Math. Struct. in Comp. Science (2015), *vol.* 25, *pp.* 1–5. © Cambridge University Press 2014
doi:10.1017/S0960129513000467

A short proof of the Schröder–Simpson Theorem

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Received 29 August 2012; revised 27 May 2013

We give a short and elementary proof of the Schröder–Simpson Theorem, which states that the space of all continuous maps from a given space X to the non-negative reals with their Scott topology is the cone-theoretic dual of the probabilistic powerdomain on X.

1. The Schröder–Simpson Theorem

A *continuous valuation* on a topological space X is a map v from the lattice $\mathcal{O}(X)$ of open subsets of X to $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{+\infty\}$ such that $v(\varnothing) = 0$, $v(U) \leqslant v(V)$ whenever $U \subseteq V$, $v(U \cup V) + v(U \cap V) = v(U) + v(V)$ for all U, V, and $v(\bigcup_{i\in I} U_i) = \sup_{i\in I} v(U_i)$ for every directed family of opens $(U_i)_{i\in I}$.

Let $\mathbf{V}_{wk}(X)$ denote the space of all continuous valuations on a topological space X, with the *weak topology*, whose subbasic open subsets are $[h > r] = \{v \mid \int_{x\in X} h(x)dv > r\}$, where h ranges over the continuous maps from X to $\overline{\mathbb{R}}^+_\sigma$, and $r \in \mathbb{R}^+$, $r > 0$; $\overline{\mathbb{R}}^+_\sigma$ is $\overline{\mathbb{R}}^+$ with the Scott topology of its natural ordering. We define integration by the Choquet formula $\int_{x\in X} h(x)dv = \int_0^{+\infty} v(h^{-1}(t, +\infty])dt$; the right-hand side is a Riemann integral.

A Riesz-like representation theorem (Kirch 1993; Tix 1995) states that $\mathbf{V}_{wk}(X)$ is isomorphic, as a cone, to the space of all continuous linear maps from $[X \to \overline{\mathbb{R}}^+_\sigma]_\sigma$ to $\overline{\mathbb{R}}^+_\sigma$. The isomorphism maps v to the continuous linear map $h \mapsto \int_{x\in X} h(x)dv$. Here, $[X \to \overline{\mathbb{R}}^+_\sigma]$ is the space of all continuous maps from X to $\overline{\mathbb{R}}^+_\sigma$, and $[X \to \overline{\mathbb{R}}^+_\sigma]_\sigma$ is this space with the Scott topology of the pointwise ordering. A map is *linear* if and only if it preserves sums and products by non-negative reals.

Schröder and Simpson (Schröder and Simpson 2005) found another representation theorem: the continuous linear maps ψ from $\mathbf{V}_{wk}(X)$ to $\overline{\mathbb{R}}^+$ are exactly those of the form $v \in \mathbf{V}_{wk}(X) \mapsto \int_{x\in X} h(x)dv$, for some unique $h \in [X \to \overline{\mathbb{R}}^+_\sigma]$. This answered a question in Heckmann (1996). Their proof is very technical. Keimel's proof (Keimel 2012) is more conceptual and leads to more general results. Our argument is elementary. To make a comparison, Schröder and Simpson start with the simple observation that if h exists, then h must be the map $x \in X \mapsto \psi(\delta_x)$, where δ_x is the Dirac mass at x. Checking that this map answers the question, however, proves to be difficult. Instead, we build h from the shape of the weak open $\psi^{-1}(1, +\infty]$ using two simple lemmas.

[†] The work reported in this paper was partially supported by the ANR programme blanc project CPP.

https://doi.org/10.1017/S0960129513000467 Published online by Cambridge University Press

2. The proof

The key to our proof is Keimel's Lemma (Keimel 2012, Lemma 5.5), which is perhaps (half of) the fundamental reason the Schröder–Simpson Theorem holds. We state it in a slightly more general form; Keimel dealt with the case of only two maps h_i, and they were simple, that is, of the form $\sum_{i=1}^n a_i \chi_{U_i}$ for some $a_i \in \mathbb{R}^+$ and opens U_i, where χ_U is the characteristic map of U. Let $\mathbf{V}_b(X)$ denote the space of all *bounded* continuous valuations v, that is, such that $v(X) < +\infty$, with the subspace topology, which we again call the weak topology.

Lemma 2.1. Let ψ be a continuous linear map from $\mathbf{V}_b(X)$ to $\overline{\mathbb{R}}^+_\sigma$, and $(h_i)_{i\in I}$ be a family of continuous maps from X to $\overline{\mathbb{R}}^+_\sigma$. The following are equivalent:

(1) $\psi(v) \geqslant \sup_{i\in I} \int_{x\in X} h_i(x)dv$ for every $v \in \mathbf{V}_b(X)$;
(2) $\psi(v) \geqslant \int_{x\in X} \sup_{i\in I} h_i(x)dv$ for every $v \in \mathbf{V}_b(X)$.

Proof. The (2) $\Rightarrow$ (1) direction is clear.

For the converse, we assume (1). We start with Keimel's case, where $I = \{1, 2\}$, and h_1, h_2 are simple. Keimel argues as follows.

We write h_1 as $\sum_{i=1}^m a_i \chi_{U_i}$ and h_2 as $\sum_{j=1}^n b_j \chi_{V_j}$, where $a_i, b_j \in \mathbb{R}^+$, U_i, V_j are open. Let C_k, $1 \leqslant k \leqslant p$, be an enumeration of the atoms of the Boolean algebra of subsets of X generated by the sets U_i and V_j, that is, the non-empty subsets of the form $\bigcap_{i=1}^m \pm_i U_i \cap \bigcap_{j=1}^n \pm_{m+j} V_j$, where each sign $\pm_i$ is $+$ or $-$, and $+A = A$, $-A = X \setminus A$. We can then write h_1 as $\sum_{k=1}^p a'_k \chi_{C_k}$ and h_2 as $\sum_{k=1}^p b'_k \chi_{C_k}$, with $a'_k, b'_k \in \mathbb{R}^+$.

Each C_k is a *crescent*, that is, the difference $A \setminus B$ of two opens A and B. The Smiley–Horn–Tarski Theorem (which is also known as the Pettis Theorem) states that each $v \in \mathbf{V}_b(X)$ extends to a unique additive function on the finite disjoint unions of crescents, defined on crescents by $v(A \setminus B) = v(A \cup B) - v(B) = v(A) - v(A \cap B)$. Additivity means that the v value of a finite disjoint union of crescents is the sum of the v values of each crescent.

We define the *restriction* $v_{|C}$ of v to C by $v_{|C}(U) = v(C \cap U)$ for every $U \in \mathcal{O}(X)$. Writing $C = A \setminus B$ with A, B open, $v_{|C}(U) = v((A \cap U) \cup B) - v(B)$, from which $v_{|C}$ is in $\mathbf{V}_b(X)$. Moreover, $v_{|C}(C) = v(C)$, and $v_{|C}(C') = 0$ for every crescent C' disjoint from C.

So $\int_{x\in X} h_1(x)dv_{|C_k} = \sum_{k'=1}^p a'_{k'} v_{|C_k}(C_{k'}) = a'_k$. Similarly, $\int_{x\in X} h_2(x)dv_{|C_k} = b'_k$. Similarly again, $\int_{x\in X} \sup(h_1(x), h_2(x))dv_{|C_k} = \sup(a'_k, b'_k)$. By assumption, we have $\psi(v_{|C_k}) \geqslant \sup(\int_{x\in X} h_1(x)dv_{|C_k}, \int_{x\in X} h_1(x)dv_{|C_k}) = \sup(a'_k, b'_k) = \int_{x\in X} \sup(h_1(x), h_2(x))dv_{|C_k}$. Since the disjoint union of C_k, $1 \leqslant k \leqslant n$, is the whole of X, $v = \sum_{k=1}^p v_{|C_k}$, so $\psi(v) = \sum_{k=1}^p \psi(v_{|C_k}) \geqslant \sum_{k=1}^p \int_{x\in X} \sup(h_1(x), h_2(x))dv_{|C_k} = \int_{x\in X} \sup(h_1(x), h_2(x))dv$.

We now prove (1) $\Rightarrow$ (2) in the case where $I = \{1, 2\}$, without assuming h_1 or h_2 simple. Every continuous map $h : X \to \overline{\mathbb{R}}^+_\sigma$ is the pointwise supremum of the countable chain $(h_K)_{K\in\mathbb{N}}$ where $h_K = \frac{1}{2^K} \sum_{i=1}^{K2^K} \chi_{h^{-1}(i/2^K,+\infty]}$ is simple. Moreover, $\sup(h_{1K}, h_{2K}) = \sup(h_1, h_2)_K$. If $\psi(v) \geqslant \sup_{i\in\{1,2\}} \int_{x\in X} h_i(x)dv$ for every $v \in \mathbf{V}_b(X)$, then for every $K \in \mathbb{N}$, $\psi(v) \geqslant \sup_{i\in\{1,2\}} \int_{x\in X} h_{iK}(x)dv$ for every $v \in \mathbf{V}_b(X)$, and by Keimel's argument above, $\psi(v) \geqslant \int_{x\in X} \sup(h_{1K}(x), h_{2K}(x))dv = \int_{x\in X} \sup(h_1, h_2)_K(x)dv$. Taking sups over $K \in \mathbb{N}$, $\psi(v) \geqslant \int_{x\in X} \sup(h_1(x), h_2(x))dv$ for every $v \in \mathbf{V}_b(X)$.

https://doi.org/10.1017/S0960129513000467 Published online by Cambridge University Press

The implication (1) $\Rightarrow$ (2) now follows easily when I is finite. For general I, our assumption (1) implies $\psi(v) \geqslant \sup_{i\in J} \int_{x\in X} h_i(x)dv$ for every $v \in \mathbf{V}_b(X)$ and for every finite subset J of I. We have seen that this implied $\psi(v) \geqslant \int_{x\in X} \sup_{i\in J} h_i(x)dv$ for every $v \in \mathbf{V}_b(X)$. Taking (directed) sups over J, and since integration is Scott continuous in the integrated function, $\psi(v) \geqslant \int_{x\in X} \sup_{i\in I} h_i(x)dv$ for every $v \in \mathbf{V}_b(X)$. □

Keimel's Separation Theorem (Keimel 2006, Theorem 9.1) states the following. Let C be a topological cone, U be an open convex subset and A be a non-empty convex subset of C disjoint from U. Then there is a continuous linear map $\Lambda : C \to \overline{\mathbb{R}}^+_\sigma$ such that $\Lambda(a) \leqslant 1$ for every $a \in A$ and $\Lambda(u) > 1$ for every $u \in U$. A cone is a commutative monoid $(C, +)$ with a multiplicative action (scalar product) of $(\mathbb{R}^+, +, \times)$. A topological cone additionally has a topology such that both addition $+$ and scalar product are (jointly) continuous. A subset A of C is convex if and only if for all $\alpha \in [0, 1]$, $a, a' \in A$, $\alpha a + (1 - \alpha)a'$ is in A. Keimel's Separation Theorem is a consequence of a sandwich theorem by Roth (Roth 2000), which relies on the Axiom of Choice. The special case where we use it ($C = \overline{\mathbb{R}}^{+n}$) is very likely to be provable without the Axiom of Choice, though.

Lemma 2.2. Let ψ be a continuous linear map from $\mathbf{V}_b(X)$ to $\overline{\mathbb{R}}^+_\sigma$, and $h_1, \ldots, h_n$ be finitely many continuous maps from X to $\overline{\mathbb{R}}^+_\sigma$. If $\bigcap_{i=1}^n [h_i > 1] \subseteq \psi^{-1}(1, +\infty]$, then there is a continuous map $h : \to \overline{\mathbb{R}}^+_\sigma$ such that $\bigcap_{i=1}^n [h_i > 1] \subseteq [h > 1] \subseteq \psi^{-1}(1, +\infty]$.

Proof. We define $F : \mathbf{V}_b(X) \to \overline{\mathbb{R}}^{+n}_\sigma$ by $F(v) = \left(\int_{x\in X} h_i(x)dv\right)_{1\leqslant i\leqslant n}$. The image of $\bigcap_{i=1}^n [h_i > 1]$ by F is included in the open convex subset $U = (1, +\infty]^n$ of $\overline{\mathbb{R}}^{+n}_\sigma$. It is easy to check that the image A of the complement of $\psi^{-1}(1, +\infty]$ by F is also convex since ψ is linear. A is non-empty, since $F(0)$ is in A. We claim that A and U are disjoint: every element $(a_i)_{1\leqslant i\leqslant n}$ of A is of the form $F(v)$ with $v \notin \psi^{-1}(1, +\infty]$, hence $v \notin \bigcap_{i=1}^n [h_i > 1]$ by assumption, so $a_i = \int_{x\in X} h_i(x)dv \leqslant 1$ for some i, which implies $(a_i)_{1\leqslant i\leqslant n} \notin U$. By Keimel's Separation Theorem, there is a continuous linear map $\Lambda : C \to \overline{\mathbb{R}}^+_\sigma$ such that $\Lambda(a) \leqslant 1$ for every $a \in A$ and $\Lambda(u) > 1$ for every $u \in U$. Let $\gamma_i = \Lambda(\vec{e}_i)$ where $\vec{e}_i$ is a tuple of zeroes, except for a 1 at position i. If we agree that $(+\infty).0 = 0.(+\infty) = 0$, then $\Lambda(\vec{c}) = \sum_i \gamma_i c_i$ for every point $\vec{c} = (c_i)_{1\leqslant i\leqslant n}$ of $\overline{\mathbb{R}}^{+n}_\sigma$ (continuity of Λ is essential here). Letting $h = \sum_{i=1}^n \gamma_i h_i$, we obtain $(\Lambda \circ F)(v) = \sum_{i=1}^n \gamma_i \int_{x\in X} h_i(x)dv = \int_{x\in X} h(x)dv$ for every $v \in \mathbf{V}_b(X)$. Every element v of $\bigcap_{i=1}^n [h_i > 1]$ is such that $F(v)$ is in U, so $(\Lambda \circ F)(v) > 1$, that is, v is in $[h > 1]$. Every element v outside $\psi^{-1}(1, +\infty]$ is such that $F(v)$ is in A, so $(\Lambda \circ F)(v) \leqslant 1$, that is, v is not in $[h > 1]$. This proves the other inclusion $[h > 1] \subseteq \psi^{-1}(1, +\infty]$. □

The following statement is equivalent, and more in the style of Lemma 2.1, but we will not use it here. Given $\psi, h_1, \ldots, h_n$ as above, if $\psi(v) \geqslant \min_{i=1}^n \int_{x\in X} h_i(x)dv$ for every $v \in \mathbf{V}_b(X)$, then there is a continuous map $h : X \to \overline{\mathbb{R}}^+_\sigma$ such that $\psi(v) \geqslant \int_{x\in X} h(x)dv \geqslant \min_{i=1}^n \int_{x\in X} h_i(x)dv$ for every $v \in \mathbf{V}_b(X)$.

Theorem 2.3 (Schröder–Simpson). Let X be a topological space and ψ be a continuous linear map from $\mathbf{V}_b(X)$ to $\overline{\mathbb{R}}^+_\sigma$. Then there is a unique continuous map $h : X \to \overline{\mathbb{R}}^+_\sigma$ such that $\psi(v) = \int_{x\in X} h(x)dv$ for every $v \in \mathbf{V}_b(X)$, and $h(x) = \psi(\delta_x)$ for every $x \in X$.

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Proof. Since ψ is continuous, $\psi^{-1}(1,+\infty]$ is open, so it has the form $\bigcup_{i\in I}\bigcap_{j\in J_i}[h_{ij} > r_{ij}]$, where each J_i is finite, h_{ij} is continuous from X to $\overline{\mathbb{R}}^+_\sigma$, and $r_{ij} \in \mathbb{R}^+$, $r_{ij} > 0$. Without loss of generality, we can assume $r_{ij} = 1$ since $[h_{ij} > r_{ij}] = [1/r_{ij}.h_{ij} > 1]$.

For each $i \in I$, Lemma 2.2 gives us a continuous map $h_i : X \to \mathbb{R}^+$ such that $\bigcap_{j\in J_i}[h_{ij} > 1] \subseteq [h_i > 1] \subseteq \psi^{-1}(1,+\infty]$. So $\psi^{-1}(1,+\infty] = \bigcup_{i\in I}\bigcap_{j\in J_i}[h_{ij} > 1] \subseteq \bigcup_{i\in I}[h_i > 1] \subseteq \psi^{-1}(1,+\infty]$, whence $\psi^{-1}(1,+\infty] = \bigcup_{i\in I}[h_i > 1]$. Let $h = \sup_{i\in I} h_i$. By Lemma 2.1, $\psi(v) \geqslant \int_{x\in X} h(x)dv$ for every $v \in \mathbf{V}_b(X)$. If the inequality were strict for some v, then $\psi(v) > t > \int_{x\in X} h(x)dv$ for some $v \in \mathbf{V}_b(X)$ and some $t > 0$. Replacing v by $1/t.v$ if necessary, we may assume $t = 1$. Then $v \in \psi^{-1}(1,+\infty] = \bigcup_{i\in I}[h_i > 1]$, so $v \in [h_i > 1]$ for some $i \in I$. But then $t = 1 > \int_{x\in X} h(x)dv \geqslant \int_{x\in X} h_i(x)dv > 1$, which gives a contradiction. □

Corollary 2.4. Let X be a topological space, and ψ be a continuous linear map from $\mathbf{V}_{wk}(X)$ to $\overline{\mathbb{R}}^+_\sigma$. Then there is a unique continuous map $h : X \to \overline{\mathbb{R}}^+_\sigma$ such that $\psi(v) = \int_{x\in X} h(x)dv$ for every $v \in \mathbf{V}_{wk}(X)$, and $h(x) = \psi(\delta_x)$ for every $x \in X$.

Proof. Every continuous valuation v is the supremum of a directed family of bounded continuous valuations v_i, $i \in I$ (Heckmann 1996, Theorem 4.2). Let $h(x) = \psi(\delta_x)$. By Theorem 2.3, and since integration is Scott continuous in the valuation, $\psi(v) = \sup_{i\in I} \psi(v_i) = \sup_{i\in I} \int_{x\in X} h(x)dv_i = \int_{x\in X} h(x)dv$. □

We can also give the following corollary to Theorem 2.3, which is also due to Schröder and Simpson (Schröder and Simpson 2005) – the following argument is also theirs. Let $\mathbf{V}_{\leqslant 1\, wk}(X)$ denote the space of all subprobability valuations, that is, of all continuous valuations v on X such that $v(X) \leqslant 1$, with the subspace topology (which is also known as the *weak* topology) from $\mathbf{V}_b(X)$. We say $\psi : \mathbf{V}_{\leqslant 1\, wk}(X) \to \overline{\mathbb{R}}^+_\sigma$ is *linear* in this case if and only if $\psi(av + bv') = a\psi(v) + b\psi(v')$ for all $v, v' \in \mathbf{V}_{\leqslant 1\, wk}(X)$ and $a, b \in [0,1]$.

Corollary 2.5. Let X be a topological space, and ψ be a continuous linear map from $\mathbf{V}_{\leqslant 1\, wk}(X)$ to $\overline{\mathbb{R}}^+_\sigma$. Then there is a unique continuous map $h : X \to \overline{\mathbb{R}}^+_\sigma$ such that $\psi(v) = \int_{x\in X} h(x)dv$ for every $v \in \mathbf{V}_{\leqslant 1\, wk}(X)$, and $h(x) = \psi(\delta_x)$ for every $x \in X$.

Proof. Define $\psi' : \mathbf{V}_b(X) \to \overline{\mathbb{R}}^+_\sigma$ by $\psi'(v) = a\psi(1/a.v)$, for any arbitrary $a > 0$ such that $v(X) \leqslant a$. This is well defined by linearity, and is linear in the usual sense. Using the equivalence between $1/a.v \in [h > r]$ and $v \in [h > ar]$, it is easy to show that ψ' is continuous. Finally, we apply Theorem 2.3 to ψ'. □

We will conclude with a remark on dual cones. Given a topological cone C, we let its *dual cone* C^* be the set of all linear continuous maps from C to $\overline{\mathbb{R}}^+_\sigma$. This has an obvious cone structure, and we topologise it with the topology induced by the inclusion of C^* into $\overline{\mathbb{R}}^{+C}_\sigma$ with the product topology: this is variously known as the weak topology or as the topology of pointwise convergence, and a subbasis of opens is given by the sets $[c > r]^*$ defined as $\{\psi \in C^* \mid \psi(c) > r\}$, $c \in C$, $r \in \mathbb{R}^+$. The Schröder–Simpson Theorem, as stated above, implies that $\mathbf{V}_{wk}(X)^*$ is isomorphic to $[X \to \overline{\mathbb{R}}^+_\sigma]$ as a cone through the map $h \in [X \to \overline{\mathbb{R}}^+_\sigma] \mapsto (v \in \mathbf{V}_{wk}(X) \mapsto \int_{x\in X} h(x)dv)$, with inverse $\psi \in \mathbf{V}_{wk}(X)^* \mapsto (x \in X \mapsto \psi(\delta_x))$. This becomes an isomorphism of *topological* cones once we equip $C = [X \to \overline{\mathbb{R}}^+_\sigma]$ with the

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coarsest topology containing the sets $[v > r]^* = \{h \in C \mid \int_{x \in X} h(x)dv > r\}$ (the confusion of notation with $[v > r]^* = \{\psi \in C^* \mid \psi(v) > r\}$ is vindicated by the isomorphism). It is fair to call this the *weak** topology on $C = [X \to \overline{\mathbb{R}}^+_\sigma]$, and we write $[X \to \overline{\mathbb{R}}^+_\sigma]_{wk*}$ for the resulting topological cone.

By the Riesz-like representation theorem mentioned in the introduction, $[X \to \overline{\mathbb{R}}^+_\sigma]^*_\sigma$ is isomorphic to $\mathbf{V}_{wk}(X)$ as a cone. The very definition of the weak* topology (and of the isomorphism) shows that the isomorphism extends to an isomorphism of *topological* cones between $[X \to \overline{\mathbb{R}}^+_\sigma]^*_{wk*}$ and $\mathbf{V}_{wk}(X)$.

We say a topological cone is *reflexive* if and only if the function that maps each $c \in C$ to $\widehat{c} = (h \in C^* \mapsto h(c)) \in C^{**}$ is an isomorphism of topological cones. The following theorem summarises this discussion.

Theorem 2.6. For every topological space X, the topological cones $\mathbf{V}_{wk}(X)$ and $[X \to \overline{\mathbb{R}}^+_\sigma]_{wk*}$ are reflexive, and dual to each other (up to isomorphisms of topological cones).

Acknowledgments

I would like to thank Alex Simpson. I would also like to thank Klaus Keimel, who spotted a mistake in my original proof of Lemma 2.2, and suggested further simplifications. As the citations in the paper make apparent, Klaus Keimel is also the discoverer of many of the results I have relied on.

References

Heckmann, R. (1996) Spaces of valuations. In: Andima, S., Flagg, R. C., Itzkowitz, G., Kong, Y., Kopperman, R. and Misra, P. (eds.) Papers on General Topology and Applications: 11th Summer Conference at the University of Southern Maine. *Annals of the New York Academy of Sciences* **806** 174–200.

Keimel, K. (2006) Topological cones: Foundations for a domain-theoretical semantics combining probability and nondeterminism. *Electronic Notes in Theoretical Computer Science* **155** 423–443.

Keimel, K. (2012) Locally convex cones and the Schröder–Simpson theorem. *Questiones Mathematicae* **35** 353–390.

Kirch, O. (1993) Bereiche und Bewertungen. Master's thesis, TH Darmstadt.

Roth, W. (2000) Hahn–Banach type theorems for locally convex cones. *Journal of the Australian Mathematical Society* **68** (1) 104–125.

Schröder, M. and Simpson, A. (2005) Probabilistic observations and valuations. Talk at Mathematical Foundations of Programming Semantics XXI, Birmingham. (Slides available at `http://homepages.inf.ed.ac.uk/als/Talks/mfps05.pdf`.)

Tix, R. (1995) *Stetige Bewertungen auf topologischen Räumen*, Diplomarbeit, TH Darmstadt.

https://doi.org/10.1017/S0960129513000467 Published online by Cambridge University Press