DESINGULARIZATIONS AND THE UNIFORM ARTIN–REES THEOREM

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ABSTRACT

It is proved that an excellent ring which is the homomorphic image of a regular ring of finite Krull dimension and having a resolution of singularities satisfies the uniform Artin–Rees property. Extensions of some results of Lipman, Sathaye and Teissier are also obtained.

1. *Introduction and background*

The main purpose of this paper is to relate the existence of a resolution of singularities to several classical problems in commutative algebra. As a consequence of our work, we will be able to give an affirmative answer, for certain classes of rings, to a conjecture made in [**8**] concerning the uniform Artin–Rees property. We will also obtain extensions of theorems of Lipman and Sathaye [**11**] and Lipman and Teissier [**12**].

We first describe the uniform Artin–Rees theorem. The usual Artin–Rees lemma states that if R is Noetherian, $N \subseteq M$ are finitely generated R-modules, and I is an ideal of R, then there exists a $k > 0$ (depending on I, M, N) such that for all $n > k$, $I^n M \cap N = I^{n-k}(I^k M \cap N)$. A weaker statement which is often used is that for all $n > k$,

$$I^n M \cap N \subseteq I^{n-k} N.$$

When this containment holds for a fixed k and for all ideals I, the pair $N \subseteq M$ is said to satisfy the uniform Artin–Rees property. When all pairs $N \subseteq M$ of finitely generated R-modules satisfy the uniform Artin–Rees property the ring R is said to have the uniform Artin–Rees property. A main question is which rings have the uniform Artin–Rees property. This problem was originally inspired by work of Eisenbud and Hochester, and later Duncan and O'Carroll. See [**3–5**, **8**, **14**, **15**] for a general discussion of this problem. The following conjecture was made in [**8**].

CONJECTURE 1.1. Let R be an excellent Noetherian ring of finite Krull dimension. Let $N \subseteq M$ be two finitely generated R-modules. Then there exists an integer $k = k(N, M)$ such that for all ideals $I \subseteq R$ and all $n \geqslant k$,

$$I^n M \cap N \subseteq I^{n-k} N.$$

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In this paper we will give an affirmative answer to this question whenever R is a homomorphic image of a regular ring of finite Krull dimension and has the property that for all primes P, the integral closure of R/P has a resolution of singularities obtained by blowing up an ideal. While it is true that the largest known class of rings satisfying this property are known to have the uniform Artin–Rees property by the main result of [**8**, (4.12)], the class of rings which have resolution of singularities is expected to include all excellent Noetherian domains. Moreover the method of proof gives considerable insight into related features of the uniform Artin–Rees problem. In particular we find generalizations of theorems of Lipman and Sathaye [**11**] and Lipman and Teissier [**12**] along the way, as well as some new results concerning the uniform annihilation of local cohomology. See Theorem 3.3 and Corollary 4.8 below. Both of these theorems and their generalizations have to do with the integral closure of ideals. Recall the following definition.

DEFINITION 1.2. Let R be a ring and let I be an ideal of R. An element $x \in R$ is *integral over* I if x satisfies an equation of the form $x^n + a_1 x^{n-1} + \ldots + a_n = 0$, where $a_j \in I^j$ for $1 \leqslant j \leqslant n$. The *integral closure* of I, denoted by $\bar{I}$, is the set of all elements integral over I. This set is an ideal.

A theorem proved by Briançon and Skoda [**1**] for convergent power series over the complex numbers and generalized to arbitrary regular local rings by Lipman and Sathaye is the following.

THEOREM 1.3 [**1**, **11**]. *Let R be a regular local ring and let I be an ideal generated by ℓ elements. Then for all $n \geqslant \ell$,*

$$\overline{I^n} \subseteq I^{n-\ell+1}.$$

A weaker generalization to pseudo-rational singularities was obtained by Lipman and Teissier [**12**]. We will discuss this generalization in Section 4.

It turns out that such theorems are closely connected to the uniform Artin–Rees theorems above. The following conjecture was made in [**8**, (1.4)].

CONJECTURE 1.4. Let R be a reduced excellent Noetherian ring of finite Krull dimension. Then there exists an integer k, depending only on R, such that for every ideal $I \subseteq R$, and all $n \geqslant k$,

$$\overline{I^n} \subseteq I^{n-k}.$$

Using the work in [**8**] it is known that an affirmative answer to Conjecture 1.4 for R/P for every prime ideal P of R implies that both Conjectures 1.1 and 1.4 hold. Hence we concentrate on Conjecture 1.4. We will prove that Conjecture 1.4 is true if R is excellent, the homomorphic image of a regular ring of finite Krull dimension and has the property that for every prime P of R, the integral closure of R/P has a resolution of singularities obtained by blowing up an ideal. We take an approach to this conjecture via the method employed by Lipman and Sathaye [**11**]. One of the main assumptions in that paper is that the base ring is regular. The difficulty is that even if we assume that our base ring is regular, we must replace the base ring R by R/P for arbitrary primes P. In particular, we cannot make any assumption about the

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singularities or depth of the rings we are working with other than assuming that they are excellent domains. In their paper [**11**], the regularity of the base ring is used in two ways. The first way simply uses the fact that regular rings are Cohen–Macaulay. The second way uses that the centers of the Rees valuations of an arbitrary ideal on the base ring give regular local rings upon localization at those centers. This is crucial in their argument. It is at this step that we use desingularizations to guarantee regular centers. It means replacing our original base ring with a Rees algebra whose associated Proj is nonsingular. Even if we assume that our original base ring is Cohen–Macaulay, the Rees ring we change to will not in general be Cohen–Macaulay; this change of rings makes the first problem quite significant. In fact, such a Rees algebra is Cohen–Macaulay essentially when the original base ring has rational singularities, a perspective which allows us to generalize work of Lipman and Teissier. To overcome the lack of Cohen–Macaulayness, we prove several theorems about the uniform annihilation of local cohomology, and prove a 'relative' version of the main result in [**11**] which takes into account this annihilation. See Theorem 3.3.

The outline of this paper is as follows: in Section 2 we prove the basic facts about annihilation of local cohomology which we will use. In Section 3, we give a proof of the relative version of the theorem of Lipman and Sathaye. In Section 4 we introduce the notion of regular centers and apply desingularizations to analyze them. In this section we give a discussion of the results of Lipman and Teissier and are able to improve some of their theorems. In particular, see Corollary 4.8. The final section of the paper proves our new uniform Artin–Rees theorem and a general Briançon–Skoda theorem.

2. *Uniform annihilation of local cohomology*

In this section we prove a few facts concerning the annihilators of local cohomology modules which are crucial for the main proof. What we are aiming for is Theorem 2.5. Let R be a Cohen–Macaulay domain, and let S be an integrally closed finitely generated domain containing R. Write $R \subseteq S = T/P$, where $T = R[X_1, \ldots, X_n]$. It is well known that if we choose a regular sequence $g_1, \ldots, g_k \in P$ which generically generates P, then the S-ideal $((g_1, \ldots, g_k) :_T P)\, S$ is reflexive. What we need is a version of this result when R is no longer Cohen–Macaulay. Throughout this paper we say elements $g_1, \ldots, g_k$ in a Noetherian ring R are *parameters* if the height of the ideal they generate is k. We need to prove that if R_c is Cohen–Macaulay, then there is a fixed power of c, say c^N, depending only on R, such that for all such S, and all parameters $g_1, \ldots, g_k \in P$ which generate P generically, the reflexive closure of the S-ideal $((g_1, \ldots, g_k) :_T P)\, S$ is multiplied into this ideal by c^N. We begin by developing several facts that we will need.

PROPOSITION 2.1. *Let R be a d-dimensional Noetherian domain which is a homomorphic image of a regular ring of finite Krull dimension. Assume that $d > 0$. There exists an ideal J in R which defines the non-Cohen–Macaulay locus of* $\operatorname{Spec}(R)$ *with the following property: for every $n \in \mathbb{N}$, and every polynomial ring $T = R[X_1, \ldots, X_n]$ over R, and for every sequence of parameters $g_1, \ldots, g_j \in T$,*

$$J((g_1, \ldots, g_{j-1}) : g_j) \subseteq (g_1, \ldots, g_{j-1}).$$

Proof. We first find an ideal $I \subseteq R$ defining the non-Cohen–Macaulay locus of R, such that for all polynomial rings $T = R[X_1, \ldots, X_n]$, and all ideals $K \subseteq T$, $IH^i_K(T) = 0$ for all $i < \operatorname{ht} K$.

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This step needs some preliminaries regarding uniform annihilation of local cohomology. The version we need is in [**2**, (9.4.13), (9.4.12)], which we set out as a remark for ease of reference.

REMARK 2.2. Assume that R is a homomorphic image of a regular (commutative Noetherian) ring for which there exists a nonzero divisor $c \in R$ such that the ring R_c is Cohen–Macaulay. Assume that either (a) R is an integral domain, or (b) R is an equidimensional local ring.

Then there exists $r \in \mathbb{N}$ such that, for all choices of $n \in \mathbb{N}$ and all choices of an ideal K of the polynomial ring $R[X_1, \ldots, X_n]$ (interpret this as R in the case when $n = 0$), we have

$$c^r H^i_K(R[X_1, \ldots, X_n]) = 0$$

for all $i < \operatorname{ht} K$.

In our case, R satisfies condition (a) of Remark 2.2, and by applying this in turn to each element of a fixed set of generators for an arbitrary ideal defining the non-Cohen–Macaulay locus of R (which is closed since R is the homomorphic image of a regular ring), we can find an ideal I as claimed.

Set $J = I^d$. We claim that J satisfies the conditions of the proposition. Suppose not. Let $g_1, \ldots, g_j$ be parameters in T and suppose that $J((g_1, \ldots, g_{j-1}) : g_j) \nsubseteq (g_1, \ldots, g_{j-1})$. We may localize at some prime Q to preserve this fact and assume that T is local. After replacing R by R_q where $q = Q \cap R$, we may assume that both R and T are local. (Of course, T is no longer a polynomial ring over R.) We denote the maximal ideal of T by the same letter Q. By construction, $IH^i_Q(T) = 0$ for $i < \dim T$. Set $\mathscr{C}_i = \operatorname{Ann}_T(H^i_Q(T))$. By [**16**, (11.2.1)], it follows that the product $\mathscr{C} = \prod_{i=0}^{D-1} \mathscr{C}_i$ annihilates the quotient module $N = ((g_1, \ldots, g_{j-1}) : g_j)/(g_1, \ldots, g_{j-1})$, where $D = \dim T$.

By [**13**, Section 23] we know that $\dim T = \dim R + \dim T/qT$, and $\operatorname{depth} T = \operatorname{depth} R + \operatorname{depth} T/qT$, where q denotes the maximal ideal of R. Since the fibers of the inclusion of R into T are Cohen–Macaulay, we see that $\operatorname{depth} T = \operatorname{depth} R + \operatorname{depth} T/qT = \operatorname{depth} R + \dim T/qT = \operatorname{depth} R + \dim T - \dim R \geqslant D - d$. Therefore $\mathscr{C}_i = T$ for $0 \leqslant i \leqslant D - d - 1$ (as $H^i_Q(T) = 0$ for $i < \operatorname{depth} T$). By assumption, $I \subseteq \mathscr{C}_i$ for all $i < D$, and it now follows that $J = I^d \subseteq \mathscr{C}$. □

LEMMA 2.3. *Let T be a Noetherian ring, and let $J \subseteq T$ be an ideal such that for all parameters $g_1, \ldots, g_j \in T$, $J((g_1, \ldots, g_{j-1}) : g_j) \subseteq (g_1, \ldots, g_{j-1})$. Then for every ideal I,*

$$JH^i_I(T/(g_1, \ldots, g_j)) = 0$$

for all parameters $g_1, \ldots, g_j \in T$ and for all $i < \operatorname{ht}((I + (g_1, \ldots, g_j))/(g_1, \ldots, g_j))$.

Proof. Observe that since $g_1, \ldots, g_j$ are parameters, the image of J in the ring $T/(g_1, \ldots, g_j)$ satisfies the same assumption as J does in T, namely for all parameters $y_1, \ldots, y_m$ in $B = T/(g_1, \ldots, g_j)$, $J((y_1, \ldots, y_i) :_B y_{i+1}) \subseteq (y_1, \ldots, y_i)$. We see this simply by lifting back the y_i to $z_i \in T$ in such a way that $g_1, \ldots, g_j, z_1, \ldots, z_m$ are parameters in T. It then suffices to prove the lemma for the case $j = 0$, in other words, to prove that for all ideals I, $JH^i_I(T) = 0$ for $i < \operatorname{ht}(I)$. We use induction on i to prove this last statement. If $\operatorname{ht}(I) = 0$ there is nothing to prove. Assume $\operatorname{ht}(I) > 0$. We first prove that $JH^0_I(T) = 0$. Let $u \in H^0_I(T)$. For some m, $I^m u = 0$, and hence there is an element $b \in I$, not in any minimal prime of T, such that $bu = 0$. Then $u \in 0 : b$, and this latter ideal is annihilated by J, since b is a parameter.

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Fix i, $0 < i < \mathrm{ht}(I)$, and choose an element $u \in H^i_I(T)$. Set $L = H^0_I(T)$. Since L is killed by a power of I, it follows that $H^k_I(L) = 0$ for $k > 0$, and hence $u \in H^i_I(T) \cong H^i_I(S)$, where $S = T/L$. Choose an element $a \in I$, not in any minimal prime of T, such that $au = 0$. Write a 'bar' to denote images after going modulo a. There is an exact sequence

$$0 \longrightarrow S \xrightarrow{a} S \longrightarrow \bar{S} \longrightarrow 0,$$

which induces a long exact sequence on local cohomology

$$\ldots \longrightarrow H^{i-1}_I(\bar{S}) \longrightarrow H^i_I(S) \xrightarrow{a} H^i_I(S) \longrightarrow \ldots.$$

The exact sequence above then proves that u comes from an element $v \in H^{i-1}_I(\bar{S})$, since $au = 0$. There is an exact sequence

$$0 \longrightarrow K \longrightarrow \bar{T} \longrightarrow \bar{S} \longrightarrow 0$$

where K is the image of L in $\bar{T}$. The long exact sequence on local cohomology gives

$$\ldots \longrightarrow H^{i-1}_I(\bar{T}) \longrightarrow H^{i-1}_I(\bar{S}) \longrightarrow H^i_I(K) \longrightarrow \ldots.$$

However, $H^k_I(K) = 0$ for all $k > 0$ as K is killed by a power of I. In addition, $\mathrm{ht}(\bar{I}) < \mathrm{ht}(I)$. By induction, $\bar{J}H^i_{\bar{I}}(\bar{T}) = 0$ for $i < \mathrm{ht}(\bar{I}) \leqslant \mathrm{ht}(I) - 1$, since our hypothesis concerning J is preserved in $\bar{T}$. It follows that J annihilates $H^{i-1}_I(\bar{S})$. In particular, $Jv = 0$. □

NOTATION 2.4. Let T be a ring and let $R = T/I$ for some ideal I of height n. Suppose that $g_1, \ldots, g_n \in I$ form a regular sequence which generate I_P after localization at an arbitrary minimal prime P of I. By $\omega_{\underline{g}}$ we denote the module $(((g_1, \ldots, g_n) : I) + I)/I$. This module depends on the choice of the g_i in general, but under suitable conditions will represent a relative canonical module for R as a T-algebra.

THEOREM 2.5. *Let R be a d-dimensional Noetherian domain which is a homomorphic image of a regular ring S of finite Krull dimension. Assume that $d > 0$. Let $T = R[X_1, \ldots, X_n]$ be a polynomial ring over R, and let $J \subseteq T$ be an ideal such that for all parameters $f_1, \ldots, f_k \in T$ and for all $0 \leqslant i \leqslant k-1$, $J((f_1, \ldots, f_i) : f_{i+1}) \subseteq (f_1, \ldots, f_i)$. Let $P \in \mathrm{Spec}(T)$ be of height h. Assume that*

(1) *$g_1, \ldots, g_h \in P$ are parameters which generate P generically;*
(2) *$\mathrm{ht}(J + (\underline{g})) \geqslant h + 2$, where $(\underline{g}) = (g_1, \ldots, g_h)$.*

Then $J^3 H^1_J(\omega_{\underline{g}}) = 0$.

Proof. Employing the notation established above in the case $I = P$, we use the short exact sequence,

$$0 \longrightarrow ((\underline{g} : P) \cap P)/(\underline{g}) \longrightarrow \underline{g} : P/(\underline{g}) \longrightarrow \omega_{\underline{g}} \longrightarrow 0. \tag{2.1}$$

We first claim that $J((\underline{g} : P) \cap P) \subseteq (\underline{g})$, that is, J annihilates the left-hand module in the above sequence.

Write a primary decomposition of $(\underline{g})$; since $\underline{g}$ generically generates P, we can write

$$(\underline{g}) = P \cap q \cap K$$

where q is the intersection of the other minimal primary components of $\underline{g}$ (all of height h), and where K is the intersection of some choice of the embedded components. In

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particular, ht $K \geqslant h+1$. If $u \in ((\underline{g}:P) \cap P)$, we claim that $Ku \subseteq (\underline{g})$ (in fact $u \in P \cap q$). It is enough to prove that $Ku \subseteq q$, and this follows since $Pu \subseteq (\underline{g})$ and P is not in any minimal prime above q. This means that $((\underline{g}:P) \cap P)/(\underline{g}) \subseteq H^0_K(T/(\underline{g}))$. As the height of K is at least $h+1$, Lemma 2.3 gives that J annihilates this module, which finishes the proof of the claim.

The long exact sequence on cohomology coming from (2.1) then shows that to finish the proof of the theorem it is enough to prove that J^2 annihilates $H^1_J((\underline{g}:P)/(\underline{g}))$. To prove this, consider the exact sequence,

$$0 \longrightarrow (\underline{g}:P)/(\underline{g}) \longrightarrow T/(\underline{g}) \longrightarrow T/\underline{g}:P \longrightarrow 0.$$

It is enough to prove that J kills both $H^0_J(T/\underline{g}:P)$ and $H^1_J(T/(\underline{g}))$.

Lemma 2.3 gives that $JH^i_J(T/(\underline{g})) = 0$ for $i = 0, 1$. We claim that this also implies that $JH^0_J(T/(\underline{g}:P)) = 0$. For suppose that $u \in T$ and $J^N u \subseteq (\underline{g}):P$ for some N. Then $J^N uP \subseteq (\underline{g})$, and then $JuP \subseteq (\underline{g})$ as $JH^0_J(T/(\underline{g})) = 0$. The claim follows, and finishes the proof of the theorem. □

COROLLARY 2.6. *Let R be an excellent domain which is the homomorphic image of a regular ring of finite Krull dimension. There exists an ideal $\mathscr{A} \subseteq R$ defining the non-Cohen–Macaulay locus of R such that for all $n \in \mathbb{N}$ and for every polynomial ring $T = R[X_1, \ldots, X_n]$, the following holds: if $S = T/P$ is a quotient domain of T with $P \cap R = 0$, so that $R \subseteq S$, $g_1, \ldots, g_h \in P$ are parameters which generate P generically, and* $\mathrm{ht}(\mathscr{A} + (\underline{g})) \geqslant h+2$, *then*

$$\mathscr{A} H^1_{\mathscr{A}}(\omega_{\underline{g}}) = 0.$$

Proof. Observe that T_P is regular as P contracts to 0 in R and hence T_P is a localization of $K[X_1, \ldots, X_n]$, where K is the fraction field of R. We apply Theorem 2.5. T is universally catenary and is equidimensional. If we prove that there is an ideal $J \subseteq R$ satisfying the condition that for all parameters $g_1, \ldots, g_j \in T$, $J((g_1, \ldots, g_{j-1}):g_j) \subseteq (g_1, \ldots, g_{j-1})$, then $\mathscr{A} = J^3$ will satisfy the conclusion of the corollary. Such an ideal J, defining the non-Cohen–Macaulay locus of R, exists by Proposition 2.1. □

3. *A generalized Lipman–Sathaye theorem*

In this section we give a new version of the theorem of Lipman and Sathaye [**11**] which they used to give an algebraic proof of the theorem of Briançon and Skoda. In the paper of Lipman and Sathaye, the base ring R is assumed to be regular. They point out that one can relax some of the conditions and assume that R is Cohen–Macaulay and assume an extra condition which we will describe below. For the applications to the uniform Artin–Rees theorem, it is crucial to remove even the assumption that R is Cohen–Macaulay. We replace this condition by using the results proved in the last section concerning the uniform annihilation of local cohomology. For the proof we simply follow one of the proofs given in [**11**], modifying it where necessary. Given the philosophy of uniform annihilation, the final result is not surprising. We will use this result to prove our main theorems in the next two sections.

We begin by establishing the notation we will be using throughout the rest of the paper.

NOTATION 3.1. Throughout, R will denote an excellent Noetherian domain of positive dimension which is the homomorphic image of a regular ring of finite Krull

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dimension. We assume that $R \subseteq S = R[X_1, \dots, X_n]/P$, where P is prime with $P \cap R = 0$. We let K be the fraction field of R, and L be the fraction field of S. We assume that L is finite and separable over K. If $g_1, \dots, g_n \in P$, we let g_X denote the determinant of the Jacobian matrix whose ijth entry is $\partial g_i/\partial X_j$. We write $T = R[X_1, \dots, X_n]$.

REMARK 3.2. Adopt the notation and assumptions of Notation 3.1.

(1) Since L is finite over K we must have $L = K[X_1, \dots, X_n]/P_K$, where P_K is the image of P in $K[X_1, \dots, X_n]$. Hence $T_P = K[X_1, \dots, X_n]_{P_K}$ is a regular local ring of dimension n. In particular, the height of P is n, the number of variables.

(2) If $g_1, \dots, g_n \in P$, then $g_X \notin P$ if and only if $(g_1, \dots, g_n) : P \not\subseteq P$, that is, if and only if $g_1, \dots, g_n$ generate P generically.

(3) Set $\mathscr{T} = \mathscr{T}_{S/R} = \{g_X \mid g_1, \dots, g_n \in P\} S$. Then $\mathscr{T}$ is generated by elements g_X such that $(g_1, \dots, g_n) : P \not\subseteq P$, and $\operatorname{ht}(g_1, \dots, g_n) \geqslant n$.

(4) The Cohen–Macaulay locus of R is open, and hence there is an ideal J defining the non-Cohen–Macaulay locus. Using the work in Section 2, we may take a power of J, say $\mathscr{A}$, that satisfies the conclusion of Proposition 2.1 and also, given assumptions (1) and (2) of Theorem 2.5, the conclusion of Theorem 2.5. We fix such an ideal $\mathscr{A}$ for the remainder of this section.

With this notation, the main result of this section, and the main technical result we use to prove our results on uniform Artin–Rees and a generalization of work of Lipman and Teissier, is the following theorem.

THEOREM 3.3. *Assume that R is a normal Noetherian domain of positive dimension with field of fractions K. There exists an ideal $\mathscr{B}$ defining the non-Cohen–Macaulay locus of R such that for all $R \subseteq S = R[X_1, \dots, X_n]/P$ with $P \cap R = 0$, a domain with fraction field L which is separable and finite over K,*

$$\mathscr{B}(\bar{S} :_L \mathscr{T}_{\bar{S}/R}) \subseteq S :_L \mathscr{T}$$

(*where $\bar{S}$ denotes the integral closure of S*) *provided that S satisfies the following:*

(#)*: For every height one prime Q of S, $R_{Q \cap R}$ is regular.*

In particular, whenever these assumptions hold,

$$\mathscr{B}\mathscr{T}\bar{S} \subseteq S.$$

In the case in which the base ring R is regular, this theorem is the main result of Lipman and Sathaye [**11**] with the ideal $\mathscr{B} = R$. However, they point out the importance of condition (#), but their proofs rely heavily on the base ring R being Cohen–Macaulay. What we have done is to replace the assumption that R is Cohen–Macaulay with the use of the uniform annihilator $\mathscr{B}$.

DEFINITION 3.4. We say that $\underline{g} = g_1, \dots, g_n \in P$ are *acceptable* if $g_X \notin P$ and $\operatorname{ht}(g_1, \dots, g_n) = n$, or equivalently, $\operatorname{ht}(g_1, \dots, g_n) = n$ and $(\underline{g}) : P \not\subseteq P$.

NOTATION 3.5. Assume that $g_1, \dots, g_n \in P$ are acceptable. We proceed as in [**11**] to define an S-homomorphism $\phi_{\underline{g}} : ((g_1, \dots, g_n) : P)/(g_1, \dots, g_n) \longrightarrow L$ as follows: if $u \in (g_1, \dots, g_n) : P$ represents a class $\bar{u} \in ((g_1, \dots, g_n) : P)/(g_1, \dots, g_n)$, define $\phi_{\underline{g}}(\bar{u}) = u'/g'_X$, where we use $(\,)'$ to denote images in L. We set $M_{\underline{g}}$ to be the image of $\phi_{\underline{g}}$.

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A key point in our generalized version of the theorem of Lipman and Sathaye is the following proposition.

PROPOSITION 3.6. *Adopt the notation and assumptions of Remark* 3.2 *and Notation* 3.5. *Let* $c \in R$ *be an element such that* R_c *is Cohen–Macaulay. Let* g *and* h *be acceptable sequences. There exists an integer* k *such that*

$$c^k M_{\underline{g}} \subseteq M_{\underline{h}} \quad \text{and} \quad c^k M_{\underline{h}} \subseteq M_{\underline{g}}.$$

Proof. By raising c to a power, we may assume that $c \in \mathscr{A}$ (see Remark 3.2(4)). Following [**11**], we define a distance between two acceptable sequences $\underline{g}$ and $\underline{h}$ in $T = R[X_1, \dots, X_n]$ by $\rho(\underline{g}, \underline{h}) = \min\{s \mid \underline{g}E_1 = (g'_1, \dots, g'_s, w_{s+1}, \dots, w_n)$ and $\underline{h}E_2 = (h'_1, \dots, h'_s, w_{s+1}, \dots, w_n)\}$ for some invertible matrices E_1 and E_2 with coefficients in the ring.

We prove the proposition by induction upon $\rho(\underline{g}, h)$. We prove the stronger claim that the integer k can be chosen less than or equal to $\rho(\underline{g}, \underline{h})$. If $\rho(\underline{g}, \underline{h}) = 0$, then it is easy to check that $M_{\underline{g}} = M_{\underline{h}}$ as there is an invertible matrix E such that $\underline{g} = \underline{h}E$.

We need to do the case $\rho(\underline{g}, \underline{h}) = 1$ separately. After multiplying by invertible matrices, we may assume without loss of generality that $\underline{g} = (y_1, \dots, y_{n-1}, g)$ and $\underline{h} = (y_1, \dots, y_{n-1}, h)$. Let $u \in (\underline{g}) : P$, and write

$$uh = \sum_{i=1}^{n-1} r_i y_i + vg. \tag{3.1}$$

It is straightforward to check that $u'/g'_X = v'/h'_X$ (see Notation 3.5).

We claim that $cv \in (\underline{h}) : P$. Multiplying (3.1) by an arbitrary element $z \in P$ yields $zuh = \sum_{i=1}^{n-1} r_i y_i z + vgz$. As $z \in P$ and $u \in (\underline{g}) : P$, there is an equation $zu = \sum_{i=1}^{n-1} s_i y_i + sg$, and upon substitution in the preceding equation one obtains

$$g(sh - vz) \in (y_1, \dots, y_{n-1}).$$

Since $\operatorname{ht}(y_1, \dots, y_{n-1}, g) = n$, the choice of c guarantees that $c(sh - vz) \in (y_1, \dots, y_{n-1})$; see the conclusion of Proposition 2.1. It follows that $cvz \in (\underline{h})$, and hence that $cv \in (\underline{h}) : P$. $M_{\underline{g}}$ is generated by elements u'/g'_X as u ranges over generators of $(\underline{g}) : P$. Since $c'u'/g'_X = c'v'/h'_X$, and $cv \in (\underline{h}) : P$, this proves that $cM_{\underline{g}} \subseteq M_{\underline{h}}$. By symmetry we obtain that $cM_{\underline{h}} \subseteq M_{\underline{g}}$. This finishes the case $\rho(\underline{g}, \underline{h}) = 1$.

Suppose that $\rho(\underline{g}, \underline{h}) = m > 1$. We may assume that $g_i = h_i$ for $i > m$. Using the prime avoidance theorem, we may choose an element $h'_1 = h_1 + \sum_{i=2}^{n} \mu_i h_i$ such that all of the ideals generated by the sequences $g_1, \dots, g_{i-1}, h'_1, g_{i+1}, \dots, g_n$ for $1 \leqslant i \leqslant m$ have height n. As $\underline{g}$ is acceptable, $(\underline{g}) : P \not\subseteq P$ and there is an element $u \in (\underline{g}) : P$ such that $u \notin P$. Write $uh'_1 = \sum_{i=1}^{n} \lambda_i g_i$. Some $\lambda_i \notin P$ for $i \leqslant m$; otherwise we would have $uh'_1 \in P^2 + (g_{m+1}, \dots, g_n) = P^2 + (h_{m+1}, \dots, h_n)\, T$ which contradicts the fact that $\underline{h}$ is acceptable and hence that $h'_1, h_2, \dots, h_n$ form a minimal basis of PT_P. Fix i between 1 and m such that $\lambda_i \notin P$, and let $\underline{h}' = (g_1, \dots, g_{i-1}, h'_1, g_{i+1}, \dots, g_n)$. Since $\lambda_i \notin P$, it follows that $(\underline{h}') : P \not\subseteq P$, and by choice, $\operatorname{ht}(\underline{h}') = n$. Hence $\underline{h}'$ is acceptable. By construction, $\rho(\underline{g}, \underline{h}') \leqslant 1$. By the case in which the sequences are at most distance 1, we know that $cM_{\underline{g}} \subseteq M_{\underline{h}'}$ and $cM_{\underline{h}'} \subseteq M_{\underline{g}}$. In addition $\rho(\underline{h}', \underline{h}) < m$. Hence there exists an integer $l \leqslant \rho(\underline{h}', \underline{h})$ such that $c^l M_{\underline{h}'} \subseteq M_{\underline{h}}$ and $c^l M_{\underline{h}} \subseteq M_{\underline{h}'}$. The proposition follows with $k = l + 1$. □

DEFINITION 3.7. For a finitely generated S-module N contained in a not necessarily finitely generated S-module M, and an ideal J, set

$$N_M^J = \{\alpha \in M \mid \text{there exists } n \text{ such that } J^n \alpha \subseteq N\}.$$

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COROLLARY 3.8. *Adopt the notation and assumptions of Remark* 3.2 *and Notation* 3.5. *Let* g *and* h *be acceptable sequences. Then* $(M_{\underline{g}})_L^{\mathscr{A}} = (M_{\underline{h}})_L^{\mathscr{A}}$.

Proof. The conclusion follows at once from Proposition 3.6. □

Adopt the notation and assumptions of Notations 3.1 and 3.5. Since R is the homomorphic image of a Cohen–Macaulay ring, the locus of primes where R is locally not Cohen–Macaulay is closed, defined by an ideal J. If $\underline{g}$ is acceptable, we set $K_{S/R} = (M_{\underline{g}})_L^J$. Corollary 3.9 proves that $K_{S/R}$ does not depend upon the choice of acceptable sequence $\underline{g}$, since $(M_{\underline{g}})_L^J = (M_{\underline{g}})_L^{\mathscr{A}}$ as J and $\mathscr{A}$ have the same radical. The notation $K_{S/R}$ is meant to suggest a relative canonical module, which is what role this module plays in the proofs. The proof of Proposition 3.6 requires a potentially unbounded power of $\mathscr{A}$ to be needed in the definition of $K_{S/R}$, since the induction used in that proof needed the power of c to be up to the distance between the two sequences $\underline{g}$ and $\underline{h}$. However, it is critical for us to bound the power of $\mathscr{A}$ used. Our work in the previous section provides us with the tools we need.

COROLLARY 3.9. *Adopt the notation and assumptions of Notations* 3.1 *and* 3.5. *Then* $K_{S/R} \subseteq (S :_L \mathscr{T}_{S/R})_L^{\mathscr{A}}$.

Proof. Let $c \in \mathscr{A}$, and let $g_X \in \mathscr{T}_{S/R}$ where $\underline{g}$ is acceptable (recall that $\mathscr{T}_{S/R}$ is generated by g_X as $\underline{g}$ runs over acceptable sequences). It suffices to prove that $c^N g_X K_{S/R} \subseteq S$ for some positive integer N. Choose any element $\alpha \in K_{S/R}$. By the definition, there exists a power N of c such that $c^N \alpha \in M_{\underline{g}}$. The definition of $M_{\underline{g}}$ then yields the desired containment. □

The next step is to compare $K_{S/R}$ and $K_{B/R}$ where B is a finite birational extension of S. This was done in [**11**] and did not require the general hypotheses of that paper. We need to recall the actual construction given in [**11**, p. 214].

Adopt the notation of Notation 3.1. Let $B = S[y] \subseteq L$ be a simple integral extension of S, where we assume that $y \notin S$. Extend the map of T onto S to an epimorphism $\phi: T[Y] \longrightarrow B$. Let Q be the kernel of ϕ. Clearly $Q \cap T = P$. Since the map is birational, Q contains an element $aY - b (a, b \in T, a \notin P)$ and also Q contains a monic polynomial $h(Y)$ with coefficients in T.

Suppose that $\underline{g}$ is an acceptable sequence for P. The sequence $\underline{g}^* = g_1, \ldots, g_n, h$ is clearly a sequence in Q having height $n+1$, and the Jacobian $g^*_{X,Y} = g_X(\partial h/\partial Y)$. If $\partial h/\partial Y \notin Q$, then $\underline{g}^*$ will be an acceptable sequence for Q. Since $a \notin P = Q \cap T$, if $\partial h/\partial Y \in Q$, then $\partial(h + aY - b)/\partial Y = \partial h/\partial Y + a \notin Q$. Since the degree of h must be at least 2, this change does not affect the fact that h is monic.

We now have the map $((\underline{g}^*): Q)/(\underline{g}^*) \longrightarrow L$ given by $v \in ((\underline{g}^*): Q)$ goes to $v'/(g^*_{X,Y})'$. We denote the image as above by $M_{\underline{g}^*}$. The following lemma is proved in [**11**, Lemmas 3.15 and 3.17].

LEMMA 3.10. $M_{\underline{g}^*} \subseteq M_{\underline{g}}$. *Precisely, for every* $v \in T[Y]$ *with* $vQ \subseteq (g_1, \ldots, g_n, h)\, T[Y]$, *there is a* $u \in T$ *with* $uP \subseteq (g_1, \ldots, g_n)\, T$ *such that*

$$v \equiv u \frac{\partial h}{\partial Y} \bmod Q.$$

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In summary we have the following proposition.

PROPOSITION 3.11. *Adopt the notation and assumptions of Notations* 3.1 *and* 3.5. *If R and S are as in Notation* 3.1, *and $S \subseteq T$ is a finitely generated birational extension, then given an arbitrary acceptable sequence $\underline{g}$ for S, there exists an acceptable sequence $\underline{h}$ for T such that $M_{\underline{h}} \subseteq M_{\underline{g}}$.*

Proof. Simply use Lemma 3.10 repeatedly. □

We next need to analyze $K_{S/R}$ when S is normal. In particular, we need to prove it is reflexive, and is equal to $S:_L \mathscr{T}_{S/R}$ when S is normal and the height 1 primes of S contract to primes at which R is regular. A preliminary proposition is needed.

PROPOSITION 3.12. *Let R be an integrally closed Noetherian domain with fraction field K. Let $N \subseteq M = K^n$ be a finitely generated R-module. Suppose that I is an ideal of height at least* 2 *such that N_P is reflexive for all primes P not containing I. Then N_M^I is a reflexive and finitely generated R-module.*

Proof. For simplicity we write N^I for N_M^I. Let C be the cokernel of the injection of N into N^I. Since every element of C is killed by a power of I, $H_I^0(C) = C$ and $H_I^i(C) = 0$ for all $i \geqslant 1$. We claim that $C \cong H_I^1(N)$ and $H_I^1(N^I) = 0$. The short exact sequence $0 \longrightarrow N \longrightarrow N^I \longrightarrow C \longrightarrow 0$ induces a long exact sequence on local cohomology, a part of which is

$$0 \longrightarrow H_I^0(C) \longrightarrow H_I^1(N) \longrightarrow H_I^1(N^I)$$

so if we prove that $H_I^1(N^I) = 0$, then the first claim follows also. Let x, y be a regular sequence in I. Since N^I is torsion-free, multiplication by x on N^I is injective with cokernel $G = N^I/xN^I$, and there is an induced long exact sequence

$$0 \longrightarrow H_I^0(G) \longrightarrow H_I^1(N^I) \xrightarrow{x} H_I^1(N^I). \tag{3.2}$$

Let $u \in G$ be a nonzero element of $H_I^0(G)$. For some l, $I^l u = 0$, and hence lifting u back to an element $v \in N^I$, $I^l v \subseteq xN^I$. Multiplying by a further power of I we obtain that $I^m v \subseteq xN$ for some integer m. Then the element $\alpha = v/x \in M = K^n$ has the property that $I^m \alpha \subseteq N$. It follows that $\alpha \in N^I$, and that $v \in xN^I$. Hence $H_I^0(G) = 0$. As every element of $H_I^1(N^I)$ is killed by a power of I, the resulting injectivity of multiplication by x in (3.2) then proves that $H_I^1(N^I) = 0$, which proves our claim.

To prove that N^I is finitely generated as an R-module, it suffices to prove that $N^I/N \cong H_I^1(N)$ is finitely generated. We use a version of Grothendieck's finiteness theorem [**2**, 9.5.2] (see also [**6**]). Set

$$s = \min_{I \not\subseteq P}\{\operatorname{depth} N_P + \operatorname{ht}((I+P)/P)\}.$$

Then $H_I^i(N)$ is finitely generated for all $i < s$. In our case it suffices to see that $s \geqslant 2$. Since $\operatorname{ht}((I+P)/P) \geqslant 1$ for all $I \not\subseteq P$, we need only to prove that depth $N_P \geqslant 1$ for all primes P for which $\operatorname{ht}((I+P)/P) = 1$. Since N is torsion-free, the depth of N_P is at least 1 for every nonzero prime P in R. If $P = 0$, then by assumption $\operatorname{ht} I \geqslant 2$. Hence N^I is finitely generated.

It remains to prove that N^I is reflexive, and this is a local property. If $I \not\subseteq P$, then $(N^I)_P = N_P$ is reflexive by assumption. If $I \subseteq P$, then the fact that $H_I^1(N^I) = 0$ proves that $\operatorname{depth}(N^I)_P \geqslant \min\{\operatorname{ht} P, 2\}$. It follows that N^I is reflexive. □

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COROLLARY 3.13. *Adopt the notation and assumptions of Notations* 3.1 *and* 3.5. *Assume that S is normal, and that for every height* 1 *prime Q of S,* $R_{Q\cap R}$ *is regular. Then* $K_{S/R}$ *is reflexive. Furthermore* $K_{S/R} = S:_L \mathscr{T}_{S/R}$.

Proof. Let J define the non-Cohen–Macaulay locus of the ring R. We first observe that $\operatorname{ht} JS \geqslant 2$. If not, there exists a height 1 prime of S, say Q, containing J. Let $q = Q \cap R$. By assumption, R_q is regular, hence Cohen–Macaulay. It follows that $J \not\subseteq q$, a contradiction.

By using Proposition 3.12 we need only to prove that $(K_{S/R})_Q$ is reflexive for every prime Q not containing JS. If Q does not contain JS, then first localizing R at $Q \cap R$, we are in a situation where we may assume that R is Cohen–Macaulay. In this case, $K_{S/R} = M_{\underline{g}}$ for every acceptable $\underline{g}$, and this case follows from [**11**, (3.13)].

To prove the second claim in the Corollary, we will prove that $K_{S/R} \subseteq S:_L \mathscr{T}_{S/R}$, and that the containment induces an isomorphism when localized at every height 1 prime Q of S. Since $K_{S/R}$ is reflexive, and clearly $S:_L \mathscr{T}_{S/R}$ is torsion-free, this implies that $K_{S/R} = S:_L \mathscr{T}_{S/R}$ using standard methods.

We first prove that $K_{S/R} \subseteq S:_L \mathscr{T}_{S/R}$. Fix any element $u \in K_{S/R}$, and fix any one of the special generators g_X for $\mathscr{T}_{S/R}$, where $\underline{g}$ is acceptable. We know that $M_{\underline{g}} \subseteq K_{S/R}$, and that there is a power of J such that $J^n u \subseteq M_{\underline{g}}$. Note that $\operatorname{ht} JS \geqslant 2$ because $R_{Q\cap R}$ is regular (and therefore is Cohen–Macaulay) for every height 1 prime Q of S. We then have $J^n u g_X \subseteq S$. Since S is normal and $\operatorname{ht} JS \geqslant 2$, this implies that $u g_X \in S$. It follows that $u \in S:_L \mathscr{T}_{S/R}$.

Finally, we prove that the containment of $K_{S/R}$ into $S:_L \mathscr{T}_{S/R}$ induces an isomorphism when localized at every height 1 prime Q of S. We may again first localize R at $Q \cap R$. Then R is regular, and $K_{S/R} = M_{\underline{g}}$ for every acceptable $\underline{g}$. The desired conclusion then is immediate from [**11**, (3.14)]. □

We are now ready for the proof of the main result of this section.

Proof of Theorem 3.3. By Remark 3.2(2), $\mathscr{T}_{S/R}$ is generated by elements g_X where $\underline{g}$ is acceptable for $S = T/P$. Fix one such $\underline{g}$. We write $\bar{S} = S[y_1, \ldots, y_l]$ and use Lemma 3.10 repeatedly. This lemma shows that there is an acceptable sequence $\underline{g}^*$ for $\bar{S}$ such that

$$M_{\underline{g}^*} \subseteq M_{\underline{g}}. \tag{3.3}$$

Let $\mathscr{A}$ be as in Remark 3.2(4). We claim that

$$\mathscr{A}^3 K_{\bar{S}/R} \subseteq M_{\underline{g}^*}. \tag{3.4}$$

Consider the short exact sequence (which defines C)

$$0 \longrightarrow M_{\underline{g}^*} \longrightarrow K_{\bar{S}/R} \longrightarrow C \longrightarrow 0.$$

The long exact sequence on local cohomology with support in $\mathscr{A}$ gives us

$$0 = H^0_{\mathscr{A}}(K_{\bar{S}/R}) \longrightarrow H^0_{\mathscr{A}}(C) = C \longrightarrow H^1_{\mathscr{A}}(M_{\underline{g}^*}) \longrightarrow \ldots,$$

where the middle equality follows from Definition 3.7. We must prove that $\mathscr{A}^3 C = 0$, and it clearly suffices to prove that $\mathscr{A}^3 H^1_{\mathscr{A}}(M_{\underline{g}^*}) = 0$. However, the definition of $M_{\underline{g}^*}$ shows that it is isomorphic to $((\underline{g}^*):_T Q + Q)/Q$, where $\bar{S} = T/Q$ with $T = R[X_1, \ldots, X_m]$, and we finish the proof of the claim with an application of Theorem 2.5; note that hypothesis (2) in the statement of Theorem 2.5 is satisfied, by the proof of Corollary 3.13.

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Combining the two containments (3.13) and (3.14), we have $\mathscr{A}^3 K_{\bar{S}/R} \subseteq M_g$. By Corollary 3.13, $K_{\bar{S}/R} = \bar{S} :_L \mathscr{T}_{\bar{S}/R}$. Hence $\mathscr{A}^3(\bar{S} :_L \mathscr{T}_{\bar{S}/R}) \subseteq M_g$. Using the definition of M_g it follows that $g_X(\mathscr{A}^3(\bar{S} :_L \mathscr{T}_{\bar{S}/R})) \subseteq S$. Since $\mathscr{T}_{S/R}$ is generated by such g_X, the conclusion now follows by setting $\mathscr{B} = \mathscr{A}^3$. □

4. *Desingularizations and a generalization of a theorem of Lipman and Teissier*

We wish to apply Theorem 3.3 for rings of the form $S = R[It, t^{-1}]$. We need to identify which ideals I have the property that the contractions of height 1 primes from the integral closure of $R[It, t^{-1}]$ have regular localizations. This property is important enough to warrant a special name.

DEFINITION 4.1. Let I be an ideal in a Noetherian ring R. We say that I has *regular centers* if for every prime P of R such that R_P is not regular, $\ell(I_P) < \dim R_P$. Here, $\ell(J)$ is the analytic spread of J.

The following proposition is well known to experts.

PROPOSITION 4.2. *Suppose that R is excellent and equidimensional. Let I be an ideal of R, not nilpotent. The primes P such that $\ell(I_P) = \dim(R_P)$ are finite in number, and are exactly the contractions to R of the minimal primes of IS for $S = R[It]$.*

Proof. Let P be a prime such that $\ell(I_P) = \dim(R_P)$. We may localize at P and assume that $P = m$, the maximal ideal of R, without loss of generality. By definition, $\ell(I) = \dim R[It]/mR[It]$. Since $\dim R[It] = \dim R + 1$ [7, (9.7)], and since R is equidimensional and excellent, it follows that the height of $mR[It]$ is 1. Hence m is the contraction of a height 1 prime in $R[It]$ which contains $IR[It]$.

Conversely, let Q be a height 1 prime in $R[It]$ containing $IR[It]$. Let $P = Q \cap R$. Localize at P, and call this ring R again. Then $\operatorname{ht}(mR[It]) \leqslant \operatorname{ht} Q = 1$. As R is equidimensional and excellent, we know that $\dim R + 1 = \dim R[It] = \dim R[It]/mR[It] + \operatorname{ht} mR[It] = \ell(I) + 1$. It follows that $\ell(I_P) = \dim(R_P)$. □

REMARK 4.3. Standard equivalences between the extended Rees algebra and the Rees algebra show that the set of primes such that $\ell(I_P) = \dim(R_P)$ can also be characterized as exactly the contractions of the minimal primes of $t^{-1}S$ to R, where $S = R[It, t^{-1}]$, the extended Rees algebra of I. Then we can also characterize this set of primes as exactly the contractions of the minimal primes of $t^{-1}\bar{S}$ to R, where $\bar{S}$ is the integral closure of S. If R is integrally closed, this is the ring $R[t^{-1}, \bar{I}t, \ldots, \overline{I^n}t^n, \ldots]$ where $\overline{I^n}$ is the integral closure of I^n in R. (See [7, (15.2)], for example.) We use here that R is universally catenary.

A key point in the application of Theorem 3.3 is that every ideal I has regular centers when extended to a Rees ring of an ideal whose blowup is regular. While the next proposition is basically an observation, it is crucial to the argument so we emphasize it.

PROPOSITION 4.4. *Let R be a Noetherian excellent domain, and suppose that J is an ideal such that $X = \operatorname{Proj}(R[Jt])$ is a desingularization of $\operatorname{Spec}(R)$. Set $S = R[Jt]$. The extended ideal IS has regular centers in S for every ideal $I \subseteq R$.*

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Proof. Let Q be a prime ideal in S such that $\ell(IS_Q) = \dim(S_Q)$. Assume that S_Q is not regular. Note that if $JtS \not\subseteq Q$, then S_Q is regular since X is nonsingular. Hence we may assume that $JtS \subseteq Q$. Let $q = Q \cap R$. We can localize R at q to compute the analytic spread, so by replacing R by R_q, we can assume that R is local and Q contains the maximal ideal m of R. In this case $Q = (m, JT)S$, and $\dim(S_Q) = \dim(R)+1$. Since IS is extended from R, $\ell(IS_Q) \leqslant \ell(I) \leqslant \dim(R)$. This contradicts our assumption. □

REMARK 4.5. It is not difficult to prove that the property that $IR[Jt]$ have regular centers for all ideals I of R forces $\mathrm{Proj}(R[Jt])$ to be a regular.

In [**12**], Lipman and Teissier introduce the notion of a pseudo-rational ring. This is a notion equivalent to rational singularities for algebras essentially of finite type over a field of characteristic 0 but which allows one to talk about 'rational singularities' even in characteristic p or mixed characteristic where resolution of singularities is not yet known. Every regular local ring is pseudo-rational by [**12**, Section 4]. Their purpose in that paper was to prove a generalization of the theorem of Briançon and Skoda, discussed in the introduction to this paper. They obtain the following generalized version of the Briançon–Skoda theorem.

THEOREM 4.6. [**12**, (2.2)]. *Let R be a Noetherian local ring and assume that the localization R_P is pseudo-rational for every prime ideal P in R. Suppose that I has a reduction J such that* $\dim R_P \leqslant \delta$ *for every associated prime P of J^n for a given positive integer n. Then*

$$\overline{I^{n+\delta-1}} \subseteq J^n.$$

In particular, if J can be generated by a regular sequence of length δ, then the above containment holds for all $n \geqslant 1$.

In related work, Lipman and Sathaye gave a proof of the Briançon–Skoda theorem, generalized to regular local rings which we have discussed above. A subtle difference between the two approaches is the following: in [**11**], it is shown that whenever R is a regular local ring and I is an ideal generated by ℓ elements, then

$$\overline{I^n} \subseteq I^{n-\ell+1}. \tag{4.1}$$

In particular for an arbitrary ideal I of analytic spread ℓ in R, we have $\overline{I^n} \subseteq J^{n-\ell+1}$ for every reduction J of I. (One can pass to $R(t)$ to guarantee that the residue field of R is infinite, and then replace any reduction J of I by a minimal reduction $K \subseteq J$. The ideal K will be generated by ℓ elements and then one can apply (4.1). Contracting back to R gives the statement.)

On the other hand, the theorem of Lipman and Teissier for pseudo-rational rings does not quite give this generality. Both theorems coalesce for ideals I primary to the maximal ideal; in this case the minimal reduction will be generated by a regular sequence, at least when the residue field is infinite. What is not clear is whether (4.1) holds in the pseudo-rational case. We are able to prove that for rational singularities over a field of characteristic 0, $\overline{I^n} \subseteq I^{n-\ell}$ when I is generated by ℓ elements. The main point is that in this case, R has a resolution of singularities obtained by blowing up an ideal J such that $R[Jt]$ is Cohen–Macaulay. In this case, our improved version of the Lipman–Sathaye theorem gives us a more general version of the Lipman–Teissier theorem as well. See Corollary 4.8. If the associated graded ring of I has a nonzero divisor in positive degree, one can even obtain (4.1).

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THEOREM 4.7. *Let A be an integrally closed Noetherian excellent ring of dimension d. Assume that J is an ideal of A such that* $\mathrm{Proj}(A[Jt])$ *is regular, and $R = A[Jt]$ is integrally closed. Let $c' \in A$ be an element such that $R_{c'}$ is Cohen–Macaulay. Then there is a fixed power $c = (c')^k$ such that for every ideal I of A generated by ℓ elements, $c\overline{I^n} \subseteq I^{n-\ell}$ for every $n \geqslant \ell$.*

Proof. Write $I = (x_1, \ldots, x_\ell)$. Let $S = R[x_1 t, \ldots, x_\ell t, t^{-1}]$. Clearly $R[t^{-1}]$ is integrally closed, and S is a birational finitely generated extension ring of $R[t^{-1}]$. We claim that the extension $R[t^{-1}] \subseteq S$ satisfies condition (#) of Theorem 3.3. Let Q be a height 1 prime of S. If $t^{-1} \notin Q$, then after localization at Q, S_Q is a localization of $R[t, t^{-1}]$ at a height 1 prime. This implies that $Q \cap R[t^{-1}]$ is also height 1, and since $R[t^{-1}]$ is integrally closed, $R[t^{-1}]_{Q \cap R[t^{-1}]}$ is regular. If $t^{-1} \in Q$, then Proposition 4.4 shows that as $IR[t^{-1}]$ has regular centers, $R[t^{-1}]_{Q \cap R[t^{-1}]}$ is regular. Hence condition (#) of Theorem 3.3 is satisfied.

We can apply Theorem 3.3 to conclude that there is an ideal $\mathscr{B}$ defining the non-Cohen–Macaulay locus of $R[t^{-1}]$ such that $\mathscr{B}\mathscr{T}_{S/R[t^{-1}]}\overline{S} \subseteq S$. Fix a power c of c' which lies in $\mathscr{B}$.

Write $S = R[t^{-1}][X_1, \ldots, X_\ell]/P$. In P are the elements $X_i t^{-1} - x_i$ for $1 \leqslant i \leqslant \ell$. In particular, $\mathscr{T}_{S/R[t^{-1}]}$ contains the element $t^{-\ell}$. Hence $ct^{-\ell}\overline{S} \subseteq S$, which forces $c\overline{I^n} \subseteq I^{n-\ell}$. □

COROLLARY 4.8. *Let A be an integrally closed Noetherian excellent ring of dimension d. Assume that J is an ideal of A such that* $\mathrm{Proj}(A[Jt])$ *is regular. Assume that $A[Jt]$ is Cohen–Macaulay. Then for every ideal I of A of analytic spread ℓ, $\overline{I^n} \subseteq I^{n-\ell}$ for all $n \geqslant \ell$. In particular, if A has rational singularities and is essentially of finite type over a field of characteristic 0, then for every ideal I of A of analytic spread ℓ,*

$$\overline{I^n} \subseteq I^{n-\ell}$$

for all $n \geqslant \ell$.

Proof. The first claim follows at once from Theorem 4.7. To prove the last claim, we only need to prove that if A has rational singularities and is essentially of finite type over a field of characteristic 0, then there exists an ideal $J \subseteq A$ such that $\mathrm{Proj}(A[Jt])$ is regular and $A[Jt]$ is Cohen–Macaulay. This fact follows at once from [**10**, Theorem 1.4.1]. □

REMARK 4.9. Let A and I be as in Corollary 4.8. Assume that there is an element $x \in I$, $x \notin I^2$ such that the leading form of x in $\mathrm{gr}_I(R)$ is a nonzero divisor. Then we can improve the conclusion of Corollary 4.8 to

$$\overline{I^n} \subseteq I^{n-\ell+1}.$$

We simply need to replace the extension $R[t^{-1}] \subseteq R[It, t^{-1}]$ used implicitly in the proof of Corollary 4.8 by the extension $R \subseteq R[I/x]$. This improves the power of I in the right-hand side of the conclusion by 1 for the ring $R[I/x]$, and we can contract back to A by using the assumption.

5. *The uniform Artin–Rees theorem and the general Briançon–Skoda theorem*

In this section we prove the uniform Artin–Rees theorem discussed in the introduction, see Theorems 5.4 and 5.5. An important definition in [**8**] is the following.

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DEFINITION 5.1. Let A be a Noetherian ring. We let $T(A)$ be the set of all elements $c \in A$ such that there exists an integer k with the property that for all ideals I of A and all $n \geqslant k$, $c\overline{I^n} \subseteq I^{n-k}$.

We need a slightly stronger version of the main results of [**8**], which we give in Theorem 5.2.

THEOREM 5.2. *Let A be a homomorphic image of a regular ring of finite Krull dimension having infinite residue fields. Assume that for all prime ideals P of A, $T(B) \neq 0$, where B is the integral closure of A/P. Then A has the uniform Artin–Rees property.*

REMARK 5.3. To apply Theorem 5.2 one needs infinite residue fields. However, it suffices to prove our main theorems after faithfully flat extension and we can then make the residue fields infinite. Replace A by $A(t)$, the localization of the polynomial ring $A[t]$ at the multiplicatively closed set consisting of all polynomials whose content is the entire ring A. Every residue field of $A(t)$ is infinite, and the extension $A \longrightarrow A(t)$ is faithfully flat.

Proof of Theorem 5.2. Let $N \subseteq M$ be any two finitely generated A-modules. We will use induction on $\dim(M/N)$ to show that $N \subseteq M$ has the uniform Artin–Rees property. By using a prime filtration of M/N together with [**8**, (2.2)], it suffices to show that every $P \subseteq A$ has the uniform Artin–Rees property, where P is prime. Fix such a prime P. By induction we may assume that the uniform Artin–Rees theorem holds for all pairs of A-modules $N \subseteq M$ such that $\dim(M/N) < \dim(A/P)$.

It suffices to prove that $T(A/P) \neq 0$ by [**8**, proof of (3.4)]. (See also [**8**, Proposition 4.5].) By replacing A by A/P, it suffices to assume that A is a domain and to prove that $T(A) \neq 0$. Under this change, we can assume that the uniform Artin–Rees property holds for all pairs of A-modules whose quotient has dimension strictly less than the dimension of A.

Let B denote the integral closure of A. As A is excellent, B is a finite A-module. Furthermore, $\dim(B/A) < \dim(A)$. By our induction, $A \subseteq B$ satisfies the uniform Artin–Rees property. Suppose that k is chosen so that for every ideal I in A,

$$I^n B \cap A \subseteq I^{n-k}A = I^{n-k}, \text{ for all } n \geqslant k.$$

By assumption $T(B) \neq 0$. Choose a nonzero element $c \in T(B)$. By taking a multiple of c we may assume that $c \in A$. There exists an integer m such that for all ideals $J \subseteq B$ and for all $n \geqslant m$, $c\overline{J^n} \subseteq J^{n-m}$. We claim that $c \in T(A)$. Let I be an ideal of A. Then for $n \geqslant m+k$ we have

$$c\overline{I^n} \subseteq c\overline{I^n B} \cap A \subseteq I^{n-m}B \cap A \subseteq I^{n-m-k}$$

proving that $c \in T(A)$ and finishing the proof of the theorem. □

We can apply this theorem and our work on the extended Lipman–Sathaye theorem to prove the following.

THEOREM 5.4. *Let A be an excellent Noetherian ring which is the homomorphic image of a regular ring of finite Krull dimension. Assume that for all primes P of A, the integral closure of A/P has a resolution of singularities obtained by blowing up an ideal. Then A satisfies the uniform Artin–Rees property.*

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Proof. We may assume that the residue fields are infinite. Using Theorem 5.2 it suffices to prove that for every prime P in A, $T(B) \neq 0$, where B is the integral closure of A/P. Fix such a P and B.

By assumption there exists an ideal $J \subseteq B$ such that $\operatorname{Proj} B[Jt]$ is a resolution of singularities of B. By replacing J by the integral closure of a high power of J we may assume that $R = B[Jt]$ is integrally closed. We use Theorem 4.7. Set $d = \dim(B)$. Given an arbitrary ideal K of B, K is integral over an ideal I generated by at most $d+1$ elements [**9**, Corollary A]. Applying Theorem 4.7 proves that there exists a nonzero $c \in B$ such that $c\overline{K^n} = c\overline{I^n} \subseteq I^{n-d-1} \subseteq K^{n-d-1}$. This proves that $T(B) \neq 0$. Applying Theorem 5.2 finishes the proof. □

We are able to give similarly an affirmative answer to the general Briançon–Skoda conjecture.

THEOREM 5.5. *Let A be a reduced excellent Noetherian ring with infinite residue fields which is the homomorphic image of a regular ring of finite Krull dimension. Assume that for all primes P of A, the integral closure of A/P has a resolution of singularities obtained by blowing up an ideal. There exists an integer k such that for every ideal I of A,*

$$\overline{I^n} \subseteq I^{n-k}.$$

Proof. We use [**8**, Proposition 3.7]. This proposition states that the conclusion of Theorem 5.5 holds provided for every prime ideal P of A, $T(A/P) \neq 0$ and $\mathrm{CM}(A/P) \neq 0$. The definition of $\mathrm{CM}(A/P)$ is not important here since the condition that $\mathrm{CM}(A/P) \neq 0$ is automatically satisfied whenever A is the homomorphic image of a regular ring of finite Krull dimension by [**8**, Proposition 4.5]. The first condition was shown in the proof of Theorem 5.4. □

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