a, \ldots), T(\mathbf{y})) \leq$$
$$\sum_{k=1}^{m-1} \frac{d(a, y_k)}{2^k} + \sum_{k=m}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon,$$

where $T(\mathbf{y}) = (y_1, y_2, y_3, \ldots)$.

We have just proved that T is continuous. Note that since all the functions f_1, f_2, $f_3, \ldots, f_m$ are surjective, also T is surjective. Finally, we prove that T is injective. Let $\mathbf{x}, \mathbf{y} \in X_F$ be such that $T(\mathbf{x}) = T(\mathbf{y})$ and let

$$T(\mathbf{x}) = (x, g_1(x), g_2(g_1(x)), \ldots) \quad \text{and} \quad T(\mathbf{y}) = (y, h_1(x), h_2(h_1(x)), \ldots),$$

where for each positive integer k, $g_k, h_k \in \{f_1, f_2, f_3, \ldots, f_n\}$. It follows that $x = y$. First, suppose that $g_1(x) = x$. Then, $\mathbf{x} \in A^\star$. Since $h_1(y) = g_1(x)$, it follows that $h_1(y) = y$ and, therefore, $\mathbf{y} \in A^\star$. Therefore, $\mathbf{x} = \mathbf{y}$. Next, suppose that $g_1(x) \neq x$. Then, $h_1(y) \neq y$ and it follows that $\mathbf{x}, \mathbf{y} \in X_F \setminus A^\star$. Since for each $x \in X$ and for all $i, j \in \{1, 2, 3, \ldots, n\}$,

$$f_i(x) = f_j(x) \quad \text{and} \quad i \neq j \Longrightarrow F(x) = \{x\},$$

it follows that for each positive integer k, $g_k = h_k$. Therefore, $\mathbf{x} = \mathbf{y}$. We have just proved that T is a continuous bijection from the compactum X_F to the metric space X_F^+. Therefore, it is a homeomorphism. □

We conclude this section by giving various examples.

Example 4.14. Let $X = [0, 1]$ and let $F = \{(x, x^2) \mid x \in [0, 1]\} \cup \{(x, x^{\frac{1}{3}}) \mid x \in [0, 1]\}$. Note that X_F^+ is the topological suspension of the Cantor set (see [**26**, p. 42] for the definition of the topological suspension). By Theorem 4.13, X_F is homeomorphic to X_F^+ and by Theorem 4.8, σ_F^+ is transitive, since for $D = (0, 1)$, the set $\mathcal{U}_F^{\oplus}(s)$ is dense in

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

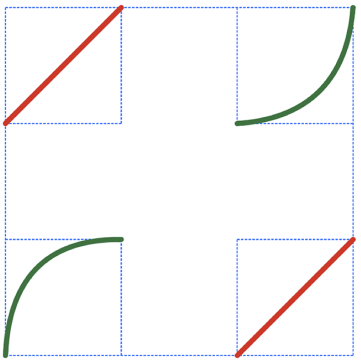


FIGURE 3. The relation F from Example 4.15.

X (by Lemma 4.10) for each $s \in D$. It follows from Theorem 4.5 that σ_F is a transitive homeomorphism on X_F.

In the following examples, transitive homeomorphisms on the Cantor fan are obtained. These are our third and fourth variant of the proof of Theorem 1.1.

Example 4.15. (Our third proof of Theorem 1.1) Let $X = [0, 1] \cup [2, 3]$, let

$$f_1(x) = \begin{cases} x^2; & x \in [0, 1], \\ (x-2)^{\frac{1}{3}} + 2; & x \in [2, 3], \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} x + 2; & x \in [0, 1], \\ x - 2; & x \in [2, 3] \end{cases}$$

for each $x \in X$, and let $F = \Gamma(f_1) \cup \Gamma(f_2)$; see Figure 3.

Let $D = (0, 1) \cup (2, 3)$. Then, D is dense in X and for each $s \in D$, $\mathrm{Cl}(\mathcal{U}_F^{\oplus}(s)) = X$ (a slight modification of Lemma 4.10 would give this result—we leave the details for the reader). Therefore, by Theorem 4.8, σ_F^+ is transitive. It follows from Theorem 4.13 that X_F is homeomorphic to X_F^+, and from Theorem 4.5 that σ_F is a transitive homeomorphism on X_F.

Next, we use this to obtain a transitive homeomorphism on the Cantor fan. For each $\alpha : \mathbb{Z} \to \{1, 2\}$ and for each $t \in X$, let $g(\alpha, t) = (\ldots, t_{-1}, t_0; t_1, t_2, \ldots)$, where $t_0 = t$ and for each integer k, $t_k = f_{\alpha(k)}(t_{k-1})$. Note that g is a homeomorphism from $C \times ([0, 1] \cup [2, 3])$ onto X_F, where $C = \{1, 2\}^{\mathbb{Z}}$ is a Cantor set. For all $\mathbf{x}, \mathbf{y} \in X_F$, we define

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \mathbf{y} \text{ or for each integer } k, \{\mathbf{x}(k), \mathbf{y}(k)\} \subseteq \{0, 2\}.$$

Note that $X_F/_\sim$ is a Cantor fan and that for all $\mathbf{x}, \mathbf{y} \in X_F$, $\mathbf{x} \sim \mathbf{y}$ if and only if $\sigma_F(\mathbf{x}) \sim \sigma_F(\mathbf{y})$. By Theorem 3.2, the function $\sigma_F^\star : X_F/_\sim \to X_F/_\sim$, defined by $\sigma_F^\star([x]) = [\sigma_F(x)]$ for each $x \in X$, is a transitive homeomorphism.

Example 4.16. (Our fourth proof of Theorem 1.1) Let $X = [-1, 1]$ and let $F = \Gamma(f_1) \cup \Gamma(f_2)$, where $f_1 : [-1, 1] \to [-1, 1]$ and $f_2 : [-1, 1] \to [-1, 1]$ are defined by

$$f_1(x) = -x \quad \text{and} \quad f_2(x) = \begin{cases} x^{\frac{1}{3}}; & x \in [-1, 0], \\ x^2; & x \in [0, 1] \end{cases}$$

for each $x \in X$; see Figure 4.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

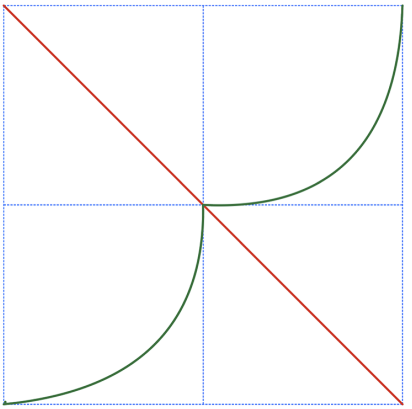


FIGURE 4. The relation F from Example 4.16.

Let $D = (-1, 0) \cup (0, 1)$. Then, D is dense in X and for each $s \in D$, $\mathrm{Cl}(\mathcal{U}_F^{\oplus}(s)) = X$ (a slight modification of Lemma 4.10 would give this result—we leave the details for the reader). Therefore, by Theorem 4.8, σ_F^+ is transitive. It follows from Theorem 4.13 that X_F is homeomorphic to X_F^+, and from Theorem 4.5 that σ_F is a transitive homeomorphism on X_F. Finally, we show that X_F is a Cantor fan by showing that X_F^+ is a Cantor fan. By [**15**, Example 2.7, p. 22], X_G^+, where $G = \{(x, x) \mid x \in [-1, 1]\} \cup \{(x, -x) \mid x \in [-1, 1]\}$, is a Cantor fan. Obviously, X_F^+ and X_G^+ are homeomorphic. Therefore, X_F^+ is a Cantor fan.

5. *Sigma transitivity*

In this section, we state and prove our second main result—Theorem 1.2. First, we introduce the notion of σ-transitivity of CR-dynamical systems and show that under certain minor conditions, the CR-dynamical system (X, F) is σ-transitive if and only if the dynamical system (X_F^+, σ_F^+) is transitive. First, we define a σ-transitive CR-dynamical system. Note that for any continuous function f on a compact metric space X, f is uniquely determined by its graph $\Gamma(f)$. Therefore, by giving the pair $(X, \Gamma(f))$, the dynamical system (X, f) is uniquely determined. Since the graph $\Gamma(f)$ is an example of a closed relation on X, this motivates the following definition, where we define CR-dynamical systems using closed relations F on X.

Definition 5.1. Let X be a compact metric space and let F be a closed relation on X. Then, we say that (X, F) is *a dynamical system with a closed relation* or *a CR-dynamical system*.

Definition 5.2. Let (X, F) be a CR-dynamical system. We say that (X, F) is *σ-transitive* if there is a countable collection $\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$ of open sets in $\prod_{i=1}^{\infty} X$ such that:

(1) for each positive integer j, there is a positive integer m_j and there are open sets $U_1^j, U_2^j, U_3^j, \ldots, U_{m_j}^j$ in X such that $U_j = U_1^j \times U_2^j \times U_3^j \times \cdots \times U_{m_j}^j \times \prod_{i=m_j+1}^{\infty} X$;

(2) for each open set V in $\prod_{i=1}^{\infty} X$ such that $V \cap X_F^+ \neq \emptyset$ and for each $\mathbf{t} \in V \cap X_F^+$, there is a positive integer j such that $\mathbf{t} \in U_j \subseteq V$;

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

(3) there is a sequence $(\mathbf{x}_j)$ of points in X_F^+ such that for each positive integer j:
   (a) $\mathbf{x}_j \in U_j$ and
   (b) there is a positive integer $\ell_j > m_j$ such that $\pi_{\ell_j}(\mathbf{x}_j) = \pi_1(\mathbf{x}_{j+1})$.

We use the following well-known observation to prove Lemma 5.2.

Observation 5.1. Let X be a compact metric space with cardinality greater than 1. Then, the topological product $\prod_{i=1}^{\infty} X$ does not contain any isolated points.

LEMMA 5.2. *Let (X, F) be a CR-dynamical system with the cardinality of X greater than 1. If (X, F) is σ-transitive with $\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$, as in Definition 5.2, then for each positive integer j, there are infinitely many positive integers k such that $U_k \subset U_j$.*

Proof. Assume (X, F) is a CR-dynamical system with the cardinality of X greater than 1, and assume (X, F) is σ-transitive with $\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$, as in Definition 5.2. Suppose j is a positive integer and $\mathbf{t} \in U_j \cap X_F^+$. Since no element of $\prod_{i=1}^{\infty} X$ is isolated in $\prod_{i=1}^{\infty} X$, there is an element $\mathbf{y}_1 \in U_j \setminus \{\mathbf{t}\}$. Now, $U_j \setminus \{\mathbf{y}_1\}$ is an open set such that $\mathbf{t} \in U_j \setminus \{\mathbf{y}_1\}$, and thus there is a positive integer $j_1 \neq j$ such that $\mathbf{t} \in U_{j_1} \subset U_j$. Similarly, $U_j \setminus \{\mathbf{y}_1, \mathbf{t}\}$ is a non-empty open set in $\prod_{i=1}^{\infty} X$. Let $\mathbf{y}_2 \in U_j \setminus \{\mathbf{y}_1, \mathbf{t}\}$. Then, there is a positive integer $j_2 \notin \{j, j_1\}$ such that $\mathbf{t} \in U_{j_2} \subset U_{j_1} \setminus \{\mathbf{y}_1, \mathbf{y}_2\} \subset U_j$. A simple induction argument finishes the proof. □

Definition 5.3. Let X be a compact metric space, let m and n be positive integers, and let $(x_1, x_2, x_3, \ldots, x_n) \in \prod_{k=1}^{n} X$ and $(y_1, y_2, y_3, \ldots, y_m) \in \prod_{k=1}^{m} X$ be such that $x_n = y_1$. Then, we define $(x_1, x_2, x_3, \ldots, x_n) \star (y_1, y_2, y_3, \ldots, y_m)$ by

$$(x_1, x_2, x_3, \ldots, x_n) \star (y_1, y_2, y_3, \ldots, y_m) = (x_1, x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_m).$$

The following observation easily follows.

Observation 5.3. Let (X, F) be a σ-transitive CR-dynamical system with $(\mathbf{x}_j)$, (m_j), (ℓ_j), and $\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$ as in Definition 5.2. Also, let $s_0 = 0$ and $s_k = \Sigma_{j=1}^{k} (\ell_j - 1)$ for each positive integer k, and let

$$\mathbf{x} = \pi_{[1,\ell_1]}(\mathbf{x}_1) \star \pi_{[1,\ell_2]}(\mathbf{x}_2) \star \pi_{[1,\ell_3]}(\mathbf{x}_3) \star \cdots .$$

Then, $\mathbf{x} \in X_F^+$ and $(\sigma_F^+)^{s_k}(\mathbf{x}) \in U_k$ for each non-negative integer k.

We also need the well-known result from Observation 5.4.

Observation 5.4. Let (X, f) be a dynamical system. The following statements are equivalent.

(1) (X, f) is transitive.
(2) There is a point $x \in X$ such that $f(x)$ is a transitive point in (X, f).

THEOREM 5.5. *Let (X, F) be a CR-dynamical system. If (X, F) is σ-transitive, then (X_F^+, σ_F^+) is transitive.*

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

Proof. Let (X, F) be a CR-dynamical system such that (X, F) is σ-transitive with $\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$, $(\mathbf{x}_j)$, (m_j), and (ℓ_j) as in Definition 5.2. Also, let $s_0 = 0$ and $s_k = \Sigma_{j=1}^{k} \ell_j - 1$ for each positive integer k, and let

$$\mathbf{x} = \pi_{[1,\ell_1]}(\mathbf{x}_1) \star \pi_{[1,\ell_2]}(\mathbf{x}_2) \star \pi_{[1,\ell_3]}(\mathbf{x}_3) \star \cdots .$$

Additionally, let $\theta = \sigma_F^+$. By Observation 5.4, it suffices to show that there is a point $\mathbf{z} \in X_F^+$ such that $\theta(\mathbf{z})$ is a transitive point in (X_F^+, θ). Note that $\ell_1 > 1$ and let $\mathbf{z} = \theta^{\ell_1 - 2}(\mathbf{x})$. We show that $\theta(\mathbf{z})$ is dense in (X_F^+, θ) by showing that $\theta^{\ell_1 - 1}(\mathbf{x})$ is a transitive point in (X_F^+, θ). To that end, let V be an open set in $\Pi_{i=1}^{\infty} X$ such that $V \cap X_F^+ \neq \emptyset$. Then, there is a positive integer n such that $U_n \subset V$. By Lemma 5.2, there is a positive integer k such that $k > n$ and $U_k \subset U_n$. Therefore, by Observation 5.3, $\theta^{s_k}(\mathbf{x}) \in U_k$. Now, $s_k - (\ell_1 - 1) > 0$ and $\theta^{s_k}(\mathbf{x}) = \theta^{s_k - (\ell_1 - 1)}(\theta^{\ell_1 - 1}(\mathbf{x})) \in U_k \subset U_n$. Thus, we have shown that $\theta^{\ell_1 - 1}(\mathbf{x})$ is a transitive point in (X_F^+, θ). □

Next, we present an example showing that there is a continuum X and a closed relation F on X such that:

(1) X_F^+ has an isolated point;
(2) (X_F^+, σ_F^+) is dense orbit transitive; and
(3) (X, F) is not σ-transitive.

Example 5.6. Let $X = [0, 1]$, let $F = \{(0, 1), (1, 1)\}$, and let $\sigma = \sigma_F^+$. Then,

$$X_F^+ = \{(0, 1, 1, \ldots), (1, 1, 1, \ldots)\}$$

and $\sigma(0, 1, 1, 1, \ldots) = \sigma(1, 1, 1, \ldots) = (1, 1, 1, \ldots)$. The point $(0, 1, 1, \ldots)$ is transitive in (X_F^+, σ) since its orbit equals X_F^+:

$$\{(0, 1, 1, \ldots), \sigma(0, 1, 1, \ldots), \sigma^2(0, 1, 1, \ldots), \ldots\} = \{(0, 1, 1, \ldots), (1, 1, 1, \ldots)\}.$$

Next, we show that (X, F) is not σ-transitive. Suppose (X, F) is σ-transitive with $\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$ and sequence (m_j) as in Definition 5.2. We show that (X, F) is not σ-transitive by showing that *it does not hold* that there are a sequence $(\mathbf{x}_j)$ of points in X_F^+ and a sequence of positive integers (ℓ_j) such that for each positive integer j, $\ell_j > m_j$, $\mathbf{x}_j \in U_j$, and $\pi_{\ell_j}(\mathbf{x}_j) = \pi_1(\mathbf{x}_j)$.

Let

$$\mathcal{U}_0 = \{U \in \mathcal{U} \mid (0, 1, 1, \ldots) \in U, (1, 1, 1, \ldots) \notin U\}$$

and let

$$\mathcal{U}_1 = \{U \in \mathcal{U} \mid (0, 1, 1, \ldots) \notin U, (1, 1, 1, \ldots) \in U\}.$$

Note that each of the collections $\mathcal{U}_0$ and $\mathcal{U}_1$ is infinite. Therefore, there is a positive integer j such that $U_j \in \mathcal{U}_1$ and $U_{j+1} \in \mathcal{U}_0$. It follows that if $\mathbf{x}_j \in U_j \cap X_F^+$, then $\mathbf{x}_j = (1, 1, 1, \ldots)$ and, therefore, for each positive integer ℓ, there is no $\mathbf{x}_{j+1} \in U_{j+1} \cap X_F^+$ such that $\pi_1(\mathbf{x}_{j+1}) = \pi_\ell(\mathbf{x}_j)$ since $\pi_\ell(\mathbf{x}_j) = 1$ while $\pi_1(\mathbf{x}_{j+1}) = 0$.

We use the well-known result from Observation 5.7 to prove the converse of Theorem 5.5.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

Observation 5.7. Let (X, f) be a dynamical system. If (X, f) is transitive, then for each transitive point x in (X, f) and for each positive integer n, $f^n(x)$ is also a transitive point in (X, f).

THEOREM 5.8. *Let (X, F) be a CR-dynamical system. If (X_F^+, σ_F^+) is transitive, then (X, F) is σ-transitive.*

Proof. Let $\sigma = \sigma_F^+$ and suppose that (X_F^+, σ) is transitive. By Observation 2.2, the dynamical system (X_F^+, σ) is dense orbit transitive. Let

$$\mathbf{x} = (x_1, x_2, x_3, \ldots) \in X_F^+$$

be a transitive point in (X_F^+, σ) and let $\mathcal{B}$ be a countable base for $\prod_{i=1}^{\infty} X$ such that for each $B \in \mathcal{B}$, there are a positive integer m and open sets $B_1, B_2, B_3, \ldots, B_m$ in X such that $B = B_1 \times B_2 \times B_3 \times \cdots \times B_m \times \prod_{i=m+1}^{\infty} X$. Also, let

$$\mathcal{U} = \{B \in \mathcal{B} \mid B \cap X_F^+ \neq \emptyset\} = \{U_1, U_2, U_3, \ldots\}.$$

For each positive integer j, let m_j be a positive integer and let $U_1^j, U_2^j, U_3^j, \ldots, U_{m_j}^j$ be open sets in X such that $U_j = U_1^j \times U_2^j \times U_3^j \times \cdots \times U_{m_j}^j \times \prod_{i=m_j+1}^{\infty} X$. Then, the following hold.

(1) Obviously, item (1) from Definition 5.2 holds.

(2) Let V be any open set in $\prod_{i=1}^{\infty} X$ and let $\mathbf{t} \in V \cap X_F^+$. Since $\mathcal{B}$ is a base for $\prod_{i=1}^{\infty} X$, there is a positive integer j such that $\mathbf{t} \in U_j \subseteq V$. Therefore, item (2) from Definition 5.2 is satisfied.

(3) We construct sequences (ℓ_j) and $(\mathbf{x}_j)$ satisfying item (3) from Definition 5.2 inductively.

(a) Let $j = 1$. Then, let k be such a positive integer that $\sigma^k(\mathbf{x}) \in U_1$. Such a positive integer k does exist since $\mathbf{x}$ is a transitive point in (X_F^+, σ). Let $\mathbf{x}_1 = \sigma_F^k(\mathbf{x})$.

(b) Let $j = 2$. Then, let ℓ_1 be a positive integer such that $\ell_1 > m_1$ and $\sigma^{\ell_1 - 1}(\mathbf{x}_1) \in U_2$. Such a positive integer ℓ_1 does exist since, by Observation 5.7, $\sigma^{m_1}(x_1)$ is also a transitive point in (X_F^+, σ). Let $\mathbf{x}_2 = \sigma^{\ell_1 - 1}(\mathbf{x}_1)$. Note that $\ell_1 > m_1$, $\pi_{\ell_1}(\mathbf{x}_1) = \pi_1(\mathbf{x}_2)$, $\mathbf{x}_1 \in U_1$, and $\mathbf{x}_2 \in U_2$.

(c) Let $j > 1$ be a positive integer and suppose that we have already constructed positive integers $\ell_1, \ell_2, \ldots, \ell_{j-1}$ and points $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_j$ in X_F^+ so that for each $k < j$, we have $\ell_k > m_k$, $\pi_{\ell_k}(\mathbf{x}_k) = \pi_1(\mathbf{x}_{k+1})$, and $\mathbf{x}_k \in U_k$, and that we also have $\mathbf{x}_j \in U_j$. Then, we construct ℓ_j and $\mathbf{x}_{j+1}$ as follows. Let ℓ_j be a positive integer such that $\ell_j > m_j$ and $\sigma^{\ell_j - 1}(\mathbf{x}_{j-1}) \in U_j$. Such a positive integer ℓ_j does exist since, by Observation 5.7, $\sigma^{m_{j-1}}(\mathbf{x}_{j-1})$ is also a transitive point in (X_F^+, σ). Let $\mathbf{x}_{j+1} = \sigma^{\ell_j - 1}(\mathbf{x}_j)$. Note that $\ell_j > m_j$, $\pi_{\ell_j}(\mathbf{x}_j) = \pi_1(\mathbf{x}_{j+1})$, and $\mathbf{x}_{j+1} \in U_{j+1}$.

The sequences (ℓ_j) and $(\mathbf{x}_j)$ were constructed in such a way that item (3) from Definition 5.2 is satisfied.

It follows that (X, F) is σ-transitive. □

Our next goal is to prove Theorem 1.2, which is one of our main results. First, we prove the following lemma that will be needed in its proof.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

LEMMA 5.9. *For each $x \in (0, 1]$, the set*

$$\{(\tfrac{1}{2})^{(3^h-1)/2^{k+1}} \cdot x^{3^h/2^k} \mid k \text{ and } h \text{ are positive integers}\}$$

is dense in $[0, 1]$.

Proof. Let $x \in (0, 1]$. Note that the set $\{(\frac{1}{2})^{(3^h-1)/2^{k+1}} \mid k \text{ and } h \text{ are positive integers}\}$ is dense in $[0, 1]$. Therefore, we assume in the rest of the proof that $x < 1$. We prove the lemma by showing that

$$\{\ln((\tfrac{1}{2})^{(3^h-1)/2^{k+1}} \cdot x^{3^h/2^k}) \mid k \text{ and } h \text{ are positive integers}\}$$

is dense in $(-\infty, 0]$. First, note that

$$\ln\left(\left(\frac{1}{2}\right)^{(3^h-1)/2^{k+1}} \cdot x^{3^h/2^k}\right) = \ln\left(\frac{1}{2}\right)^{(3^h-1)/2^{k+1}} + \ln x^{3^h/2^k} = \frac{3^h-1}{2^{k+1}} \cdot \ln\frac{1}{2} + \frac{3^h}{2^k} \cdot \ln x$$

and let $a = \ln \frac{1}{2}$ and $b = \ln x$. Then, $a < 0$ and $b < 0$, and it follows that

$$\ln\left(\left(\frac{1}{2}\right)^{(3^h-1)/2^{k+1}} \cdot x^{3^h/2^k}\right) = \frac{3^h-1}{2^{k+1}} \cdot a + \frac{3^h}{2^k} \cdot b = \frac{3^h}{2^k} \cdot \frac{a}{2} + \frac{3^h}{2^k} \cdot b - \frac{a}{2^{k+1}} = \frac{3^h}{2^k} \cdot \left(\frac{a}{2} + b\right) - \frac{a}{2^{k+1}}.$$

To conclude the proof, we show that for all $a < 0$ and $b < 0$,

$$\left\{\frac{3^h}{2^k} \cdot \left(\frac{a}{2} + b\right) - \frac{a}{2^{k+1}} \;\middle|\; k \text{ and } h \text{ are positive integers}\right\}$$

is dense in $(-\infty, 0]$. Let $c < d \le 0$ and let $d' = (c + d)/2$. A similar argument as in the proof of Lemma 4.10 proves that the set $\{3^h/2^k \mid h \text{ and } k \text{ are positive integers}\}$ is dense in $[0, \infty)$. Let k and h be positive integers such that

$$\frac{d'}{(a/2)+b} < \frac{3^h}{2^k} < \frac{c}{(a/2)+b} \quad \text{and} \quad 0 < \frac{-a}{2^{k+1}} < \frac{d-c}{2}$$

(note that $1/((a/2) + b) < 0$). Then, $c < (3^h/2^k)((a/2) + b) < d'$, and it follows that

$$c < c - \frac{a}{2^{k+1}} < \frac{3^h}{2^k}\left(\frac{a}{2} + b\right) - \frac{a}{2^{k+1}} < d' - \frac{a}{2^{k+1}} < \frac{c+d}{2} + \frac{d-c}{2} = d.$$

This completes the proof. □

Definition 5.4. We use I to denote $I = [0, 1]$ and H to denote

$$H = \{(x, \sqrt{x}) \mid x \in [0, 1]\} \cup \{(x, \tfrac{1}{2}x^3) \mid x \in [0, 1]\}.$$

We also use f_0 and f_1 to denote the functions $f_0, f_1 : I \to I$ that are defined by $f_0(x) = \frac{1}{2}x^3$ and $f_1(x) = \sqrt{x}$ for any $x \in I$.

THEOREM 5.10. *I_H^+ is a Cantor fan.*

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

Proof. It follows from [2, Example 1, p. 7] that I_G^+, where $G = \{(x, x) \mid x \in I\} \cup \{(x, \frac{1}{2}x) \mid x \in I\}$ is a Cantor fan. Obviously, the function $\varphi : I_G^+ \to I_H^+$, defined by

$$\varphi(x_1, x_2, x_3, \ldots) = (x_1, g_1(x_1), g_2(g_1(x_1)), g_3(g_2(g_1(x_1))), \ldots)$$

for any $(x_1, x_2, x_3, \ldots) \in I_G^+$, where for each positive integer k, $g_k = f_0$ if $x_{k+1} = \frac{1}{2}x_k$ and $g_k = f_1$ if $x_{k+1} = x_k$, is a homeomorphism. □

Observation 5.11. For each $x \in (0, 1]$, $f_1^k(f_0^h(x)) = (\frac{1}{2})^{(3^h-1)/2^{k+1}} \cdot x^{3^h/2^k}$.

THEOREM 5.12. *The shift map σ_H^+ is a transitive continuous surjection.*

Proof. σ_H^+ is a continuous surjection. To show that it is transitive, we show that the CR-dynamical system (I, H) is σ-transitive. To do this, let $\mathcal{B}$ be a countable base for $\prod_{i=1}^{\infty} I$ such that for each $B \in \mathcal{B}$, there are a positive integer m and open sets B_1, B_2, $B_3, \ldots, B_m$ in I such that $B = B_1 \times B_2 \times B_3 \times \cdots \times B_m \times \prod_{i=m+1}^{\infty} I$. Also, let

$$\mathcal{U} = \{B \in \mathcal{B} \mid B \cap I_H^+ \neq \emptyset\} = \{U_1, U_2, U_3, \ldots\}.$$

For each positive integer j, let m_j be a positive integer and let $U_1^j, U_2^j, U_3^j, \ldots, U_{m_j}^j$ be open sets in I such that $U_j = U_1^j \times U_2^j \times U_3^j \times \cdots \times U_{m_j}^j \times \prod_{i=m_j+1}^{\infty} I$. Then, the following hold.

(1) Obviously, item (1) from Definition 5.2 is satisfied.

(2) Let V be any open set in $\prod_{i=1}^{\infty} I$ and let $\mathbf{t} \in V \cap I_H^+$. Since $\mathcal{B}$ is a base for $\prod_{i=1}^{\infty} I$ and since $\mathbf{t} \in I_H^+$, there is a positive integer j such that $\mathbf{t} \in U_j \subseteq V$. Therefore, item (2) from Definition 5.2 is satisfied.

(3) Let j be a positive integer and let $\mathbf{x}'_j = (x_1^j, x_2^j, x_3^j, \ldots) \in (U_j \cap I_H^+) \setminus \{(0, 0, 0, \ldots)\}$. Also, let $(g_1^{(j)}, g_2^{(j)}, g_3^{(j)}, \ldots) \in \{f_0, f_1\}^{\mathbb{N}}$ be such a sequence that for each positive integer n, $x_{n+1}^j = g_n^{(j)}(x_n^j)$. Next, let $V_{m_j-1}^j$ be an open set in $[0, 1]$ such that $x_{m_j-1}^j \in V_{m_j-1}^j \subseteq U_{m_j-1}^j$ and for each $x \in V_{m_j-1}^j$, $g_{m_j-1}^{(j)}(x) \in U_{m_j}^j$. Note that such a set $V_{m_j-1}^j$ does exist since $g_{m_j-1}^{(j)}$ is continuous. Next, let $V_{m_j-2}^j$ be an open set in I such that $x_{m_j-2}^j \in V_{m_j-2}^j \subseteq U_{m_j-2}^j$ and for each $x \in V_{m_j-2}^j$, $g_{m_j-2}^{(j)}(x) \in V_{m_j-1}^j$. Again, such a set $V_{m_j-2}^j$ does exist since $g_{m_j-2}^{(j)}$ is continuous. We continue inductively. Finally, suppose that we have already constructed the sets $V_{m_j}^j, V_{m_j-1}^j, V_{m_j-2}^j, \ldots, V_2^j$. Then, we construct the set V_1^j as follows. Let V_1^j be an open set in $[0, 1]$ such that $x_1^j \in V_1^j \subseteq U_1^j$ and for each $x \in V_1^j$, $g_1^{(j)}(x) \in V_2^j$. Such a set V_1^j does exist since $g_1^{(j)}$ is continuous.

Next, we define the sequence $(\mathbf{x}_k)$ of points in I_H^+ such that item (3) from Definition 5.2 is satisfied as follows.

- Let $x_1 \in V_1^1 \cap (0, 1]$ and let $y_1 = g_{m_1-1}^{(1)}(\ldots g_3^{(1)}(g_2^{(1)}(g_1^{(1)}(x_1))) \ldots)$. Then, $y_1 \in V_{m_1}^1 \cap (0, 1]$. By Lemma 5.9, the set

$$\{(\tfrac{1}{2})^{(3^h-1)/2^{k+1}} \cdot y_1^{3^h/2^k} \mid k \text{ and } h \text{ are positive integers}\}$$

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

is dense in $[0, 1]$. Therefore, there are positive integers k_1 and h_1 such that $(\frac{1}{2})^{(3^{h_1}-1)/2^{k_1+1}} \cdot y_1^{3^{h_1}/2^{k_1}} \in V_1^2$. Fix such positive integers k_1 and h_1. Then, let

$$\begin{aligned}\mathbf{y}_1 = (&x_1, g_1^{(1)}(x_1), g_2^{(1)}(g_1^{(1)}(x_1)), \ldots, g_{m_1-2}^{(1)}(\ldots g_3^{(1)}(g_2^{(1)}(g_1^{(1)}(x_1))) \ldots),y_1,\\ &f_0(y_1), f_0^2(y_1), f_0^3(y_1), \ldots, f_0^{h_1}(y_1),\\ &f_1(f_0^{h_1}(y_1)), f_1^2(f_0^{h_1}(y_1)), f_1^3(f_0^{h_1}(y_1)), \ldots, f_1^{k_1}(f_0^{h_1}(y_1))),\end{aligned}$$

let $\ell_1 = m_1 + h_1 + k_1$, and let $\mathbf{x}_1 \in I_H^+$ be any point such that $\pi_{[1,\ell_1]}(\mathbf{x}_1) = \mathbf{y}_1$. Note that $\mathbf{y}_1$ is an element of the finite Mahavier product $I_H^{m_1+h_1+k_1-1}$. Also, note that $\mathbf{x}_1 \in U_1$ and that $\pi_{\ell_1}(\mathbf{x}_1) \in V_1^2 \cap (0, 1]$.

- Let $x_2 = \pi_{\ell_1}(\mathbf{x}_1)$ and let $y_2 = g_{m_2-1}^{(2)}(\ldots g_3^{(2)}(g_2^{(2)}(g_1^{(2)}(x_2))) \ldots)$. Then, $y_2 \in V_{m_2}^2 \cap (0, 1]$. By Lemma 5.9, the set

$$\{(\tfrac{1}{2})^{(3^h-1)/2^{k+1}} \cdot y_2^{3^h/2^k} \mid k \text{ and } h \text{ are positive integers}\}$$

is dense in I. Therefore, there are positive integers k_2 and h_2 such that $(\frac{1}{2})^{(3^{h_1}-1)/2^{k_1+1}} \cdot y_2^{3^{h_1}/2^{k_1}} \in V_1^3$. Fix such positive integers k_2 and h_2. Then, let

$$\begin{aligned}\mathbf{y}_2 = (&x_2, g_1^{(2)}(x_2), g_2^{(2)}(g_1^{(2)}(x_2)), \ldots, g_{m_2-2}^{(2)}(\ldots g_3^{(2)}(g_2^{(2)}(g_1^{(2)}(x_2))) \ldots),y_2,\\ &f_0(y_2), f_0^2(y_2), f_0^3(y_2), \ldots, f_0^{h_2}(y_2),\\ &f_1(f_0^{h_2}(y_2)), f_1^2(f_0^{h_2}(y_2)), f_1^3(f_0^{h_2}(y_2)), \ldots, f_1^{k_2}(f_0^{h_2}(y_2))),\end{aligned}$$

let $\ell_2 = m_2 + h_2 + k_2$, and let $\mathbf{x}_2 \in I_H^+$ be any point such that $\pi_{[1,\ell_2]}(\mathbf{x}_2) = \mathbf{y}_2$. Note that $\mathbf{y}_2$ is an element of $I_H^{m_2+h_2+k_2-1}$. Also, note that $\mathbf{x}_2 \in U_2$ and that $\pi_{\ell_2}(\mathbf{x}_2) \in V_1^3 \cap (0, 1]$.

Continuing inductively, we construct the sequence $(\mathbf{x}_k)$ such that item (3) from Definition 5.2 is satisfied.

It follows that (X, F) is σ-transitive. □

Note that σ_H^+ from Theorem 5.12 is not a homeomorphism. Next, we show that the inverse limit $\varprojlim(I_H^+, \sigma_H^+)$ is homeomorphic to the Lelek fan. We need the following definition.

Observation 5.13. Observe that for any $\mathbf{x} \in I_H \setminus \{(\ldots, 0, 0, 0; 0, 0, \ldots)\}$, there are unique sequences $(a_1, a_2, a_3, \ldots), (b_1, b_2, b_3, \ldots) \in \{f_0, f_1\}^{\mathbb{N}}$ and a unique point $t \in (0, 1]$ such that

$$\mathbf{x} = (\ldots, a_2^{-1}(a_1^{-1}(t)), a_1^{-1}(t); t, b_1(t), b_2(b_1(t)), b_3(b_2(b_1(t))), \ldots).$$

Definition 5.5. Let g_0 and g_1 be the functions $g_0, g_1 : [0, \infty) \to [0, \infty)$, defined by $g_0(x) = \frac{1}{2}x^3$ and $g_1(x) = \sqrt{x}$ for any $x \in [0, \infty)$. For any $\mathbf{h} = (h_1, h_2, h_3, \ldots) \in \{g_0, g_1\}^{\mathbb{N}}$, we use

$$\overrightarrow{\mathbf{h}}(t) = (t, h_1(t), h_2(h_1(t)), h_3(h_2(h_1(t))), \ldots)$$

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

and

$$\overleftarrow{\mathbf{h}}(t) = (\ldots, h_3^{-1}(h_2^{-1}(h_1^{-1}(t))), h_2^{-1}(h_1^{-1}(t)), h_1^{-1}(t))$$

for any $t \in I$.

Observation 5.14. Observe that for any $\mathbf{x} \in I_H \setminus \{(\ldots, 0, 0, 0; 0, 0, \ldots)\}$, there are unique sequences $\mathbf{a}, \mathbf{b} \in \{g_0, g_1\}^{\mathbb{N}}$ and a unique point $t \in (0, 1]$ such that

$$\mathbf{x} = \overleftarrow{\mathbf{a}}(t) \oplus \overrightarrow{\mathbf{b}}(t).$$

Next, we show that I_H is a fan by showing that I_H is a subcontinuum of a Cantor fan.

LEMMA 5.15. *I_H is a subcontinuum of a Cantor fan.*

Proof. Let g_0 and g_1 be the functions $g_0, g_1 : [0, \infty) \to [0, \infty)$, defined by $g_0(x) = \frac{1}{2}x^3$ and $g_1(x) = \sqrt{x}$ for any $x \in [0, \infty)$. For all $\mathbf{a}, \mathbf{b} \in \{g_0, g_1\}^{\mathbb{N}}$, let

$$A_{\mathbf{a},\mathbf{b}} = \{\overleftarrow{\mathbf{a}}(t) \oplus \overrightarrow{\mathbf{b}}(t) \mid t \in [0, 1]\}.$$

Note that for all $\mathbf{a}, \mathbf{b} \in \{g_0, g_1\}^{\mathbb{N}}$, $A_{\mathbf{a},\mathbf{b}}$ is an arc from $(\ldots, 0, 0, 0; 0, 0, \ldots)$ to $\mathbf{e}_{\mathbf{a},\mathbf{b}} = \overleftarrow{\mathbf{a}}(1) \oplus \overrightarrow{\mathbf{b}}(1)$ in the Hilbert cube

$$\cdots \times [0, \sqrt[3]{2\sqrt[3]{2}}] \times [0, \sqrt[3]{2}] \times [0, 1] \times [0, 1] \times [0, 1] \times \cdots.$$

Also, note that for all $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}' \in \{g_0, g_1\}^{\mathbb{N}}$,

$$(\mathbf{a}, \mathbf{b}) \neq (\mathbf{a}', \mathbf{b}') \Longrightarrow A_{\mathbf{a},\mathbf{b}} \cap A_{\mathbf{a}',\mathbf{b}'} = \{(\ldots, 0, 0, 0; 0, 0, \ldots)\}.$$

Let $X = \bigcup_{(\mathbf{a},\mathbf{b}) \in \{g_0,g_1\}^{\mathbb{N}}} A_{\mathbf{a},\mathbf{b}}$ and let $E(X) = \{\mathbf{e}_{\mathbf{a},\mathbf{b}} \mid (\mathbf{a}, \mathbf{b}) \in \{g_0, g_1\}^{\mathbb{N}}\}$. Next, we show that $E(X)$ is homeomorphic to the Cantor set $\{g_0, g_1\}^{\mathbb{N}} \times \{g_0, g_1\}^{\mathbb{N}}$ (here, $\{g_0, g_1\}$ is equipped with the discrete topology while the sets $\{g_0, g_1\}^{\mathbb{N}}$ and $\{g_0, g_1\}^{\mathbb{N}} \times \{g_0, g_1\}^{\mathbb{N}}$ are equipped with the product topologies). Let $\varphi : \{g_0, g_1\}^{\mathbb{N}} \times \{g_0, g_1\}^{\mathbb{N}} \to E(X)$ be defined by $\varphi(\mathbf{a}, \mathbf{b}) = \mathbf{e}_{\mathbf{a},\mathbf{b}}$ for any $(\mathbf{a}, \mathbf{b}) \in \{g_0, g_1\}^{\mathbb{N}} \times \{g_0, g_1\}^{\mathbb{N}}$. Note that φ is a bijection. To see that φ is a homeomorphism, we only need to see that it is continuous (since every continuous bijection from a compact space to a metric space is a homeomorphism). To see that φ is continuous, let $(\mathbf{a}, \mathbf{b}) \in \{g_0, g_1\}^{\mathbb{N}} \times \{g_0, g_1\}^{\mathbb{N}}$ and let $(\mathbf{a}_n, \mathbf{b}_n)$ be a sequence in $\{g_0, g_1\}^{\mathbb{N}} \times \{g_0, g_1\}^{\mathbb{N}}$ such that $\lim_{n\to\infty}(\mathbf{a}_n, \mathbf{b}_n) = (\mathbf{a}, \mathbf{b})$. Since the sequences $(\mathbf{a}_n)$ and $(\mathbf{b}_n)$ are coordinate-wise converging to $\mathbf{a}$ and $\mathbf{b}$, respectively, it follows that $\lim_{n\to\infty} \mathbf{e}_{\mathbf{a}_n,\mathbf{b}_n} = \mathbf{e}_{\mathbf{a},\mathbf{b}}$. Therefore, φ is a homeomorphism and it follows that X is a Cantor fan. Since I_H is a subcontinuum of X, it follows that I_H is a subcontinuum of a Cantor fan. □

LEMMA 5.16. *Let $\mathbf{x} \in I_H \setminus \{(\ldots, 0, 0, 0; 0, 0, \ldots)\}$. If there is an integer k such that $\mathbf{x}(k) = 1$, then $\mathbf{x} \in E(I_H)$.*

Proof. Let $x \in (0, 1]$ and let $\mathbf{a}, \mathbf{b} \in \{f_0, f_1\}^{\mathbb{N}}$ such that $\mathbf{x} = \overleftarrow{\mathbf{a}}(x) \oplus \overrightarrow{\mathbf{b}}(x)$. Suppose that $\mathbf{x} \notin E(I_H)$. Let A be the maximal arc in I_H with one end-point being $(\ldots, 0, 0, 0; 0, 0, \ldots)$ such that $\mathbf{x} \in A$. Also, let $\pi_0(A) = [0, e]$. It follows that $x \leq e$. Since the point $\mathbf{x}$ is not an end-point of I_H, it follows that $x < e$. Let $\mathbf{e} = \overleftarrow{\mathbf{a}}(e) \oplus \overrightarrow{\mathbf{b}}(e)$.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

Note that for each positive integer k, $\mathbf{x}(k) < \mathbf{e}(k)$. If there is an integer k such that $\mathbf{x}(k) = 1$, then $\mathbf{e}(k) > 1$, which is a contradiction. Therefore, for each integer k, $\mathbf{x}(k) < 1$. This completes the proof. □

THEOREM 5.17. *The two-sided Mahavier product I_H is a Lelek fan.*

Proof. Our strategy is to show that the set of endpoints of I_H is dense in I_H. Let $\mathbf{x} = (\ldots, x_{-2}, x_{-1}; x_0, x_1, x_2, \ldots) \in I_H$ be any point and let $\varepsilon > 0$. Also, let m be a positive integer such that $\sum_{k=m}^{\infty} 1/2^k < \varepsilon/3$. For each integer $k \in \{-m, -m+1, -m+2, \ldots, m-2, m-1, m\}$, let $g_k \in \{f_0, f_1\}$ be such that $x_{k+1} = g_k(x_k)$. Next, let for each integer $k \in \{-m, -m+1, -m+2, \ldots, m-2, m-1, m\}$, V_k be an open set in I such that

$$\mathbf{x} \in \left(\prod_{k=-\infty}^{-m-1} I\right) \times V_{-m} \times V_{-m+1} \times \cdots \times V_{m-1} \times V_m \times \left(\prod_{i=m+1}^{\infty} I\right) \subseteq B\left(\mathbf{x}, \frac{\varepsilon}{3}\right).$$

Without loss of generality, we may assume that for each integer $k \in \{-m, -m+1, -m+2, \ldots, m-2, m-1\}$, $g_k(V_k) \subseteq V_{k+1}$.

By Lemma 5.9, the set $\{(\frac{1}{2})^{(3^h-1)/2^{k+1}} \mid k \text{ and } h \text{ are positive integers}\}$ is dense in $[0, 1]$ and by Observation 5.11, $f_1^k(f_0^h(1)) = (\frac{1}{2})^{(3^h-1)/2^{k+1}}$. Let h_0 and k_0 be positive integers such that $f_1^{k_0}(f_0^{h_0}(1)) \in V_{-m}$. We define $\mathbf{e}$ as follows:

(1) $\mathbf{e}(-m-h_0-k_0) = 1$;
(2) for each $k \leq -m-k_0-h_0$, let $\mathbf{e}(k-1) = f_1^{-1}(\mathbf{e}(k))$;
(3) for each $k \in \{-m-k_0-h_0, -m-k_0-h_0+1, -m-k_0-h_0+2, \ldots, -m-h_0-1\}$, let $\mathbf{e}(k+1) = f_0(\mathbf{e}(k))$;
(4) for each $k \in \{-m-h_0, -m-h_0+1, -m-h_0+2, \ldots, -m-1\}$, let $\mathbf{e}(k+1) = f_1(\mathbf{e}(k))$;
(5) for each $k \in \{-m, -m+1, -m+2, \ldots, m-2, m-1\}$, let $\mathbf{e}(k+1) = g_k(\mathbf{e}(k))$;
(6) for each $k \geq m$, let $\mathbf{e}(k+1) = f_0(\mathbf{e}(k))$.

By Lemma 5.16, $\mathbf{e} \in E(I_H)$ and it follows from its construction that

$$D_H(\mathbf{x}, \mathbf{e}) = \sum_{k=-\infty}^{-m-1} \frac{|x_k - \mathbf{e}(k)|}{2^{|k|}} + \sum_{k=-m}^{m} \frac{|x_k - \mathbf{e}(k)|}{2^{|k|}} + \sum_{k=m+1}^{\infty} \frac{|x_k - \mathbf{e}(k)|}{2^k} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

We have just proved that the set of end-points of I_H is dense in I_H, therefore, I_H is a Lelek fan. □

We conclude this paper by giving a proof of our second main result, Theorem 1.2.

Proof. Let $X = I_H^+$. We have proved in Theorem 5.10 that X is a Cantor fan. Let $f = \sigma_H^+$. By Theorem 5.12, the mapping f is a transitive continuous surjection. It follows from Theorem 4.1 that the inverse limit $\varprojlim\{X, f\}$ is homeomorphic to the two-sided Mahavier product I_H, which is by Theorem 5.17 a Lelek fan. Therefore, the inverse limit $\varprojlim\{X, f\}$ is a Lelek fan. In addition, by Theorem 2.3, the shift map on $\varprojlim\{X, f\}$ is a transitive homeomorphism. □

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

To our knowledge, there are two known fans that admit a transitive homeomorphism, the Lelek fan and the Cantor fan. The Cantor fan may be the 'simplest' non-degenerate one-dimensional continuum (that is, not a simple closed curve) that admits a transitive homeomorphism. Therefore, the following open problems are a good place to finish the paper.

Problem 5.18. Are the Cantor fan and the Lelek fan the only fans (see [2, Definition 9, p. 5] for the definition of a fan) that admit a transitive homeomorphism?

Problem 5.19. Is there 'a simpler' non-degenerate one-dimensional continuum (that is, not a simple closed curve) that admits a transitive homeomorphism?

Acknowledgements. The authors thank the anonymous referees for careful reading. Their comments helped us to improve our paper significantly. This work is supported in part by the Slovenian Research Agency (research projects J1-4632, BI-US/22-24-086, BI-HR/23-24-011, and BI-US/22-24-094, and research program P1-0285).

References

[1] E. Akin. *General Topology of Dynamical Systems (Graduate Studies in Mathematics Series, 1)*. American Mathematical Society, Providence, RI, 1993.
[2] I. Banič, G. Erceg and J. Kennedy. The Lelek fan as the inverse limit of intervals with a single set-valued bonding function whose graph is an arc. *Mediterr. J. Math.* **20** (2023), 1–24.
[3] I. Banič, G. Erceg and J. Kennedy. A transitive homeomorphism on the Lelek fan. *J. Difference Equ. Appl.* **29** (2023), 393–418.
[4] M. Barge and J. Martin. Chaos, periodicity, and snakelike continua. *Trans. Amer. Math. Soc.* **289** (1985), 355–365.
[5] J. Boroński and J. Kupka. New chaotic planar attractors from smooth zero entropy interval maps. *Adv. Difference Equ.* **232** (2015), 11 pp.
[6] J. Boroński, P. Minc and S. Štimac. On conjugacy between natural extensions of one-dimensional maps. *Ergod. Th. & Dynam. Sys.* **43** (2022), 2915–2937.
[7] J. Boroński and P. Oprocha. On indecomposability in chaotic attractors. *Proc. Amer. Math. Soc.* **143** (2015), 3659–3670.
[8] J. Boroński and P. Oprocha. On dynamics of the Sierpiński carpet. *C. R. Math. Acad. Sci. Paris* **356** (2018), 340–344.
[9] W. D. Bula and L. Overseegen. A characterization of smooth Cantor bouquets. *Proc. Amer. Math. Soc.* **108** (1990), 529–534.
[10] W. J. Charatonik. The Lelek fan is unique. *Houston J. Math.* **15** (1989), 27–34.
[11] J. Činč and P. Oprocha. Parametrized family of pseudo-arc attractors: physical measures and prime end rotations. *Proc. Lond. Math. Soc. (3)* **125** (2022), 318–357.
[12] M. Handel. A pathological area preserving C^∞ diffeomorphism of the plane. *Proc. Amer. Math. Soc.* **86** (1982), 163–168.
[13] F. He and J. Liu. Invariant measures and uniform positive entropy property for inverse limits. *Appl. Math. J. Chinese Univ. Ser. B.* **14** (1999), 265–272.
[14] L. C. Hoehn and C. Mouron. Hierarchies of chaotic maps on continua. *Ergod. Th. & Dynam. Sys.* **34** (2014), 1897–1913.
[15] W. T. Ingram. *An Introduction to Inverse Limits with Set-Valued Functions*. Springer, New York, 2012.
[16] J. Kennedy. A transitive homeomorphism on the pseudoarc which is semiconjugate to the tent map. *Trans. Amer. Math. Soc.* **326** (1991), 773–793.
[17] S. Kolyada and L. Snoha. Topological transitivity. *Scholarpedia* **4**(2) (2009), 5802.
[18] A. Lelek. On plane dendroids and their end-points in the classical sense. *Fund. Math.* **49** (1960/1961), 301–319.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press

[19] S. Li. Dynamical properties of the shift maps on the inverse limit spaces. *Ergod. Th. & Dynam. Sys.* **12** (1992), 95–108.
[20] V. Martínez-de-la-Vega, J. M. Martínez-Montejano and C. Mouron. Mixing homeomorphisms and indecomposability. *Topology Appl.* **254** (2019), 50–58.
[21] P. Minc and W. R. R. Transue. A transitive map on [0, 1] whose inverse limit is the pseudoarc. *Proc. Amer. Math. Soc.* **111** (1991), 1165–1170.
[22] C. Mouron. Tree-like continua do not admit expansive homeomorphisms. *Proc. Amer. Math. Soc.* **130** (2002), 3409–3413.
[23] C. Mouron. Expansive homeomorphisms and indecomposable subcontinua. *Topology Appl.* **126** (2002), 13–28.
[24] C. Mouron. Positive entropy homeomorphisms of chainable continua and indecomposable subcontinua. *Proc. Amer. Math. Soc.* **139** (2011), 2783–2791.
[25] C. Mouron. Mixing sets, positive entropy homeomorphisms and non-Suslinean continua. *Ergod. Th. & Dynam. Sys.* **36** (2016), 2246–2257.
[26] S. B. Nadler. *Continuum Theory. An Introduction*. Marcel Dekker, Inc., New York, 1992.
[27] P. Oprocha. The Lelek fan admits a completely scrambled weakly mixing homeomorphism. *Bull. Lond. Math. Soc.* **57**(2) (2025), 315–656.
[28] G. T. Seidler. The topological entropy of homeomorphisms on one-dimensional continua. *Proc. Amer. Math. Soc.* **4** (1990), 1025–1030.

https://doi.org/10.1017/etds.2025.6 Published online by Cambridge University Press