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Bundle structures on manifolds

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This note gives homotopy-theoretic criteria in the metastable range for an $(n+l)$-manifold with boundary to be diffeomorphic to the total space of an l-disc bundle over a closed n-manifold and for two such structures to be equivalent. The results are similar to some theorems of (8) and the main technique used in the proofs is surgery on a map ((1), pp. 42–46, (7), (10)). All manifolds will be smooth.

1. *Preliminaries.* Let ξ be any closed l-disc bundle with $\partial\xi$ the associated $(l-1)$-sphere bundle and $T\xi = \xi/\partial\xi$ the *Thom space*. For a bundle map $f\colon \eta \to \xi$ or a map $f\colon (X, A) \to (\xi, \partial\xi)$, let the *Thom map* Tf be the composition of f with the collapsing map that takes $\partial\xi$ to the point ∞. All bundles will be oriented and the same symbol will be used for a bundle as for its total space. Hereafter we shall fix a particular oriented, closed l-disk bundle $\pi\colon \xi \to X$ over a finite, simply connected, CW-complex X that satisfies Poincaré duality in dimension n ((1), (7)), and M will denote a simply connected $(n+l)$-manifold with simply connected boundary ∂M. Now given a map

$$f\colon (M, \partial M) \to (\xi, \partial\xi),$$

we say that M has an *ξ-bundle structure* (ξ_1, f_1, ψ_1) *associated with* f provided there exist a closed l-disc bundle $\xi_1 \to N_1$ over a closed n-manifold, a bundle map $f_1\colon \xi_1 \to \xi$ covering a homotopy equivalence of the base spaces, and a diffeomorphism

$$\psi_1\colon (M, \partial M) \to (\xi_1, \partial\xi_1)$$

so that Tf is homotopic rel ∂M to $Tf_1 \circ \psi_1$. Two ξ-bundle structures (ξ_1, ψ_1, f_1) and (ξ_2, ψ_2, f_2) on $(n+l)$-manifolds M_1 and M_2 respectively are *equivalent* if there is a diffeomorphism $g\colon (M_1, \partial M_1) \to (M_2, \partial M_2)$ for which $\psi_2 \circ g \circ \psi_1^{-1}\colon \xi_1 \to \xi_2$ is a bundle map covering a diffeomorphism of the base spaces and $Tf_1 \circ \psi_1$ is homotopic rel ∂M_1 to $Tf_2 \circ \psi_2 \circ g$. It is easy to check that this is an equivalence relation.

2. *Existence of bundle structures.*

THEOREM 1. *Suppose there is a map* $f\colon (M, \partial M) \to (\xi, \partial\xi)$ *which induces an isomorphism* $f_*\colon H_*(M, \partial M; Z) \to H_*(\xi, \partial\xi; Z)$. *Furthermore, assume* $2(n+l) \geqslant 3(n+1)$ $(n \geqslant 5)$, *and*

(*a*) n *is odd, or*

(*b*) $n = 4j$, *there is a bundle* $\pi'\colon \xi' \to X$ *such that the stable normal bundle of* M *is stably equivalent to the induced bundle* $(\pi \circ f)^* \xi'$, *and*

$$\text{index } X = L_j(\overline{p}_1(\xi \oplus \xi'), \ldots, \overline{p}_j(\xi \oplus \xi')),$$

where $\overline{p}_i$ *is the dual Pontryagin class and* L_j *is the Hirzebruch polynomial.*

Then M *has an* ξ*-bundle structure associated with* f.

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Remarks. Any two bundle structures on M are fibre homotopy equivalent ((5)). As usual when $n = 4k+2$ the difficulty with the present method is lack of knowledge about an appropriate Kervaire invariant (see (6) or (2)) which can be attached to the homotopy class of f and is the obstruction to doing middle dimensional surgery.

COROLLARY 1. *Suppose M satisfies Poincaré duality in dimension n via the fundamental class μ_M, and there is a map $h\colon (M, \partial M) \to (D^l, S^{l-1})$ for which h^* (generator of $H^l(D^l, S^{l-1})) \in H^l(M, \partial M)$ is dual to μ_M with respect to the usual Lefschetz duality of $(M, \partial M)$. If $2(n+l) \geqslant 3(n+1)$, $n \geqslant 5$, n is odd, or $n = 4j$ and index M as a $4j$ Poincaré duality space is equal to $L_j(p_1(\tau_M), \ldots, p_j(\tau_M))$, then M is diffeomorphic to $N \times D^l$ for some smooth n-manifold N.*

Proof. Let $X = M$, $\xi \to X$ be $D^l \times M \to M$, ξ' be the stable normal bundle of M, and $f\colon (M, \partial M) \to (\xi, \partial\xi)$ be $f(x) = (x, h(x))$. Apply Theorem 1.

3. *Equivalence of bundle structures.* Before stating the equivalence theorem we need to introduce a special case of the concept of the connected sum of two vector bundles and bundle maps. Let (ξ, f_1, ψ_1) be a bundle structure on M and $\sigma\colon D^l \times \Sigma \to \Sigma$ be the canonical trivial bundle over a homotopy sphere Σ. The *connected sum* $(\xi_1 \# \sigma, f_1, \psi_1)$ is obtained as follows: Take two orientation preserving bundle maps $k_1\colon D^l \times D^n \to \xi_1$ and $k_2\colon D^l \times D^n \to \sigma$ covering, respectively, orientation preserving and reversing imbeddings of D^n in N(the base space of ξ_1) and Σ; let $g\colon \sigma \to \xi_1$ be a bundle map covering a homotopically trivial map $\Sigma \to X$. Change f and g by a bundle homotopy until $fk_1 = gk_2$, and then paste the spaces and maps together via the identifications

$$k_1(v, tu) = k_2(v, (1-t)u)$$

for $|u| = 1$ and $0 < t < 1$. This construction is well defined up to equivalence of ξ-bundle structures and $f \# g$ covers a homotopy equivalence $N \# \Sigma \to X$.

THEOREM 2. *Let (ξ_1, f_1, ψ_1) and (ξ_2, f_2, ψ_2) be two bundle structures on M for which $Tf_1 \circ \psi_1$ is homotopic* rel ∂M *to $Tf_2 \circ \psi_2$ $(n \geqslant 4)$. Then*

(a) if $n = 2j$ and $2(n+l) \geqslant 3(n+1)$, (ξ_1, f_1, ψ_1) is equivalent to (ξ_2, f_2, ψ_2),

(b) if $n = 2j-1$ $(j \neq 3, 7)$ and $2(n+l) > 3(n+1)$, (ξ_1, f_1, ψ_1) is equivalent to

$$(\xi_2 \# \sigma, f_2, \psi_2),$$

where σ is the canonical trivial bundle over $\Sigma \in \theta^n(\partial\pi)$.

4. *Proofs.* The main point is to indicate how the arguments of (7) and (10) apply to our situation.

Proof of Theorem 1. The isomorphism condition on f_* implies that $f\colon M \to \xi$ is a homotopy equivalence and therefore M is an n-dimensional Poincaré duality space. By imbedding X in some Euclidean space and taking a smooth regular neighbourhood ((4)) we may assume $\xi \to X$ to be a differentiable bundle. Deform Tf rel ∂M until it is transverse regular to the zero section X. Then $N = Tf^{-1}(X)$ is a closed, oriented n-submanifold of int M; $(Tf|N)^*\xi = \nu$, the normal bundle of N in M; and the inclusion $i\colon N \to M$ as a map of n-dimensional Poincaré duality spaces has degree one ((1)). Thus $i_*\colon H_*(N) \to H_*(M)$ is onto and $\ker i_* = \ker (Tf|N)_*$ satisfies Poincaré duality and the universal coefficient theorem. Following (7, 10) the plan is to do surgery on the map

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$Tf|N$ to kill $\ker i_*$ so as to produce a new map $Tf_0\colon (M, \partial M) \to (T\xi, \infty)$ which is homotopic rel ∂M to Tf; is transverse regular to X; and has $N_0 = Tf_0^{-1}(X)$ for which $i\colon N_0 \to M$ is a homotopy equivalence. Suppose momentarily that this has been accomplished. Take $Tf_0\colon M \to T\xi$ to be a bundle map on the normal bundle ν_0 of N_0 so that $M - \nu_0$ goes to ∞ ((9), p. 69). Since M, ∂M, and N_0 are simply connected, $l \geqslant 3$, and $i\colon N_0 \to M$ is a homotopy equivalence, we have an h-cobordism $M - \operatorname{int} \nu_0$ between ∂M and $\partial \nu_0$. By Smale's Theorem ((11)), $M - \operatorname{int} \nu_0$ is a product $\partial \nu_0 \times I$, so there is an isotopy

$$h_t\colon (M, \partial M) \to (M, M - \operatorname{int} \nu_0)$$

with $h_1(M) = \nu_0$. h_1 is the required diffeomorphism for the bundle structure and $Tf_0 \circ h_t$ provides the homotopy between Tf_0 and $Tf_0 \circ h_1$.

How to get Tf_0. Suppose i is k-connected and that $\alpha \in \pi_k(N)$ for $k = 0, 1$ or $\alpha \in (\operatorname{Ker} i_*)_k$ for $2 \leqslant k \leqslant [\frac{1}{2}n]$ can be represented by an imbedded sphere $g\colon S^k \to M$ with trivial normal bundle. Then we shall show how the methods of (3) as generalized in [(10), Section 3] can be adapted to the situation $N \subset M$ instead of $N \subset S^{n+l}$ to obtain

Lemma A. *There is an imbedding* $(V; \partial_- V, \partial_+ V) \subset (\operatorname{int} M \times I; \operatorname{int} M \times 0, \operatorname{int} M \times 1)$ *of the trace* V *of a spherical modification of* N *by* α *and a homotopy*

$$H\colon (M \times I, \partial M \times I) \to (T\xi, \infty)$$

which is transverse regular on X *and satisfies* $V = H^{-1}(X)$ *and* $H|M \times 0 = Tf$.

In the low dimensions $k = 0$ or 1 the condition $\pi_0(M) = \pi_1(M) = 0$ enables us to apply the technique of [(10), Section 3]; so take N to be simply connected and let $2 \leqslant k \leqslant [\frac{1}{2}n]$. Since $i_*\colon H_j(N) \to H_j(M)$ is always onto ((1)) and is 1-1 for $j \leqslant k-1$, the exact homology sequence of the pair (M, N) implies $H_j(M, N) = 0$ for $j \leqslant k$. By the Relative Hurewicz Theorem, $H_{k+1}(M, N) = \pi_{k+1}(M, N)$; so $(\operatorname{Ker} i_*)_k$ is precisely the image of $\pi_{k+1}(M, N)$ under $\partial\colon \pi_{k+1}(M, N) \to \pi_k(N)$ followed by the Hurewicz homomorphism $\pi_k(N) \to H_k(N)$. It follows that $g\colon S^k \to N$ deforms to a point in M. Thus $\nu|g(S^k) = (Tf|S^k)^* \xi$ is trivial. Choose a framing $f_1, \ldots, f_l$ and extend g to $g'\colon D^{k+1} \to M$ for which $g'|S^k \times [0, \epsilon)$ is an imbedding tangent to f_1 along $g(S^k \times 0)$. Since $2k + 3 \leqslant n + l$, g' can be an imbedding and $k + 1 + n < n + l$ insures that N and $g'(D^{k+1})$ can be put in general position so that $g'(D^{k+1}) \cap N = g(S^k)$. Now proceed as in (10).

Use Lemma A until $H|M \times 1$ and $V \cap M \times 1$ are successful candidates for Tf_0 and N_0: For $k < [\frac{1}{2}n]$ there is, as usual, no obstruction to representing α by an imbedded sphere with trivial normal bundle and in this case surgery kills α. When n is odd and $k = [\frac{1}{2}n]$, α can still be so represented and surgery can be done on suitable α's to make $(\operatorname{Ker} i_*)_k$ vanish ((7)). If $n = 4j$ and $k = 2j$, $(\operatorname{Ker} i_*)_k$ may be killed provided it has a symplectic basis of imbedded spheres with trivial normal bundles (see (7) and (6), Section 7). But this is insured by the index condition of (b) which implies that

$$\operatorname{index} (\operatorname{Ker} i_*)_k = \operatorname{index} N - \operatorname{index} X = 0.$$

Proof of Theorem 2. Let $H\colon M \times I \to T\xi$ be the homotopy between $Tf_1 \circ \psi_1$ and $Tf_2 \circ \psi_2$. Make H transverse regular on X keeping $H|M \times 0 \cup M \times 1$ fixed. Then $V = H^{-1}(X)$ is a cobordism between the zero sections N_1 and N_2 of the two bundle structures on M. $H^*\xi = \nu$, the normal bundle of $(V, \partial V) \subset (M \times I, M \times \partial I)$. Now

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$V \subset M \times I$ plays the role of $N \subset M$ in Theorem 1 and the object here is to do surgery on V until it is an h-cobordism. The equivalence is then produced using Smale's h-cobordism theorem for pairs on $(M \times I, V)$ in a fashion similar to the existence proof. V will be an h-cobordism if $i\colon V \to M \times I$ is a homotopy equivalence; that is, if $\operatorname{Ker} i_* = \operatorname{Ker}(H|V)_* = 0$. Use surgery techniques of Lemma A and Theorem 1 to make $(\operatorname{Ker} i_*)_k = 0$ for $k < [\frac{1}{2}(n+1)]$.

Let $k = [\frac{1}{2}(n+1)]$. When $n+1$ is odd the condition $2(n+l) \geqslant 3(n+1)$ again suffices to find the imbedded disc with boundary sphere representing α needed for the surgery. $(\operatorname{Ker} i_*)_k$ can be killed ((7)). When $n+1 = 2k$ the method of Theorem 1 requires $2(n+l) > 3(n+1)$ to find the disc, but now there is an obstruction to finding suitable spheres with trivial normal bundles. For k even it is index $(\operatorname{Ker} i_*)$ and for k odd it is a Kervaire invariant (see (6), Section 8). In either case there is a $(k-1)$-connected, stably parallelizable $2k$-manifold with $\partial W = \Sigma \in \theta^{2k-1}(\partial\pi)$ such that W has index or Kervaire invariant equal to the negative of the obstruction (see (6) or (2)). Imbed

$$(W, \partial W) \subset (D^{n+l+1}, \partial D^{n+l+1})$$

with trivial normal bundle and take the connected sum along the boundary of the pairs (D^{n+l+1}, W) and (ν, V)—where the 'boundary' of (ν, V) is $(\operatorname{int}\nu|N_2, N_2)$. This produces a cobordism $V' \subset M \times I$ between $N_1 \subset M \times 0$ and $N_2 \# \Sigma \subset M \times 1$. Furthermore, the normal bundle of $W \subset M \times I$ is trivial, so $H\colon \nu \to \xi$ extends to a bundle map $H'\colon \nu' \to \xi$ of the normal bundle ν' of $V' \subset M \times I$. The obstruction now vanishes, so there is an h-cobordism between N_1 and $N_2 \# \Sigma$. The ξ-bundle structure $(\xi_2 \# \sigma, f_2, \psi_2)$ on M comes from the normal bundle to $N_2 \# \Sigma$ in $M \times 1$.

REFERENCES

(1) BROWDER, W. Homotopy type of differentiable manifolds. *Colloquium of Algebraic Topology* (Aarhus, 1962).

(2) BROWN, E. H. JR and PETERSON, F. P. The Kervaire invariant of $(8K+2)$-manifolds. *Bull. Amer. Math. Soc.* **71** (1965), 190–193.

(3) HAEFLIGER, A. Knotted $(4K-1)$ spheres in $6K$-space. *Ann. of Math.* (2) **75** (1962), 452–466.

(4) HIRSCH, M. W. Smooth regular neighborhoods. *Ann. of Math.* (2), **76** (1962), 524–530.

(5) HIRSCH, M. W. *On the fibre homotopy type of normal bundles of manifolds*, mimeographed notes, Cambridge Univ., Cambridge, England.

(6) KERVAIRE, M. and MILNOR, J. Groups of homotopy spheres, I. *Ann. of Math.* (2), **77** (1963), 504–537.

(7) LASHOF, R. *Some theorems of Browder and Novikov on homotopy equivalent manifolds with applications*, printed notes, Univ. of Chicago, Chicago, Illinois.

(8) LEVINE, J. On differentiable imbeddings of simply connected manifolds. *Bull. Amer. Math. Soc.* **69** (1963), 806–809.

(9) MILNOR, J. *Differential topology*, mimeographed notes (1958), Princeton Univ., Princeton N.J.

(10) WALL, C. T. C. *An extension of results of Novikov and Browder*, mimeographed notes. Cambridge Univ., Cambridge, England.

(11) WALL, C. T. C. *Differential topology*, mimeographed notes, Cambridge Univ., Cambridge, England.

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