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Homogenization Approach to Water Transport in Plant Tissues with Periodic Microstructures

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Abstract. Water flow in plant tissues takes place in two different physical domains separated by semipermeable membranes: cell insides and cell walls. The assembly of all cell insides and cell walls are termed *symplast* and *apoplast*, respectively. Water transport is pressure driven in both, where osmosis plays an essential role in membrane crossing. In this paper, a microscopic model of water flow and transport of an osmotically active solute in a plant tissue is considered. The model is posed on the scale of a single cell and the tissue is assumed to be composed of periodically distributed cells. The flow in the symplast can be regarded as a viscous Stokes flow, while Darcy's law applies in the porous apoplast. Transmission conditions at the interface (semipermeable membrane) are obtained by balancing the mass fluxes through the interface and by describing the protein mediated transport as a surface reaction. Applying homogenization techniques, macroscopic equations for water and solute transport in a plant tissue are derived. The macroscopic problem is given by a Darcy law with a force term proportional to the difference in concentrations of the osmotically active solute in the symplast and apoplast; i.e. the flow is also driven by the local concentration difference and its direction can be different than the one prescribed by the pressure gradient.

Keywords and phrases: plant tissues, osmotic pressure, water flow, homogenisation, two-scale convergence, flows in porous media

Mathematics Subject Classification: 35B27, 35K61, 74Qxx, 76M50, 76S05, 76D07

1. Introduction

Plant tissues are in general composed of two domains separated by selective membranes: apoplast and symplast. The apoplast is composed of cell walls and intercellular spaces, while the symplast is constituted by all protoplasts which can be connected by plasmodesmata. Therefore, the path of water and solutes is threefold: apoplastic, symplastic and transcellular, [35,36]. A first quantitative model of water transport in plants was proposed by van den Honert, [18,41]. The idea was to describe water flow in analogy to the flow of electric current through a resistor network. This phenomenological approach is still contemporary,

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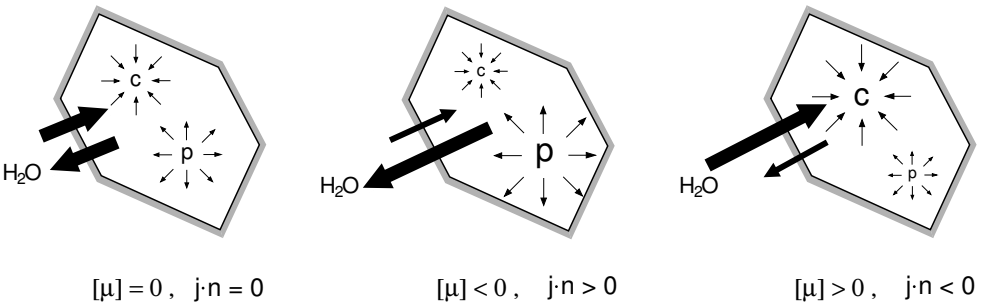


FIGURE 1. Scheme of water flux into and out of a single cell. The flux is proportional to the jump in chemical potential $[\mu]$, which depends on internal pressure p and solute concentration c.

[31], and is also used in engineering to describe water supply networks. Pressure assumes the role of the electric potential and, hence, pressure gradients produce a flux proportional to the hydraulic conductivity. This pressure driven flux was extended to include osmotically driven fluxes (diffusional fluxes) and the concept of water potential was introduced, [31]. The general concept adapted from nonequilibrium thermodynamics is that, differences in water potential produce equilibrating forces which drive the water fluxes, [40]. This relation will be presented below in more detail. Plant biologists have used this concept to describe water uptake of single cells (Fig. 1), e.g. during cell expansion [31].

Besides van den Honert's approach, less has been undertaken to extend and apply the concept for whole tissues. An interesting question is how the concepts should be used in continuum models of tissues. As will become clear later, the central problem is to find suitable transmission conditions, which describe the fluxes through the plasma membranes, and thus, between the apoplast and symplast. Another interesting task is to obtain simplified models for situations where the cell scale is small compared to the tissue or organ considered. Take for example functional structural plant models, in which often macroscopic sections of organs are combined and simulated, [43]. Whilst less complexity is the principle of a simplified model, sufficient information on the microstructure should still be part of it. Fortunatelly, plant tissues tend to be sufficiently periodic and periodic homogenization lends to treat the problem, [9].

A model of water and solute fluxes in plant tissues needs a precise description of transport processes between the cell inside (symplast) and the cell wall (apoplast) (Fig. 2). This transition takes place through semipermeable membranes and represents the connection between two different physical domains. These domains are the porous cell wall described best by a Darcy law and the cell cytoplasm in which a viscous flow applies. One of the main mathematical problems is that, in contrast to [4, 21], the free fluid and porous media domains do not interact directly, as the membrane separates the domains and controls actively and passively the fluxes of water and solutes. Thus the continuity of normal forces and the Beavers-Joseph-Saffman transmission condition between free fluid and porous medium do not apply. A similar situation to the one here is found in models of early atherosclerotic lesions [5] or low-density lipoproteins transfer through arterial walls, [32, 37]. Kedem-Katchalsky transmission conditions are used by those authors to couple the Navier-Stokes equations for blood flow in the arterial lumen with a Darcy law in the arterial wall. However, the dependence of fluid flow across the membrane on the oncotic pressure difference – proportional to the jump in lipoproteins concentration – was neglected by those authors to simplify the analysis of the model equations. Besides, transport of the solute in plant cells is mediated by proteins and can take place opposite to the gradient in chemical potential by usage of energy (e.g. ATPase pumps) and has to be based on a surface reaction mechanism.

Coupled free fluid and porous media problems have received an increasing attention during the last years from the mathematical and the numerical point of view. Well-posedness analysis and numerical

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algorithms for coupled Stokes-Darcy and Navier-Stokes-Darcy problems with Beavers-Joseph-Saffman transmission condition between free fluid and porous medium were addressed in [15,25,34] and references therein. Along many results on the homogenization of Stokes and Navier-Stokes problems and derivation of Darcy law, [3, 19, 21, 26, 38], the multiscale analysis for a Stokes-Darcy system modeling water flow in a vuggy porous media with Beavers-Joseph-Saffman transmission condition was considered in [4]. The macroscopic equations were derived using formal asymptotic expansion and two-scale convergence method. A formal asymptotic expansion was also applied successfully to define the transport velocity of auxin in a plant tissue, [7].

New transmission conditions at the cell-membrane-cell wall interface and the coupling between the flow velocity and solute concentrations via transmission conditions reflecting the osmotic nature of the water flow through a semipermeable membrane distinguish our model from the problem studied in [4]. Additional technical difficulties are introduced due to the distinction between symplastic and apoplastic velocities and the presence of plasmodesmata populated cell wall pieces. To show the existence of a unique solution of the microscopic model we apply the abstract theory of mixed problems (saddle-point problems), [17, 25], where the coercivity in the divergence-free space and the inf-sup condition ensure the existence of a unique solution of the coupled Stokes-Darcy model. The methods of the two-scale convergence and unfolding operator are used to derive macroscopic equations for the fluid flow and for the transport of osmotic active solutes. A generalized Darcy law with a force term given by the difference of solute concentrations in apoplast and symplast defines the macroscopic water velocity. Two initial-boundary-value problems for the concentration of solutes in symplast and in apoplast, respectively, coupled via ordinary differential equations for the transporter concentrations, describe the dynamics of solute concentrations in a plant tissue.

The paper is organized as follows. A thorough introduction to non-advective water fluxes is given. These concepts are then used to derive a biophysical model for transport of water and osmotically active solutes through a cell membrane, and to obtain transmission conditions between the symplast and apoplast (Section 2). Based on this biophysical model, a microscopic model for transport in a plant tissue is formulated (Section 3). Well-posedness and a priori estimates for solutions of the microscopic model are shown (Section 4), followed by derivation of averaged macroscopic equations for water and solute transport defined on the scale of a plant tissue (Section 5). At last, some results on two-scale convergence and periodic unfolding method are formulated (Appendix).

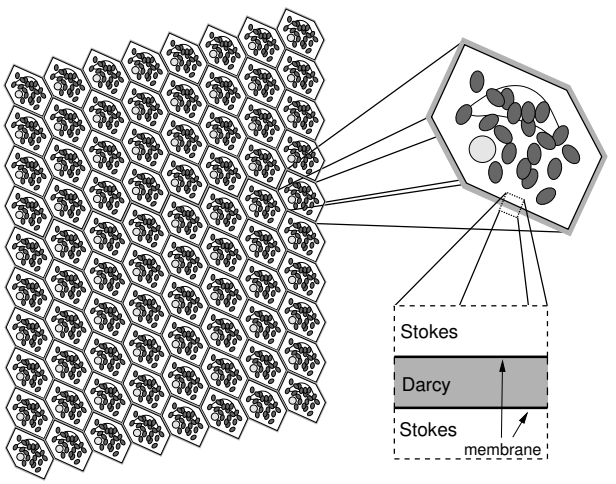


FIGURE 2. Scheme of a typical periodic plant tissue, for example lamellae of the moss *Plagiomnium affine*.

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1.1. Water fluxes and chemical potential

As noted above, water fluxes in cells are known to be driven by gradients in chemical potential, [31]. Biologist use the concept of water potential Ψ, which is the chemical potential of water per unit specific volume – i.e. expressed in pressure units instead of energy per particle. The chemical potential μ_i of a species i is a measure of how much the internal energy of a thermodynamical system changes when the number of particles of species i is varied, [23]. In equilibrium, the entropy of the system is maximal and consequently the temperature T, pressure p and the chemical potentials μ_i are constant in abscence of external fields. Consider now a situation in which the system is not in equilibrium. In this case μ_i will not be constant and diffusion fluxes arise, which drive the system into equilibrium. If the system is near to equilibrium, the diffusion fluxes can be assumed to be proportional to the gradients in chemical potentials $\nabla\mu_i$, [24]. Irreversible thermodynamics proposes that for small gradients, the diffusion mass flux density $\boldsymbol{j}_i$ of a species i is given by a weighted sum of the gradients of all chemical potentials.

Diffusion is a spontaneous molecular equilibration process which does not produce bulk flows in closed systems. The so called mass constraint applies when the system is not divided by a membrane, [16]

$$\boldsymbol{j} = \sum_i \boldsymbol{j}_i = 0 \,, \tag{1.1}$$

which states that diffusion cannot produce a macroscopic movement of the mixture. Condition (1.1) is not valid when the domain is separated by a semipermeable membrane, as such membranes allow the movement of the solvent but not of all solutes. A consequence is that in such a case the system tends to a local equilibrium: entropy reaches a local maximum and removing the membrane would allow a further increase. To account for semipermeable membranes, the concepts were extended to include a so called *reflection coefficient*, [31]. This coefficient is a measure of how much solutes are reflected by the membrane. A value of one means that all solutes are retained, while all solutes cross freely for a value of zero. The reflection coefficient lies in general between 0 and 1 for a real membrane, [31].

For the sake of simplicity, we will consider here and in the sequel the case of a binary mixture of a solvent ($i = 1$) and a solute ($i = 2$). Following the notation of [24], we introduce the mass fraction of the solute as a concentration

$$c := \rho_2/\rho \,, \tag{1.2}$$

where ρ_i, for $i = 1, 2$ is the mass density of species i and $\rho = \rho_1 + \rho_2$ the mass density of the mixture. Consequently the "concentration" of the solvent is $1 - c$. This duality of the concentrations allows to introduce one chemical potential

$$\mu = \frac{\mu_2}{m_2} - \frac{\mu_1}{m_1} \,, \tag{1.3}$$

instead of two, [24], where m_i is the mass of one particle of species i, for $i = 1, 2$. Note that m_1 and m_2 are needed to obtain a chemical potential density in units energy per mixture mass. The flux density of the solute is then $\boldsymbol{j}_2 = -\alpha\nabla\mu$ for $\alpha > 0$. Using the mass constraint (1.1), delivers for that case the flux density $\boldsymbol{j}_1 = \alpha\nabla\mu$ of the solvent. The combined chemical potential cannot be used in the case of a semipermeable membrane and the approach needs to be extended by a reflection coefficient. For this purpose, we introduce two diffusion driving potentials

$$\begin{aligned} \tilde{\mu}_1 &:= \frac{\mu_1}{m_1} - (1-\varsigma)\frac{\mu_2}{m_2} \quad \text{and} \quad \tilde{\mu}_2 := (1-\varsigma)\left(\frac{\mu_2}{m_2} - \frac{\mu_1}{m_1}\right) \,, \\ \tilde{\mu}_1 &= -(1-\varsigma)\mu + \varsigma\frac{\mu_1}{m_1} \quad \text{and} \quad \tilde{\mu}_2 = (1-\varsigma)\mu \,, \end{aligned} \tag{1.4}$$

where $0 \le \varsigma \le 1$ is a reflection coefficient. Setting $\varsigma = 0$ renders $\tilde{\mu}_2 = -\tilde{\mu}_1 = \mu$, which is consistent with [24], while setting $\varsigma = 1$ gives $\tilde{\mu}_1 = \mu_1/m_1$ and $\tilde{\mu}_2 = 0$. The corresponding mass flux densities are simply

$$\begin{aligned} \boldsymbol{j}_i &= -\alpha\,\nabla\tilde{\mu}_i \,, && \text{for } i = 1, 2 \,, \\ \boldsymbol{j} &= -\alpha\,\nabla(\tilde{\mu}_1 + \tilde{\mu}_2) = -\alpha\,\varsigma\,\nabla\mu_1 \,, \end{aligned} \tag{1.5}$$

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where $\alpha > 0$ is a coefficient related to the permeability towards the solvent. The expression for $\boldsymbol{j}$ shows that reflection ($\varsigma \neq 0$) is a must to produce bulk fluxes, and is actually the mechanism exploited by plant cells, [31]. Figure 1 presents a scheme of water fluxes through a membrane of a single cell with a semipermeable membrane ($\varsigma = 1$, $\tilde{\mu}_2 = 0$ and $\boldsymbol{j}_2 = 0$).

Often in mathematical models, diffusion is assumed to be driven only by concentration gradients, which is equivalent to setting $\tilde{\mu}_2 = \tilde{\mu}_2(c)$. For this case, Eqs. (1.5) deliver Fick's law with reflection, [24]

$$\boldsymbol{j}_2 \approx -\rho(1-\varsigma) D \nabla c \,, \tag{1.6}$$

where $D > 0$ is the usual diffusion coefficient [see Eq. (1.8)]. This approach ignores that the chemical potential depends also on pressure p and temperature T, which implies that gradients in p and T produce also fluxes. This dependency is normally assumed to be small, although the pressure term is known to be important in sedimentation processes, [42], and in cases where the pressure is nonharmonic and concentration gradients are small, [6]. Fick's law (1.6) is a fairly good approximation in a homogeneous domain, but is inapplicable for obtaining transmission conditions at semipermeable interfaces. Large pressure differences can arise across a membrane and the contributions of pressure and concentration driven diffusion are of comparable magnitude.

To account for pressure driven diffusion, the potentials $\tilde{\mu}_i$ are set to depend on concentration and pressure. For simplicity, the dependence on temperature is neglected, as large temperature gradients are not usual in plant tissues. Eqs. (1.5) render

$$\begin{aligned} \boldsymbol{j}_2 &= -\rho\,(1-\varsigma)\big(D\,\nabla c + G\,\nabla p\big)\,, \\ \boldsymbol{j} &= \rho\,\varsigma\big(\mathcal{D}\,\nabla c - \mathcal{G}\,\nabla p\big) \qquad \text{and} \\ \boldsymbol{j}_1 &= -\boldsymbol{j}_2 + \boldsymbol{j}\,, \end{aligned} \tag{1.7}$$

where

$$D = \frac{\alpha}{\rho}\frac{\partial \mu}{\partial c} > 0\,, \qquad G = \frac{\alpha}{\rho}\frac{\partial \mu}{\partial p}\,, \qquad \mathcal{D} = -\frac{\alpha}{\rho}\frac{\partial \mu_1}{\partial c} > 0\,, \qquad \mathcal{G} = \frac{\alpha}{\rho}\frac{\partial \mu_1}{\partial p} > 0\,. \tag{1.8}$$

The diffusion coefficients D and $\mathcal{D}$ are positive, while the barodiffusion coefficient G has no definite sign, [24]. Assuming incompressibility of the solvent has as a consequence that $\mathcal{G}$ is positive, [31]. These signs concord with what is known from biology: diffusion fluxes across the mebrane follow the concentration gradient and are oriented against the pressure gradient (see Fig. 1).

2. Biophysical model

2.1. Mass conservation

The total flux density of a species and the mixture is given by the combination of the contributions of advection $\rho_i\,\boldsymbol{v}$, diffusion $\boldsymbol{j}_i$ and membrane transport via transporting proteins $\boldsymbol{a}_i$

$$\begin{aligned} \boldsymbol{J}_i &= \rho_i\,\boldsymbol{v} + \boldsymbol{j}_i + \boldsymbol{a}_i\,, \qquad i = 1,2\,, \\ \boldsymbol{J} &= \rho\,\boldsymbol{v} + \boldsymbol{j} + \boldsymbol{a}\,, \end{aligned} \tag{2.1}$$

where $\boldsymbol{j} := \sum_i \boldsymbol{j}_i$ and $\boldsymbol{a} := \sum_i \boldsymbol{a}_i$. Consequently, conservation of the species and total mass are given by

$$\partial_t \rho_i + \operatorname{div}(\rho_i\,\boldsymbol{v} + \boldsymbol{j}_i + \boldsymbol{a}_i) = 0\,, \qquad i = 1,2\,, \tag{2.2a}$$

$$\partial_t \rho + \operatorname{div}(\rho\,\boldsymbol{v} + \boldsymbol{j} + \boldsymbol{a}) = 0\,. \tag{2.2b}$$

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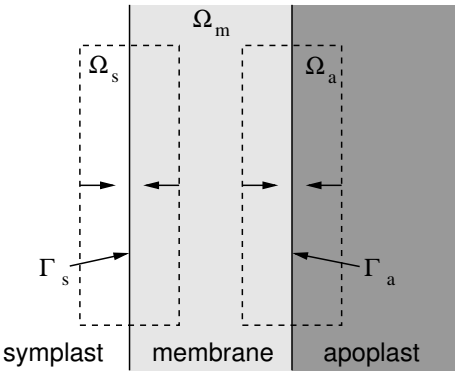


FIGURE 3. Scheme of the method used to obtain the transmission conditions. Small domains Ω_s and Ω_a enclosing the interface symplast-membrane and membrane-apoplast, respectively, are reduced to the surfaces Γ_s and Γ_a. The membrane is represented by the domain Ω_m.

First, consider the laws in a compartment without an interface. Transport via proteins takes place only on membranes. Therefore, $\boldsymbol{a}_i$ are zero inside a domain Ω that does not contain a membrane, and the mass constraint (1.1) applies. Conservation of solute and mass in Ω follows thus

$$\partial_t \rho_i + \operatorname{div}\left(\rho_i \boldsymbol{v} + \boldsymbol{j}_i\right) = 0 \qquad \text{in } \Omega \times (0,\infty)\,, \qquad i = 1,2\,, \tag{2.3a}$$

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{v}) = 0 \qquad \text{in } \Omega \times (0,\infty)\,. \tag{2.3b}$$

Consider now the case where a membrane is present. Shall Ω_s be a domain enclosing an arbitrary piece Γ_s of the interface between the symplast and the membrane (Fig. 3). Shall the membrane be represented by the domain Ω_m. Note that the derivation of the conditions for the interface between membrane and apoplast is conducted in the same manner and will not be given here explicitly. Because of the membrane, $\varsigma \neq 0$ and $\boldsymbol{a}_i \neq 0$ for $i = 1,2$. Integration of (2.2a) over Ω_s and application of Gauß' law gives

$$\partial_t \int_{\Omega_s} \rho_i \, dx + \int_{\partial\Omega_s} (\rho_i \boldsymbol{v} + \boldsymbol{j}_i + \boldsymbol{a}_i) \cdot \boldsymbol{n} \, d\gamma = 0\,.$$

The membrane can be assumed to not allow advective fluxes, so that on $\partial\Omega_s \cap \Omega_m$ we have $\boldsymbol{v} = 0$. On $\partial\Omega_s \backslash \overline{\Omega}_m$ there is no protein mediated transport so that $\boldsymbol{a} = 0$ there. The thickness of Ω_s is reduced to zero such that the interface is kept inside ($\Omega_s \to \Gamma_s$). The first integral tends to zero, while the second tends to an integral over Γ_s

$$\int_{\Gamma_s} \left(-\rho_i \boldsymbol{v} + \overline{\boldsymbol{j}}_i - \boldsymbol{j}_i + \boldsymbol{a}_i \right) \cdot \boldsymbol{n} \, d\gamma = 0\,,$$

where $\overline{\boldsymbol{j}}_i$ is the diffusion flux in Ω_m. Note that $\boldsymbol{n}$ points from the symplast to the apoplast. Because Ω_s and Γ_s were chosen arbitrarily, the integrand has to be zero. A similar approach can be applied to the mass flux of the mixture, where $\boldsymbol{j} = 0$ in Ω is used. We obtain the conditions

$$\begin{aligned} (\rho_i \boldsymbol{v} + \boldsymbol{j}_i) \cdot \boldsymbol{n} &= (\boldsymbol{a}_i + \overline{\boldsymbol{j}}_i) \cdot \boldsymbol{n} \qquad \text{on } (\Gamma_s \cup \Gamma_a) \times (0,\infty)\,, \\ \rho \boldsymbol{v} \cdot \boldsymbol{n} &= (\boldsymbol{a} + \overline{\boldsymbol{j}}) \cdot \boldsymbol{n} \qquad \text{on } (\Gamma_s \cup \Gamma_a) \times (0,\infty)\,, \end{aligned} \tag{2.4}$$

where $\overline{\boldsymbol{j}} = \overline{\boldsymbol{j}}_1 + \overline{\boldsymbol{j}}_2$. The fluxes in Ω were written on the left hand side, while the fluxes in Ω_m are on right hand side. The diffusion fluxes $\overline{\boldsymbol{j}}_i$ are normally assumed to be constant in the membrane (constant gradients), [31]. Protein mediated transport can be considered to be a chemical reaction allowing buffering of species, and hence, the fluxes $\boldsymbol{a}_{i,\mathrm{I}}$ and $\boldsymbol{a}_{i,\mathrm{II}}$ through Γ_s and Γ_a, respectively, have not necessarily to be equal. Expressions for $\boldsymbol{a}_{i,\mathrm{I}}$ and $\boldsymbol{a}_{i,\mathrm{II}}$ will be developed in Section 2.2.

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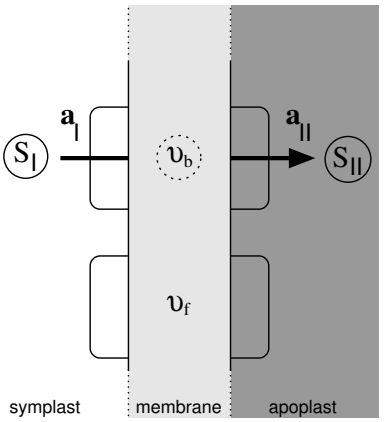


FIGURE 4. Depiction of the mechanism assumed for protein mediated transport, where a transport direction from symplast to apoplast is shown exemplarily.

The membrane Ω_m is very thin compared to the rest and is usually reduced to an interface denoted here as Γ_m. An expression for the membrane's buffering effect is obtained by integration of (2.2a) over Ω_m, application of Gauß' law and reduction of Ω_m to Γ_m. This process is equivalent to subtraction of the transmission conditions on Γ_s from those on Γ_a. Finally, the transmission conditions on the reduced membrane interface Γ_m are in total

$$(\rho_i \boldsymbol{v} + \boldsymbol{j}_i) \cdot \boldsymbol{n} = (\boldsymbol{a}_{i,\mathrm{I}} + \overline{\boldsymbol{j}}_i) \cdot \boldsymbol{n} \qquad \text{on } \Gamma_m \times (0,\infty)\,, \qquad i = 1,2\,, \tag{2.5a}$$

$$\rho\, \boldsymbol{v} \cdot \boldsymbol{n} = (\boldsymbol{a}_{\mathrm{I}} + \overline{\boldsymbol{j}}) \cdot \boldsymbol{n} \qquad \text{on } \Gamma_m \times (0,\infty)\,, \tag{2.5b}$$

$$[\rho_i \boldsymbol{v} + \boldsymbol{j}_i] \cdot \boldsymbol{n} = [\boldsymbol{a}_i] \cdot \boldsymbol{n} \qquad \text{on } \Gamma_m \times (0,\infty)\,, \qquad i = 1,2\,, \tag{2.5c}$$

$$[\rho\, \boldsymbol{v}] \cdot \boldsymbol{n} = [\boldsymbol{a}] \cdot \boldsymbol{n} \qquad \text{on } \Gamma_m \times (0,\infty)\,, \tag{2.5d}$$

where the assumptions $\overline{\boldsymbol{j}}_i = \text{const}$, and $\overline{\boldsymbol{j}} = \text{const}$ in the membrane were used, and $[\cdot]$ is defined as the jump across the membrane (i.e. $[\boldsymbol{a}_i] = \boldsymbol{a}_{i,\mathrm{II}} - \boldsymbol{a}_{i,\mathrm{I}}$).

2.2. Protein mediated transport

Most solute fluxes through membranes are mediated by transporting proteins, [31]. Transport can be either passive or active with usage of energy (e.g. ATPase pumps). We model this processes as surface reactions, in which the solute on one side (S_{I}) reacts with the transporter, builds a complex (S–T), which decays by transporting the solute to the other side (S_{II})

$$S_{\mathrm{I}} + T \underset{k_2}{\overset{k_1}{\rightleftharpoons}} S{-}T \underset{k_4}{\overset{k_3}{\rightleftharpoons}} S_{\mathrm{II}} + T\,.$$

The concentrations of the solute on either sides (S_{I} and S_{II}) are denoted by c_{I} and c_{II}, respectively. The concentrations of free and bound transporters are denoted with ϑ_f and ϑ_b, respectively. In general, the density of the mixture can differ between sides and we denote these here with ρ_{I} and ρ_{II}. The above simple reaction mechanism produces fluxes on each side

$$\begin{aligned} \boldsymbol{a}_{\mathrm{I}} &= (k_1\, \rho_{\mathrm{I}}\, c_{\mathrm{I}}\, \vartheta_f - k_2\, \vartheta_b)\, \boldsymbol{n}\,, \\ \boldsymbol{a}_{\mathrm{II}} &= (k_3\, \vartheta_b - k_4\, \rho_{\mathrm{II}}\, c_{\mathrm{II}}\, \vartheta_f)\, \boldsymbol{n}\,, \end{aligned} \tag{2.6}$$

where $\boldsymbol{n}$ is the normal of the membrane pointing out of the cell. These expressions correspond to the rate law of the above reaction mechanism. See Fig. 4 for a depiction of this mechanism.

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The concentration of free and bound transporters are obtained by pointwise reactions. The rate of change of free transporters due to binding of solute and decay of the complex is given by

$$[\boldsymbol{a}] \cdot \boldsymbol{n} = (\boldsymbol{a}_{\mathrm{II}} - \boldsymbol{a}_{\mathrm{I}}) \cdot \boldsymbol{n} = -(k_1\, \rho_{\mathrm{I}}\, c_{\mathrm{I}} + k_4\, \rho_{\mathrm{II}}\, c_{\mathrm{II}})\, \vartheta_f + (k_2 + k_3)\, \vartheta_b \ . \tag{2.7}$$

The net transport direction varies in time and depends on the solute concentrations and coefficients k_1 to k_4. By focusing on a quasistationary situation in which $\boldsymbol{a}_{\mathrm{II}} = \boldsymbol{a}_{\mathrm{I}} := \boldsymbol{a}$, the following net flux and concentration of free transporters are obtained

$$\begin{aligned} \boldsymbol{a} &= (k_1\, k_3\, \rho_{\mathrm{I}}\, c_{\mathrm{I}} - k_2\, k_4\, \rho_{\mathrm{II}}\, c_{\mathrm{II}})/(k_2 + k_3)\ \vartheta_f\, \boldsymbol{n} \ , \\ \vartheta_f &= (k_2 + k_3)/(k_2 + k_3 + k_1\, \rho_{\mathrm{I}}\, c_{\mathrm{I}} + k_4\, \rho_{\mathrm{II}}\, c_{\mathrm{II}})\ \vartheta_0 \ , \end{aligned} \tag{2.8}$$

where $\vartheta_0(t,x) = \vartheta_f(t,x) + \vartheta_b(t,x)$ is the total amount of transporters. We will focus here on the special case where either $k_1 = 0$ or $k_4 = 0$, i.e. on the case of a perfect uni-directional transporter. Using the assumption of quasistationarity, the mechanism is found to follow Michaelis-Menten kinetics. Exemplary for an efflux transporter ($k_4 = 0$)

$$\boldsymbol{a} = \rho_{\mathrm{I}}\, c_{\mathrm{I}}\, k_1\, k_3/(k_2 + k_3 + k_1\, \rho_{\mathrm{I}}\, c_{\mathrm{I}})\ \boldsymbol{n} \ . \tag{2.9}$$

The applicability of the assumption of quasistationarity is limited. Transporters are in general subjected to regulation, which reduces or increases the amount of transporters depending on the status of the system. The total amount of transporters varies as follows

$$\partial_t \vartheta_0 = R(t, x, \vartheta_0) \ , \tag{2.10}$$

where $R(t,x,\vartheta_0)$ is a function describing the rate of regulation. R is in general composed of a positive production term and a negative degradation term. Negative degradation is often assumed to be proportional to ϑ_0 to prevent negative concentrations. The production term will depend on the local or global solute concentration and regulation is used to control that concentration. A general analysis of robust homeostatic control of a species based on concentrations is presented in [30], and an example for zinc homeostasis in yeast and plant cells is found in [11–13]. We will not consider details of regulation here.

Assume that only free transporters are regulated and by using Eq. (2.7), the following system describing the pointwise dynamics of the free and bound transporters is obtained

$$\begin{aligned} \partial_t \vartheta_f &= R(t,x,\vartheta_f) - (k_1\, \rho_{\mathrm{I}}\, c_{\mathrm{I}} + k_4\, \rho_{\mathrm{II}}\, c_{\mathrm{II}})\, \vartheta_f + (k_2 + k_3)\, \vartheta_b \ , \\ \partial_t \vartheta_b &= (k_1\, \rho_{\mathrm{I}}\, c_{\mathrm{I}} + k_4\, \rho_{\mathrm{II}}\, c_{\mathrm{II}})\, \vartheta_f - (k_2 + k_3)\, \vartheta_b \ , \\ \vartheta_0(t,x) &= \vartheta_f(t,x) + \vartheta_b(t,x) \ . \end{aligned} \tag{2.11}$$

In a general situation several influx and efflux transporters might exist. The fluxes generalize then into

$$\boldsymbol{a}_{i,\mathrm{I}} = \sum_\alpha (k^i_{1,\alpha}\, \rho_{i,\mathrm{I}}\, \vartheta^i_{f,\alpha} - k^i_{2,\alpha}\, \vartheta^i_{b,\alpha})\, \boldsymbol{n} \ , \quad \text{and} \quad \boldsymbol{a}_{i,\mathrm{II}} = \sum_\alpha (k^i_{3,\alpha}\, \vartheta^i_{b,\alpha} - k^i_{4,\alpha}\, \rho_{i,\mathrm{II}}\, \vartheta^i_{f,\alpha})\, \boldsymbol{n} \ , \tag{2.12}$$

where $\rho_i = \rho\, c_i$, with $i = 1, 2$, and all $\vartheta^i_{f,\alpha}$ and $\vartheta^i_{b,\alpha}$ fulfill an equation system equivalent to Eq. (2.11).

2.3. Model

Inserting the diffusion fluxes of Eq. (1.7) into Eqs. (2.3a) and (2.3b), and using $\rho_2 = \rho c$ delivers a system describing species and mass conservation in a compartment Ω with $\varsigma = 0$

$$\begin{aligned} \partial_t(\rho\, c) + \operatorname{div}\left(\rho\, c\, \boldsymbol{v} - \rho\, D\, \nabla c - \rho\, G\, \nabla p\right) &= 0 \qquad \text{in } \Omega \times (0,\infty) \ , \\ \partial_t \rho + \operatorname{div}(\rho\, \boldsymbol{v}) &= 0 \qquad \text{in } \Omega \times (0,\infty) \ . \end{aligned} \tag{2.13}$$

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The gradients in the membrane can be assumed to be constant and can be expressed by a jump across the membrane. Therefore, the diffusion fluxes in the membrane $\overline{\boldsymbol{j}}_2$ and $\overline{\boldsymbol{j}}$ are functions of the jumps in concentration and pressure

$$\overline{\boldsymbol{j}}_2([c],[p]) = -\frac{\rho}{h}\big(D\,[c] + G\,[p]\big)\,\boldsymbol{n}\,, \quad \text{and} \quad \overline{\boldsymbol{j}}([c],[p]) = \frac{\rho}{h}\varsigma\big(\mathcal{D}\,[c] - \mathcal{G}\,[p]\big)\,\boldsymbol{n}\,, \tag{2.14}$$

where the gradients in Eqs. (1.7) were approximated by a jump across the membrane of thickness h. The expressions in Eq. (2.14) are related to the Kedem-Katchalsky equations, [5].

Using Eqs. (1.7) in (2.5a) and (2.5c), the transmission conditions on the interface Γ_m for the solute are obtained

$$\big(\rho\, c\, \boldsymbol{v} - \rho\, D\, \nabla c - \rho\, G\, \nabla p\big)\cdot \boldsymbol{n} = \big(\boldsymbol{a}_{2,\mathrm{I}} + \overline{\boldsymbol{j}}_2([c],[p])\big)\cdot \boldsymbol{n} \qquad \text{on } \Gamma_m \times (0,\infty)\,, \tag{2.15a}$$

$$\big[\rho\, c\, \boldsymbol{v} - \rho\, D\, \nabla c + \rho\, G\, \nabla p\big]\cdot \boldsymbol{n} = [\boldsymbol{a}_2]\cdot \boldsymbol{n} \qquad \text{on } \Gamma_m \times (0,\infty)\,, \tag{2.15b}$$

where $\boldsymbol{a}_2$ and $\overline{\boldsymbol{j}}_2$ are given by Eqs. (2.12) and (2.14), respectivelly.

Water is almost incompressible ($\rho_1 \approx$ const) and compressibility arises here due to the solute influencing the mixture's density $\rho = \rho_1 + \rho_2 = (1-c)\,\rho + c\,\rho$. Using the assumption of a small concentration c, in a first approximation $\rho \approx \rho_1 \approx$ const. Also, barodiffusion in Ω can be assumed to be small compared to the concentration driven flux, i.e. it is assumed to play only a role in transport through the membrane. Viscosity can be assumed to dominate the flow due to the small characteristic scale of a few microns, and the flow in the symplast (cell inside) is assumed to be a Stokes flow. The apoplast (cell wall) is porous suggesting a Darcy flow. The equations describing flow, mass and species conservation are hence

$$\partial_t \boldsymbol{v} - \frac{\eta}{\rho_1}\Delta \boldsymbol{v} + \frac{1}{\rho_1}\nabla p = 0 \qquad \text{in symplast}\,, \tag{2.16a}$$

$$\boldsymbol{v} + K\,\nabla p = 0 \qquad \text{in apoplast}\,, \tag{2.16b}$$

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \text{in symplast and apoplast}\,, \tag{2.16c}$$

$$\partial_t c + \operatorname{div}\left(\boldsymbol{v}\, c - D\,\nabla c\right) = 0 \qquad \text{in symplast and apoplast}\,, \tag{2.16d}$$

where $\eta > 0$ is the dynamic viscosity and $K > 0$ is the permeability of the apoplast.

Diffusive permeation of the solute can be assumed to be small compared against protein mediated transport. We will also assume that only the solute is subjected to transport via proteins. Assuming that the structure in the membrane is oriented normally, the velocity at the interface can be assumed to be perpendicular to it ($\boldsymbol{v}\times\boldsymbol{n} = 0$). Together with incompressibility, this condition has as a consequence that the normal viscous stress $(\boldsymbol{\sigma}'\boldsymbol{n})\cdot\boldsymbol{n}$ is zero on a flat boundary, i.e. the boundary experiences only shear stress. The viscous stress tensor of an incompressible viscous fluid is defined as

$$\boldsymbol{\sigma}' = 2\,\eta\,\mathrm{S}\boldsymbol{v}\,, \tag{2.17}$$

with $\mathrm{S}\boldsymbol{v}$ the symmetric velocity gradient

$$\mathrm{S}\boldsymbol{v} = \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathsf{T}}\right)\,, \tag{2.18}$$

and is related to the full stress tensor $\boldsymbol{\sigma} = -p\,\boldsymbol{I} + \boldsymbol{\sigma}' = -p\,\boldsymbol{I} + 2\eta\,\mathrm{S}\boldsymbol{v}$. Assuming that the effect of corners is small, it is possible to set $(\boldsymbol{\sigma}'\boldsymbol{n})\cdot\boldsymbol{n} = 0$ on $\Gamma_m\times(0,\infty)$. Water fluxes mediated by active water transporters are probably small compared to the fluxes via passive transporters (aquaporins): $\|\boldsymbol{a}_{\mathrm{I}}\| \ll \|\overline{\boldsymbol{j}}\|$. Moreover, by adding $(\boldsymbol{\sigma}'\boldsymbol{n})\cdot\boldsymbol{n} = 0$ only on the Stokes flow side, Eq. (2.5b) becomes

$$\kappa\big((\boldsymbol{\sigma}'\boldsymbol{n})\cdot\boldsymbol{n} + [p]\big) = \delta[c] - \boldsymbol{v}\cdot\boldsymbol{n} \qquad \text{on } \Gamma_m\times(0,\infty)\,, \tag{2.19}$$

where $\kappa := \varsigma\mathcal{G}/h$, $\delta := \varsigma\mathcal{D}/h$. Note that the normal and the jump $[\cdot]$ are oriented from the Stokes to the Darcy side. The physical meaning of this condition becomes clear by setting $\kappa = \delta = 0$

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{for } \kappa = \delta = 0 \qquad \text{on } \Gamma_m\times(0,\infty)\,,$$

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i.e. the net mass flux is zero for zero permeability ($\alpha = 0$) or for a nonreflecting membrane ($\varsigma = 0$). Also, a higher concentration in the cell, $[c] < 0$, means that water is driven into the cell, $\boldsymbol{v} \cdot \boldsymbol{n} < 0$ for $[p] = 0$, while a higher pressure in the cell, $[p] < 0$, results in water flowing out of the cell, $\boldsymbol{v} \cdot \boldsymbol{n} > 0$ for $[c] = 0$. Compare Fig. 1.

Finally, the following conditions are good approximations on the interface Γ_m

$$\rho_1 (c_{\mathrm{l}} \boldsymbol{v}_{\mathrm{l}} - D_2 \nabla c_{\mathrm{l}}) \cdot \boldsymbol{n} \approx \boldsymbol{a}_{2,\mathrm{l}} \cdot \boldsymbol{n} \qquad \text{on } \Gamma_m \times (0,\infty)\,, \tag{2.20a}$$

$$\rho_1 [c\, \boldsymbol{v} - D_2 \nabla c] \cdot \boldsymbol{n} \approx [\boldsymbol{a}_2] \cdot \boldsymbol{n} \qquad \text{on } \Gamma_m \times (0,\infty)\,, \tag{2.20b}$$

$$\kappa\big(2\,\eta\,(\mathrm{S}\boldsymbol{v}\,\boldsymbol{n}) \cdot \boldsymbol{n} + [p]\big) \approx \delta[c] - \boldsymbol{v}_{\mathrm{l}} \cdot \boldsymbol{n} \qquad \text{on } \Gamma_m \times (0,\infty)\,, \tag{2.20c}$$

$$[\boldsymbol{v}] \cdot \boldsymbol{n} \approx 0 \qquad \text{on } \Gamma_m \times (0,\infty)\,, \tag{2.20d}$$

$$\boldsymbol{v} \times \boldsymbol{n} = 0 \qquad \text{on } \Gamma_m \times (0,\infty)\,. \tag{2.20e}$$

3. Mathematical formulation of microscopic model

Let Ω be a cube in $\mathbb{R}^3$ representing a plant tissue, and $\varepsilon > 0$ be a parameter denoting the ratio between the size of a single cell and the size of the considered plant tissue Ω. The microscopic structure of a plant tissue is reflected in the difference between the cell wall $\Omega_a^\varepsilon \subset \Omega$ and the symplast inside the cells $\Omega_z^\varepsilon \subset \Omega$. In the cell wall domain we shall distinguish between the cell wall apoplast Ω_{aw}^ε and parts of the cell wall Ω_{as}^ε occupied by both plasmodesmata, that belong to cell symplast, and cell wall apoplast. This partition is a strong simplification of the true geometry of a plant tissue, however, it accounts for the basic structures. See Ref. [14] for more anatomical details of plant cells and tissues.

We define a unit cell $Y = \overline{Y}_z \cup \overline{Y}_a$, where Y_z is an open domain with a smooth boundary, representing the part occupied by symplast inside a cell, and Y_a is the cell wall, with $\overline{Y}_a = \overline{Y}_{aw} \cup \overline{Y}_{as}$ and Y_{as} is a domain comprising both plasmodesmata and cell wall apoplast. We define also $\overline{Y}_s = \overline{Y}_z \cup \overline{Y}_{as}$. The corresponding boundaries are denoted by $\Gamma_z = \partial Y_{aw} \cap \partial Y_z$, $\Gamma_{aw} = \partial Y_{aw} \cap \partial Y_{as}$, $\Gamma_a = \partial Y_a \setminus \partial Y$, $\Gamma_s = \partial Y_s \setminus \partial Y$, $\Gamma_{as} = \partial Y_{as} \cap \partial Y_z$, $\Gamma_{zs} = \Gamma_z \cup \Gamma_{as} = \partial Y_z$, see Fig. 5.

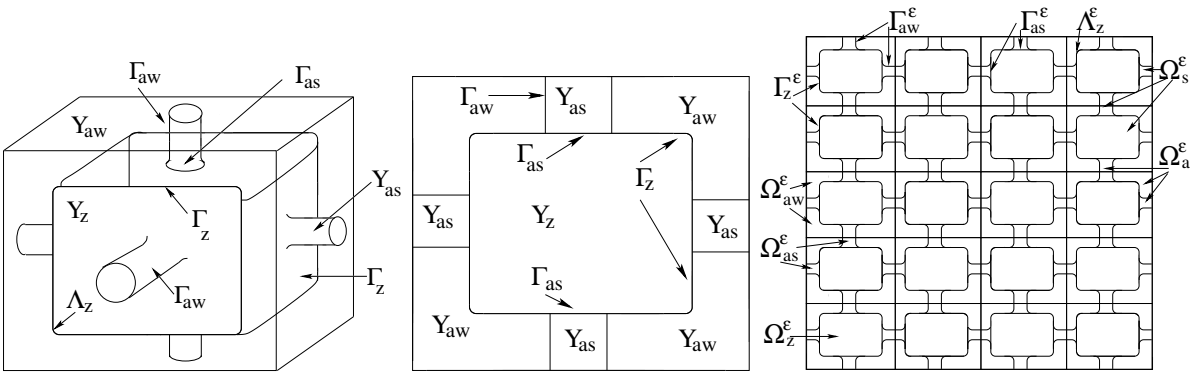


FIGURE 5. Left: unit cell Y. Center: 2D cross-section of a unit cell Y. Right: 2D cross-section of a plant tissue Ω.

Then considering translations of the unit cell given by $Y_j^k = Y_j + k$ for $j = a, z, s, aw$ or as, and $\Gamma_i^k = \Gamma_i + k$ for $i = z, aw, as$ or zs, with $k \in \mathbb{Z}^3$, we can define the domains comprising the microstructure of a plant tissue $\Omega_j^\varepsilon = \cup\{\varepsilon Y_j^k : \varepsilon Y^k \subset \Omega, k \in \mathbb{Z}^3\}$, $j = a, z, s, aw$ or as. The microscopic boundaries are given by $\Gamma_i^\varepsilon = \cup\{\varepsilon \Gamma_i^k : \varepsilon Y^k \subset \Omega, k \in \mathbb{Z}^3\}$, $i = z, aw, as$, or zs. We notice that Ω_s^ε and Ω_a^ε are connected Lipschitz domains. The domain Ω_a^ε represent porous medium of cell walls, whereas Ω_{sp}^ε is introduced

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to depict parts of cell walls, modelled also as a porous medium, comprising both cell wall apoplast and many thin channels of plasmodesmata, Fig. 5.

In order to simplify the formulation of the microscopic model we shall consider in the following the doubled notation for Ω^ε_{as}, i.e. $\Omega^\varepsilon_{ap} := \Omega^\varepsilon_{as}$ and $\Omega^\varepsilon_{sp} := \Omega^\varepsilon_{as}$.

For the velocity of water we shall consider the stationary version of Stokes equation (2.16a) inside the cells and Darcy equation (2.16b) in the cell walls apoplast and parts of cell walls comprising plasmodesmata, and incompressibility (2.16c) of all flows

$$\begin{aligned} -\varepsilon^2\eta\Delta \boldsymbol{v}^\varepsilon_z + \nabla p^\varepsilon_z &= 0 \quad \text{in } \Omega^\varepsilon_z\times(0,T)\,,\\ \operatorname{div} \boldsymbol{v}^\varepsilon_z &= 0 \quad \text{in } \Omega^\varepsilon_z\times(0,T)\,,\\ \boldsymbol{v}^\varepsilon_i + K_{i,\varepsilon}\nabla p^\varepsilon_i &= 0 \quad \text{in } \Omega^\varepsilon_i\times(0,T), \quad i=sp,aw,ap\,,\\ \operatorname{div} \boldsymbol{v}^\varepsilon_i &= 0 \quad \text{in } \Omega^\varepsilon_i\times(0,T), \quad i=sp,aw,ap\,, \end{aligned} \tag{3.1}$$

where $\eta>0$ is the viscosity constant, and the permeability tensors are given by Y-periodic functions K_i, i.e. $K_{i,\varepsilon}(x)=K_i(x/\varepsilon)$ for $x\in\Omega^\varepsilon_i$, with $i=sp,aw,ap$.

The concentration of an osmotically active solute in different partitions of a plant cell is assumed to follow the conservation Eq. (2.16d) with a production/degradation term

$$\partial_t c^\varepsilon_i + \operatorname{div}(H_M(\boldsymbol{v}^\varepsilon_i)c^\varepsilon_i - D^\varepsilon_i(t,x)\nabla c^\varepsilon_i) = F^\varepsilon_i(t,x,c^\varepsilon_i) \qquad \text{in } \Omega^\varepsilon_i\times(0,T), \quad i=z,sp,aw,ap\,, \tag{3.2}$$

where

$$H_M(\boldsymbol{\xi})_j = \begin{cases} \xi_j & \text{for } |\xi_j|\le M,\\ M\operatorname{sgn}(\xi_j) & \text{for } |\xi_j|>M, \end{cases} \qquad j=1,2,3\,,$$

for some $M>0$, and $\boldsymbol{\xi}\in\mathbb{R}^3$. The velocities $\boldsymbol{v}^\varepsilon_i$ need to be cut-off for technical reasons for the rigorous analysis of the macroscopic model. Assuming bounded velocities in tissues is biologically and physically sensible. The diffusion coefficients D^ε_i and the production/degradation terms F^ε_i, with $i=z,sp,aw,ap$, are determined by Y-periodic functions $D_i(t,y)$ and $F_i(t,y,\xi)$, i.e. $D^\varepsilon_i(t,x)=D_i(t,\frac{x}{\varepsilon})$ and $F^\varepsilon_i(t,x,\xi)=F_i(t,\frac{x}{\varepsilon},\xi)$ for $\xi\in\mathbb{R}$ and $(t,x)\in(0,T)\times\Omega^\varepsilon_i$. The symplast and apoplast are coupled by diffusion fluxes and via protein mediated transport of solute through the cell membrane described by the following boundary conditions based on Eqs. (2.20a) and (2.20b)

$$\begin{aligned} (D^\varepsilon_z\nabla c^\varepsilon_z - H_M(\boldsymbol{v}^\varepsilon_z)c^\varepsilon_z)\cdot\boldsymbol{n} &= (D^\varepsilon_{sp}\nabla c^\varepsilon_{sp} - H_M(\boldsymbol{v}^\varepsilon_{sp})c^\varepsilon_{sp})\cdot\boldsymbol{n}\\ &\quad + \varepsilon\beta^\varepsilon_a(t,x)\vartheta^\varepsilon_{b,a} - \varepsilon\alpha^\varepsilon_s(t,x)c^\varepsilon_z\vartheta^\varepsilon_{f,s} \quad \text{on } \Gamma^\varepsilon_{as}\times(0,T)\,,\\ (D^\varepsilon_z\nabla c^\varepsilon_z - H_M(\boldsymbol{v}^\varepsilon_z)c^\varepsilon_z)\cdot\boldsymbol{n} &= \varepsilon\beta^\varepsilon_a(t,x)\vartheta^\varepsilon_{b,a} - \varepsilon\alpha^\varepsilon_s(t,x)\,c^\varepsilon_z\,\vartheta^\varepsilon_{f,s} \quad \text{on } \Gamma^\varepsilon_z\times(0,T)\,,\\ (D^\varepsilon_{aw}\nabla c^\varepsilon_{aw} - H_M(\boldsymbol{v}^\varepsilon_{aw})c^\varepsilon_{aw})\cdot\boldsymbol{n} &= \varepsilon\alpha^\varepsilon_a(t,x)c^\varepsilon_{aw}\vartheta^\varepsilon_{f,a} - \varepsilon\beta^\varepsilon_s(t,x)\vartheta^\varepsilon_{b,s} \quad \text{on } \Gamma^\varepsilon_z\times(0,T)\,,\\ (D^\varepsilon_{ap}\nabla c^\varepsilon_{ap} - H_M(\boldsymbol{v}^\varepsilon_{ap})c^\varepsilon_{ap})\cdot\boldsymbol{n} &= \varepsilon\alpha^\varepsilon_a(t,x)c^\varepsilon_{ap}\vartheta^\varepsilon_{f,a} - \varepsilon\beta^\varepsilon_s(t,x)\vartheta^\varepsilon_{b,s} \quad \text{on } \Gamma^\varepsilon_{as}\times(0,T)\,, \end{aligned} \tag{3.3}$$

where $\boldsymbol{n}$ denotes the outer normal vector to $\partial\Omega^\varepsilon_z$ and the protein mediated flux $\boldsymbol{a}_{2,\mathrm{I}}$ was expressed by relation (2.12) with α^ε_l and β^ε_l, for $l=a,s$, are related to the reaction rate coefficients k_i, for $i=1,2,3$. We define the apoplastic and symplastic concentrations c^ε_a and c^ε_s, respectively, as follows

$$c^\varepsilon_a(t,x) = \begin{cases} c^\varepsilon_{aw}(t,x) & \text{in } \Omega^\varepsilon_{aw}\times(0,T)\,,\\ c^\varepsilon_{ap}(t,x) & \text{in } \Omega^\varepsilon_{as}\times(0,T)\,, \end{cases} \quad \text{and} \quad c^\varepsilon_s(t,x) = \begin{cases} c^\varepsilon_z(t,x) & \text{in } \Omega^\varepsilon_z\times(0,T)\,,\\ c^\varepsilon_{sp}(t,x) & \text{in } \Omega^\varepsilon_{as}\times(0,T)\,. \end{cases} \tag{3.4}$$

The dynamics of the transporter concentrations are modelled by following ordinary differential equations

$$\begin{aligned} \partial_t\vartheta^\varepsilon_{f,l} &= R^\varepsilon_l(t,x,\vartheta^\varepsilon_{f,l}) - \alpha^\varepsilon_l(t,x)c^\varepsilon_l\vartheta^\varepsilon_{f,l} + \beta^\varepsilon_l(t,x)\vartheta^\varepsilon_{b,l} - \gamma^\varepsilon_{f,l}(t,x)\vartheta^\varepsilon_{f,l} \quad \text{on } \Gamma^\varepsilon_{zs}\times(0,T)\,,\\ \partial_t\vartheta^\varepsilon_{b,l} &= \alpha^\varepsilon_l(t,x)c^\varepsilon_l\vartheta^\varepsilon_{f,l} - \beta^\varepsilon_l(t,x)\vartheta^\varepsilon_{b,l} - \gamma^\varepsilon_{b,l}(t,x)\vartheta^\varepsilon_{b,l} \quad \text{on } \Gamma^\varepsilon_{zs}\times(0,T)\,, \end{aligned} \tag{3.5}$$

with $l=a,s$, and $\gamma^\varepsilon_{j,l}$, for $j=f,b$ are rates of decay of corresponding transporter concentration and R^ε_l are production/degradation terms representing the genetic regulation of the transporters. The

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main difference to Eq. (2.11) is the inclusion of the decay terms, which model a possible instability of the transporting proteins. The coefficients and R_l^ε are given by Y-periodic functions R_l, α_l, β_l and $\gamma_{j,l}$, defined on $\Gamma_{zs} \times (0,T)$, i.e. $R_l^\varepsilon(t,x,\xi) = R_l(t,\frac{x}{\varepsilon},\xi)$ and $\alpha_l^\varepsilon(t,x) = \alpha_l(t,\frac{x}{\varepsilon})$, $\beta_l^\varepsilon(t,x) = \beta_l(t,\frac{x}{\varepsilon})$, $\gamma_{j,l}^\varepsilon(t,x) = \gamma_{j,l}(t,\frac{x}{\varepsilon})$, for $\xi \in \mathbb{R}$ and $(x,t) \in \Gamma_{zs}^\varepsilon \times (0,T)$, with $l = a,s$ and $j = f,b$.

In contiguous apoplast and symplast in plasmodesmata populated regions, we pose continuity and zero-flux boundary conditions – based on Eqs. (2.20a) and (2.20b) taking into consideration that no transporters are present ($\boldsymbol{a}_{2,\mathrm{I}} = 0$ and $[\boldsymbol{a}_2] = 0$) – and set the concentration to be continuous

$$\begin{aligned}
(D_{aw}^\varepsilon \nabla c_{aw}^\varepsilon - H_M(\boldsymbol{v}_{aw}^\varepsilon)c_{aw}^\varepsilon)\cdot \boldsymbol{n}_{aw} &= (D_{ap}^\varepsilon \nabla c_{ap}^\varepsilon - H_M(\boldsymbol{v}_{ap}^\varepsilon)c_{ap}^\varepsilon)\cdot \boldsymbol{n}_{aw} && \text{on } \Gamma_{aw}^\varepsilon \times (0,T)\,,\\
(D_{sp}^\varepsilon \nabla c_{sp}^\varepsilon - H_M(\boldsymbol{v}_{sp}^\varepsilon)c_{sp}^\varepsilon)\cdot \boldsymbol{n}_{aw} &= 0 && \text{on } \Gamma_{aw}^\varepsilon \times (0,T)\,,\\
c_{aw}^\varepsilon &= c_{ap}^\varepsilon && \text{on } \Gamma_{aw}^\varepsilon \times (0,T)\,,\\
c_z^\varepsilon &= c_{sp}^\varepsilon && \text{on } \Gamma_{as}^\varepsilon \times (0,T)\,,
\end{aligned} \tag{3.6}$$

where $\boldsymbol{n}_{aw}$ is the outer normal vector to $\partial\Omega_{aw}^\varepsilon$. Initial conditions for solute and transporter concentrations are posed as follows

$$\begin{aligned}
c_l^\varepsilon(0,x) &= c_l^0(x) && \text{in } \Omega_l^\varepsilon\,,\\
\vartheta_{j,l}^\varepsilon(0,x) &= \vartheta_{j,l}^{\varepsilon,0}(x) && \text{on } \Gamma_{zs}^\varepsilon\,,
\end{aligned} \tag{3.7}$$

where $j = f,b$ and $l = a,s$. The initial conditions for transporter concentrations are defined as $\vartheta_{j,l}^{\varepsilon,0}(x) = \vartheta_{j,l}^0(x,x/\varepsilon)$ for $x \in \Gamma_{zs}^\varepsilon$, where $\vartheta_{j,l}^0(x,y) = \vartheta_{1j,l}^0(x)\vartheta_{2j,l}^0(y)$ for $(x,y) \in \Omega \times \Gamma_{zs}$, and $\vartheta_{2j,l}^0$ are Y-periodic, with $j = f,b$ and $l = a,s$.

Following conditions, based on (2.20c), (2.20d) and (2.20e), are considered for the velocity of water on internal boundaries

$$\left.\begin{aligned}
\kappa_1\big(-2\varepsilon^2\eta\,(\mathrm{S}\boldsymbol{v}_z^\varepsilon\boldsymbol{n})\cdot\boldsymbol{n} + p_z^\varepsilon - p_{aw}^\varepsilon\big) &= \varepsilon\delta_1(c_z^\varepsilon - c_{aw}^\varepsilon) + \varepsilon\boldsymbol{v}_{aw}^\varepsilon\cdot\boldsymbol{n}\\
\boldsymbol{v}_z^\varepsilon\cdot\boldsymbol{n} = \boldsymbol{v}_{aw}^\varepsilon\cdot\boldsymbol{n}\,,\qquad \boldsymbol{v}_z^\varepsilon\times\boldsymbol{n} &= 0
\end{aligned}\right\} \quad \text{on } \Gamma_z^\varepsilon \times (0,T)\,,$$

$$\left.\begin{aligned}
-2\varepsilon^2\eta\,(\mathrm{S}\boldsymbol{v}_z^\varepsilon\boldsymbol{n})\cdot\boldsymbol{n} + p_z^\varepsilon &= p_{sp}^\varepsilon\\
\kappa_2\big(-2\varepsilon^2\eta\,(\mathrm{S}\boldsymbol{v}_z^\varepsilon\boldsymbol{n})\cdot\boldsymbol{n} + p_z^\varepsilon - p_{ap}^\varepsilon\big) &= \varepsilon\delta_2(c_z^\varepsilon - c_{ap}^\varepsilon) + \varepsilon\boldsymbol{v}_{ap}^\varepsilon\cdot\boldsymbol{n}\\
\boldsymbol{v}_z^\varepsilon\cdot\boldsymbol{n} = \boldsymbol{v}_{sp}^\varepsilon\cdot\boldsymbol{n} + \boldsymbol{v}_{ap}^\varepsilon\cdot\boldsymbol{n}\,,\qquad \boldsymbol{v}_z^\varepsilon\times\boldsymbol{n} &= 0
\end{aligned}\right\} \quad \text{on } \Gamma_{as}^\varepsilon \times (0,T)\,, \tag{3.8}$$

$$p_{aw}^\varepsilon = p_{ap}^\varepsilon\,,\qquad \boldsymbol{v}_{aw}^\varepsilon\cdot\boldsymbol{n}_{aw} = \boldsymbol{v}_{ap}^\varepsilon\cdot\boldsymbol{n}_{aw}\,,\qquad \boldsymbol{v}_{sp}^\varepsilon\cdot\boldsymbol{n}_{aw} = 0 \qquad \text{on } \Gamma_{aw}^\varepsilon \times (0,T)\,.$$

On the external boundary $\partial\Omega$ we assume zero-flux boundary conditions for concentrations

$$\begin{aligned}
(H_M(\boldsymbol{v}_{aw}^\varepsilon)c_{aw}^\varepsilon - D_{aw}^\varepsilon\nabla c_{aw}^\varepsilon)\cdot\boldsymbol{n}_{ex} &= 0 && \text{on } (\partial\Omega\setminus\partial\Omega_{as}^\varepsilon)\times(0,T)\,,\\
(H_M(\boldsymbol{v}_i^\varepsilon)c_i^\varepsilon - D_i^\varepsilon\nabla c_i^\varepsilon)\cdot\boldsymbol{n}_{ex} &= 0 && \text{on } (\partial\Omega\cap\partial\Omega_{as}^\varepsilon)\times(0,T)\,,
\end{aligned} \tag{3.9}$$

where $i = ap, sp$ and $\boldsymbol{n}_{ex}$ is the outer normal vector to $\partial\Omega$. We further prescribe symplastic and apoplastic normal velocities

$$\begin{aligned}
\boldsymbol{v}_{aw}^\varepsilon\cdot\boldsymbol{n}_{ex} &= v_D && \text{on } (\partial\Omega_{aw}^\varepsilon\cap\partial\Omega)\times(0,T)\,,\\
\boldsymbol{v}_{ap}^\varepsilon\cdot\boldsymbol{n}_{ex} &= v_D && \text{on } (\partial\Omega_{as}^\varepsilon\cap\partial\Omega)\times(0,T)\,,\\
\boldsymbol{v}_{sp}^\varepsilon\cdot\boldsymbol{n}_{ex} &= 0 && \text{on } (\partial\Omega_{as}^\varepsilon\cap\partial\Omega)\times(0,T)\,.
\end{aligned} \tag{3.10}$$

To simplify notations, we define similar to (3.4) apoplastic and symplastic flow variables

$$v_a^\varepsilon(t,x) = \begin{cases} v_{aw}^\varepsilon(t,x) & \text{in } \Omega_{aw}^\varepsilon\times(0,T)\,,\\ v_{ap}^\varepsilon(t,x) & \text{in } \Omega_{as}^\varepsilon\times(0,T)\,,\end{cases} \quad \text{and} \quad v_s^\varepsilon(t,x) = \begin{cases} v_z^\varepsilon(t,x) & \text{in } \Omega_z^\varepsilon\times(0,T)\,,\\ v_{sp}^\varepsilon(t,x) & \text{in } \Omega_{as}^\varepsilon\times(0,T)\,,\end{cases}$$

$$p_a^\varepsilon(t,x) = \begin{cases} p_{aw}^\varepsilon(t,x) & \text{in } \Omega_{aw}^\varepsilon\times(0,T)\,,\\ p_{ap}^\varepsilon(t,x) & \text{in } \Omega_{as}^\varepsilon\times(0,T)\,,\end{cases} \quad \text{and} \quad p_s^\varepsilon(t,x) = \begin{cases} p_z^\varepsilon(t,x) & \text{in } \Omega_z^\varepsilon\times(0,T)\,,\\ p_{sp}^\varepsilon(t,x) & \text{in } \Omega_{as}^\varepsilon\times(0,T)\,.\end{cases}$$

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The corresponding diffusion coefficients and permeability tensors are given by

$$D_a(t,y)=\begin{cases} D_{aw}(t,y) \text{ in } Y_{aw}\times(0,T)\,,\\ D_{ap}(t,y) \text{ in } Y_{as}\times(0,T)\,,\end{cases} \quad \text{and} \quad D_s(t,y)=\begin{cases} D_z(t,y) \text{ in } Y_z\times(0,T)\,,\\ D_{sp}(t,y) \text{ in } Y_{as}\times(0,T)\,,\end{cases}$$

$$K_a(y)=\begin{cases} K_{aw}(y) \text{ in } Y_{aw}\times(0,T)\,,\\ K_{ap}(y) \text{ in } Y_{as}\times(0,T)\,.\end{cases}$$

The production/degradation terms are defined by

$$F_a^\varepsilon(t,x,c_a^\varepsilon)=\begin{cases} F_{aw}^\varepsilon(t,x,c_{aw}^\varepsilon) \text{ in } \Omega_{aw}^\varepsilon\times(0,T)\,,\\ F_{ap}^\varepsilon(t,x,c_{ap}^\varepsilon) \text{ in } \Omega_{as}^\varepsilon\times(0,T)\,,\end{cases} \quad \text{and} \quad F_s^\varepsilon(t,x,c_s^\varepsilon)=\begin{cases} F_z^\varepsilon(t,x,c_z^\varepsilon) \text{ in } \Omega_z^\varepsilon\times(0,T)\,,\\ F_{sp}^\varepsilon(t,x,c_{sp}^\varepsilon) \text{ in } \Omega_{as}^\varepsilon\times(0,T)\,.\end{cases}$$

We assume that $\delta_i\neq 0$ and $\kappa_i\neq 0$, for $i=1,2$, and define $\kappa^\varepsilon(x)=\kappa(x/\varepsilon)$, $\delta^\varepsilon(x)=\delta(x/\varepsilon)$ for $x\in\Gamma_{zs}^\varepsilon$, where κ, δ are Y-periodic functions given by $\kappa(y)=\kappa_1^{-1}$ on Γ_z, $\kappa(y)=\kappa_2^{-1}$ on Γ_{as} and $\delta(y)=\delta_1\kappa_1^{-1}$ on Γ_z, $\delta(y)=\delta_2\kappa_2^{-1}$ on Γ_{as}.

Assumption 1. – Tensors $K_a\in L^\infty(Y_a)^{3\times 3}$, $K_{sp}\in L^\infty(Y_{as})^{3\times 3}$ are symmetric and uniformly elliptic, i.e. $(K_l(y)\xi,\xi)\geq k_l|\xi|^2$ for $k_l>0$, $\xi\in\mathbb{R}^3$, a.a. $y\in Y_l$, where $l=a,sp$ and $Y_{sp}:=Y_{as}$.
– Diffusion coefficients $D_l,\partial_t D_l\in L^\infty((0,T)\times Y_l)^{3\times 3}$ are symmetric and uniformly elliptic, i.e. $(D_l(t,y)\xi,\xi)\geq d_l|\xi|^2$ for $d_l>0$, $\xi\in\mathbb{R}^3$, a.a. $(t,y)\in(0,T)\times Y_l$, where $l=a,s$.
– Production/degradation $F_l:(0,T)\times Y_l\times\mathbb{R}\to\mathbb{R}$ is differentiable in t, measurable in y, F_l and $\partial_t F_l$ are sublinear in ξ, F_l is Lipschitz continuous in ξ uniformly in (t,y), and $F_l(t,y,\xi_-)\xi_-\leq C|\xi_-|^2$ for $(t,y)\in(0,T)\times Y_l$, where $\xi_-=\min\{\xi,0\}$ and $l=a,s$.
– Functions $R_l(\cdot,\cdot,\xi)\in C([0,T];L^\infty(\Gamma_{zs}))$ for all $\xi\in\mathbb{R}$ are uniformly Lipschitz continuous in ξ, and nonnegative for nonnegative ξ and $(x,t)\in\Gamma_{zs}\times[0,T]$, with $l=a,s$.
– Coefficients α_l, β_l, $\gamma_{j,l}\in C([0,T];L^\infty(\Gamma_{zs}))$ are nonnegative and $\partial_t\alpha_l,\partial_t\beta_l\in L^\infty((0,T)\times\Gamma_{zs})$, where $j=f,b$ and $l=a,s$.
– Initial conditions $c_l^0\in H^2(\Omega)$ and $\vartheta_{1j,l}^0\in L^\infty(\Omega)$, $\vartheta_{2j,l}^0\in L^\infty(\Gamma_{zs})$ are nonnegative, $l=a,s$, $j=f,b$.
– Boundary condition $v_D\in H^{1/2}(\partial\Omega)$ is given by $v_D=V_D\cdot\boldsymbol{n}_{ex}$ on $\partial\Omega$, where $V_D\in H^1(\Omega)$ and $\operatorname{div}V_D=0$ in Ω.

By Lemma 4 in [38], there exists a restriction operator $R_{Y_a}\in\mathcal{L}(H^1(Y),H^1_{\Gamma_{zs}}(Y_a))$ with the properties

$$\begin{cases} R_{Y_a}\psi=\psi & \text{in a neighborhood of } \partial Y\,,\\ \psi=0 \text{ on } \Gamma_{zs} & \Longrightarrow\ R_{Y_a}\psi=\psi\,,\\ \operatorname{div}\psi=0 & \Longrightarrow\ \operatorname{div}R_{Y_a}\psi=0\,.\end{cases} \tag{3.11}$$

Here $H^1_{\Gamma_{zs}}(Y_a)=\{\psi\in H^1(Y_a):\ \psi=0 \text{ on } \Gamma_{zs}\}$. For $\psi\in H^1(\Omega)$ we define $\psi_j^\varepsilon(y)=\psi(\varepsilon y)$ for $y\in Y^j=(Y+k^j)$, with $j=1,\dots,J$, where $J\in\mathbb{N}$ such that $\Omega=\cup_{j=1}^J\varepsilon Y^j$, and $k^j\in\mathbb{Z}^3$. Then the operator defined as $\tilde R_{\Omega_a^\varepsilon}^\varepsilon\psi(x)=(R_{Y_a}\psi_j^\varepsilon)(x/\varepsilon)$ for $x\in\varepsilon Y_a^j$, with $j=1,\dots,J$, has the properties that $\tilde R_{\Omega_a^\varepsilon}\in\mathcal{L}(H^1(\Omega),H^1_{\Gamma_{zs}^\varepsilon}(\Omega_a^\varepsilon))$ and

$$\begin{cases} \psi=0 \text{ on } \Gamma_{zs}^\varepsilon & \Longrightarrow\ \tilde R_{\Omega_a^\varepsilon}^\varepsilon\psi=\psi\,,\\ \operatorname{div}\psi=0 & \Longrightarrow\ \operatorname{div}\tilde R_{\Omega_a^\varepsilon}^\varepsilon\psi=0\,,\\ \|\tilde R_{\Omega_a^\varepsilon}^\varepsilon\psi\|_{L^2(\Omega_a^\varepsilon)} & \leq\ C\big(\|\psi\|_{L^2(\Omega)}+\varepsilon\|\nabla\psi\|_{L^2(\Omega)}\big)\,,\\ \varepsilon\|\nabla\tilde R_{\Omega_a^\varepsilon}^\varepsilon\psi\|_{L^2(\Omega_a^\varepsilon)} & \leq\ C\big(\|\psi\|_{L^2(\Omega)}+\varepsilon\|\nabla\psi\|_{L^2(\Omega)}\big)\,,\end{cases} \tag{3.12}$$

where $H^1_{\Gamma_{zs}^\varepsilon}(\Omega_a^\varepsilon)=\{\psi\in H^1(\Omega_a^\varepsilon):\ \psi=0 \text{ on } \Gamma_{zs}^\varepsilon\}$.

We define $V_D^\varepsilon=\tilde R_{\Omega_a^\varepsilon}V_D$ and, using the assumptions on V_D, obtain that $\operatorname{div}V_D^\varepsilon=0$ in Ω_a^ε and $\|V_D^\varepsilon\|_{L^2(\Omega_a^\varepsilon)}\leq C\|V_D\|_{H^1(\Omega)}$.

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For a $\sigma > 0$, a Lipschitz domain Σ, and for $\psi, \varphi \in L^2((0,\sigma) \times \Sigma)$ we shall denote

$$\langle \psi, \varphi \rangle_\Sigma = \int_\Sigma \psi\, \varphi\, dx \qquad \text{and} \qquad \langle \psi, \varphi \rangle_{\Sigma,\sigma} = \int_0^\sigma \int_\Sigma \psi\, \varphi\, dx dt\ .$$

For a bounded Lipschitz domain Σ we shall use the notion of the space

$$H(\mathrm{div}, \Sigma) = \{ \boldsymbol{v} \in L^2(\Sigma) \quad \text{such that} \quad \mathrm{div}\, \boldsymbol{v} \in L^2(\Sigma) \}\ ,$$

provided with the norm

$$\|\boldsymbol{v}\|^2_{H(\mathrm{div},\Sigma)} = \|\boldsymbol{v}\|^2_{L^2(\Sigma)} + \|\,\mathrm{div}\, \boldsymbol{v}\|^2_{L^2(\Sigma)}\ .$$

We introduce also

$$\begin{aligned} V^\varepsilon_{as} &= \{ v \in H(\mathrm{div}, \Omega^\varepsilon_{as}) :\ v \cdot \boldsymbol{n} = 0 \ \text{ on } \partial\Omega^\varepsilon_{as} \cap (\partial\Omega^\varepsilon_{aw} \cup \partial\Omega) \}\ , \\ V^\varepsilon_{a} &= \{ v \in H(\mathrm{div}, \Omega^\varepsilon_{a}) :\ v \cdot \boldsymbol{n} \in L^2(\Gamma^\varepsilon_{zs}),\, v \cdot \boldsymbol{n} = 0 \ \text{ on } \partial\Omega \}\ , \\ V^\varepsilon_{z} &= \{ v \in H^1(\Omega^\varepsilon_{z}) :\ v \times \boldsymbol{n} = 0 \ \text{ on } \Gamma^\varepsilon_{zs} \}\ . \end{aligned}$$

For $\phi \in V^\varepsilon_{as}$, since $\phi \in H(\mathrm{div}, \Omega^\varepsilon_{as})$ we have $\phi \cdot \boldsymbol{n} \in H^{-1/2}(\partial\Omega^\varepsilon_{as})$ and together with $\phi \cdot \boldsymbol{n} = 0$ on $\partial\Omega^\varepsilon_{as} \setminus \Gamma^\varepsilon_{as}$ obtain $\phi \cdot \boldsymbol{n} \in H^{-1/2}(\Gamma^\varepsilon_{as})$, [15]. Considering the geometrical structure of Ω^ε_{as} and the fact that $\phi \cdot \boldsymbol{n} = 0$ on Γ^ε_{aw}, we can extend ϕ by zero from Ω^ε_{as} to Ω^ε_{a} with $\mathrm{div}\, \phi \in L^2(\Omega^\varepsilon_{a})$ and shall use the same notation for the extension as for the original function.

We denote the spaces

$$\begin{aligned} \mathcal{V}^\varepsilon = \{ (v_1, v_2, v_3) \in V^\varepsilon_z \times V^\varepsilon_a \times V^\varepsilon_{as} : \langle (v_1 - v_2) \cdot \boldsymbol{n}, \phi \rangle_{\Gamma^\varepsilon_z} = 0 \ \text{for } \phi \in L^2(\Gamma^\varepsilon_z)\ , \\ \langle (v_1 - v_2 - v_3) \cdot \boldsymbol{n}, \psi \rangle_{H^{-1/2}(\Gamma^\varepsilon_{as}), H^{1/2}(\Gamma^\varepsilon_{as})} = 0 \ \text{ for } \psi \in H^{1/2}(\Gamma^\varepsilon_{as}) \}\ , \\ \mathcal{P}^\varepsilon = \{ (p_1, p_2, p_3) \in L^2(\Omega^\varepsilon_z) \times L^2(\Omega^\varepsilon_a) \times L^2(\Omega^\varepsilon_{as}) : \langle p_1, 1 \rangle_{\Omega^\varepsilon_z} + \langle p_2, 1 \rangle_{\Omega^\varepsilon_a} + \langle p_3, 1 \rangle_{\Omega^\varepsilon_{as}} = 0 \}\ , \end{aligned}$$

with the norms

$$\begin{aligned} \|v\|_{\mathcal{V}^\varepsilon} &= \|v_1\|_{H(\mathrm{div},\Omega^\varepsilon_z)} + \varepsilon \|\nabla v_1\|_{L^2(\Omega^\varepsilon_z)} + \|v_2\|_{H(\mathrm{div},\Omega^\varepsilon_a)} + \varepsilon^{1/2} \|v_2 \cdot \boldsymbol{n}\|_{L^2(\Gamma^\varepsilon_{zs})} + \|v_3\|_{H(\mathrm{div},\Omega^\varepsilon_{as})}\ , \\ \|p\|_{\mathcal{P}^\varepsilon} &= \|p_1\|_{L^2(\Omega^\varepsilon_z)} + \|p_2\|_{L^2(\Omega^\varepsilon_a)} + \|p_3\|_{L^2(\Omega^\varepsilon_{as})}\ . \end{aligned}$$

Notice that for $\psi \in \mathcal{V}^\varepsilon$ due to the assumptions on the normal components at the boundaries we have $\mathrm{div}\, \varphi \in L^2(\Omega)$, where $\varphi = \psi_1$ in Ω^ε_z, $\varphi = \psi_2$ in Ω^ε_{aw} and $\varphi = \psi_2 + \psi_3$ in Ω^ε_{as}. We shall consider also $L^2(0,T;\mathcal{V}^\varepsilon)$ and $L^2(0,T;\mathcal{P}^\varepsilon)$ with the norms

$$\begin{aligned} \|v\|_{L^2(0,T;\mathcal{V}^\varepsilon)} &= \|v_1\|_{L^2(0,T;H(\mathrm{div},\Omega^\varepsilon_z))} + \varepsilon \|\nabla v_1\|_{L^2((0,T)\times\Omega^\varepsilon_z)} + \|v_2\|_{L^2(0,T;H(\mathrm{div},\Omega^\varepsilon_a))} \\ &\quad + \varepsilon^{1/2} \|v_2 \cdot \boldsymbol{n}\|_{L^2((0,T)\times\Gamma^\varepsilon_{zs})} + \|v_3\|_{L^2(0,T;H(\mathrm{div},\Omega^\varepsilon_{as}))}\ , \\ \|p\|_{L^2(0,T;\mathcal{P}^\varepsilon)} &= \|p_1\|_{L^2((0,T)\times\Omega^\varepsilon_z)} + \|p_2\|_{L^2((0,T)\times\Omega^\varepsilon_a)} + \|p_3\|_{L^2((0,T)\times\Omega^\varepsilon_{as})}\ . \end{aligned}$$

The corresponding divergence-free space is denoted by

$$\mathcal{V}^\varepsilon_d = \{ (v_1, v_2, v_3) \in \mathcal{V}^\varepsilon :\ \mathrm{div}\, v_1 = 0 \ \text{ in } \Omega^\varepsilon_z\ ,\ \mathrm{div}\, v_2 = 0 \ \text{ in } \Omega^\varepsilon_a\ ,\ \mathrm{div}\, v_3 = 0 \ \text{ in } \Omega^\varepsilon_{as} \}\ .$$

4. Well-posedness and a priori estimates

We start with a weak formulation of the microscopic model (3.1)-(3.10).

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Definition 4.1. Weak solution of (3.1)-(3.10) are velocity field $(\boldsymbol{v}_z^\varepsilon, \boldsymbol{v}_a^\varepsilon - V_D^\varepsilon, \boldsymbol{v}_{sp}^\varepsilon) \in L^2(0,T;\mathcal{V}^\varepsilon)$, and pressure $p^\varepsilon = (p_z^\varepsilon, p_a^\varepsilon, p_{sp}^\varepsilon) \in L^2(0,T;\mathcal{P}^\varepsilon)$, satisfying equations (3.1) with boundary conditions (3.8) and (3.10) in the variational formulation

$$\begin{aligned}\varepsilon^2\langle 2\eta\,\mathrm{S}\,\boldsymbol{v}_z^\varepsilon, \mathrm{S}\,\psi_1\rangle_{\Omega_z^\varepsilon,T} - \langle p_z^\varepsilon, \operatorname{div}\psi_1\rangle_{\Omega_z^\varepsilon,T} + \langle K_{a,\varepsilon}^{-1}(x)\boldsymbol{v}_a^\varepsilon, \psi_2\rangle_{\Omega_a^\varepsilon,T} - \langle p_a^\varepsilon, \operatorname{div}\psi_2\rangle_{\Omega_a^\varepsilon,T}\\ +\langle K_{sp,\varepsilon}^{-1}(x)\boldsymbol{v}_{sp}^\varepsilon, \psi_3\rangle_{\Omega_{as}^\varepsilon,T} - \langle p_{sp}^\varepsilon, \operatorname{div}\psi_3\rangle_{\Omega_{as}^\varepsilon,T} + \varepsilon\langle \kappa^\varepsilon(x)\boldsymbol{v}_a^\varepsilon\cdot\boldsymbol{n} + \delta^\varepsilon(x)(c_s^\varepsilon - c_a^\varepsilon), \psi_2\cdot\boldsymbol{n}\rangle_{\Gamma_{zs}^\varepsilon,T} = 0\,,\\ \langle \operatorname{div}\boldsymbol{v}_z^\varepsilon, q_1\rangle_{\Omega_z^\varepsilon,T} + \langle \operatorname{div}\boldsymbol{v}_a^\varepsilon, q_2\rangle_{\Omega_a^\varepsilon,T} + \langle \operatorname{div}\boldsymbol{v}_{as}^\varepsilon, q_3\rangle_{\Omega_{as}^\varepsilon,T} = 0\,,\end{aligned} \tag{4.1}$$

for $\psi = (\psi_1,\psi_2,\psi_3) \in L^2(0,T;\mathcal{V}^\varepsilon)$ and $q = (q_1,q_2,q_3) \in L^2(0,T;\mathcal{P}^\varepsilon)$, and functions $c_l^\varepsilon \in L^\infty((0,T)\times\Omega_l^\varepsilon)$, $c_l^\varepsilon \in L^2(0,T;H^1(\Omega_l^\varepsilon))\cap H^1(0,T;L^2(\Omega_l^\varepsilon))$, for $l = a,s$, satisfying equations (3.2) with boundary conditions (3.3), (3.6), and (3.9) in the weak form

$$\begin{aligned}\langle \partial_t c_a^\varepsilon, \varphi_1\rangle_{\Omega_a^\varepsilon,T} + \langle D_a^\varepsilon(t,x)\nabla c_a^\varepsilon - H_M(\boldsymbol{v}_a^\varepsilon)c_a^\varepsilon, \nabla\varphi_1\rangle_{\Omega_a^\varepsilon,T} = \varepsilon\langle \beta_s^\varepsilon(t,x)\vartheta_{b,s}^\varepsilon - \alpha_a^\varepsilon(t,x)c_a^\varepsilon\vartheta_{f,a}^\varepsilon, \varphi_1\rangle_{\Gamma_{zs}^\varepsilon,T}\\ +\langle F_a^\varepsilon(t,x,c_a^\varepsilon), \varphi_1\rangle_{\Omega_a^\varepsilon,T}\,,\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\langle \partial_t c_s^\varepsilon, \varphi_2\rangle_{\Omega_s^\varepsilon,T} + \langle D_s^\varepsilon(t,x)\nabla c_s^\varepsilon - H_M(\boldsymbol{v}_s^\varepsilon)c_s^\varepsilon, \nabla\varphi_2\rangle_{\Omega_s^\varepsilon,T} = \varepsilon\langle \beta_a^\varepsilon(t,x)\vartheta_{b,a}^\varepsilon - \alpha_s^\varepsilon(t,x)c_s^\varepsilon\vartheta_{f,s}^\varepsilon, \varphi_2\rangle_{\Gamma_{zs}^\varepsilon,T}\\ +\langle F_s^\varepsilon(t,x,c_s^\varepsilon), \varphi_2\rangle_{\Omega_s^\varepsilon,T}\,,\end{aligned} \tag{4.3}$$

for all $\varphi_1 \in L^2(0,T;H^1(\Omega_a^\varepsilon))$, $\varphi_2 \in L^2(0,T;H^1(\Omega_s^\varepsilon))$, with $c_l^\varepsilon(0,x) = c_l^0(x)$ a.e. in Ω_l^ε, and transporter concentrations $\vartheta_{j,l}^\varepsilon \in W^{1,\infty}(0,T;L^\infty(\Gamma_{zs}^\varepsilon))$, with $j = f,b$ and $l = a,s$, satisfying ordinary differential equations (3.5) a.e. on $\Gamma_{zs}^\varepsilon\times(0,T)$ together with initial conditions (3.7) a.e. on Γ_{zs}^ε.

In following we shall use the notation $\bar V_D^\varepsilon = (0, V_D^\varepsilon, 0)$ and $(\boldsymbol{v}^\varepsilon - \bar V_D^\varepsilon) = (\boldsymbol{v}_z^\varepsilon, \boldsymbol{v}_a^\varepsilon - V_D^\varepsilon, \boldsymbol{v}_{sp}^\varepsilon)$.

4.1. Existence and estimates for $(\boldsymbol{v}_s^\varepsilon, p_s^\varepsilon)$ and $(\boldsymbol{v}_a^\varepsilon, p_a^\varepsilon)$

First we shall prove Korn's type inequality satisfied by functions from the space

$$\begin{aligned}\tilde{\mathcal{V}}^\varepsilon = \{(v_1,v_2,v_3) \in H^1(\Omega_z^\varepsilon)\times V_a^\varepsilon\times V_{as}^\varepsilon : \langle (v_1 - v_2)\cdot\boldsymbol{n}, \phi\rangle_{\Gamma_z^\varepsilon} = 0 \text{ for } \phi \in L^2(\Gamma_z^\varepsilon)\,,\\ \langle (v_1 - v_2 - v_3)\cdot\boldsymbol{n}, \psi\rangle_{H^{-1/2}(\Gamma_{as}^\varepsilon), H^{1/2}(\Gamma_{as}^\varepsilon)} = 0 \ \text{ for } \psi \in H^{1/2}(\Gamma_{as}^\varepsilon)\}\,.\end{aligned}$$

Lemma 4.2. *For $\boldsymbol{\psi} \in \tilde{\mathcal{V}}^\varepsilon$ we have the following Korn's type inequality*

$$\begin{aligned}\|\boldsymbol{\psi}_1\|_{L^2(\Omega_z^\varepsilon)} + \varepsilon\|\nabla\boldsymbol{\psi}_1\|_{L^2(\Omega_z^\varepsilon)} \le C\big(\varepsilon\|\,\mathrm{S}\,\boldsymbol{\psi}_1\|_{L^2(\Omega_z^\varepsilon)} + \varepsilon^{1/2}\|\boldsymbol{\psi}_1\times\boldsymbol{n}\|_{L^2(\Gamma_{zs}^\varepsilon)} + \|\boldsymbol{\psi}_2\|_{L^2(\Omega_a^\varepsilon)}\\ +\|\boldsymbol{\psi}_3\|_{L^2(\Omega_{as}^\varepsilon)} + \varepsilon\|\operatorname{div}\boldsymbol{\psi}_1\|_{L^2(\Omega_z^\varepsilon)} + \varepsilon\|\operatorname{div}\boldsymbol{\psi}_2\|_{L^2(\Omega_a^\varepsilon)} + \varepsilon\|\operatorname{div}\boldsymbol{\psi}_3\|_{L^2(\Omega_{as}^\varepsilon)}\big)\,.\end{aligned} \tag{4.4}$$

Proof. The proof follows the same lines as in [4]. First we show the estimate for $\hat{\boldsymbol{\psi}} \in \mathcal{V}(Y)$, where $\mathcal{V}(Y) = \{\boldsymbol{\psi}_1 \in H^1(Y_z)$, $\boldsymbol{\psi}_2 \in \mathcal{V}_a(Y), \boldsymbol{\psi}_3 \in \mathcal{V}_{as}(Y), \langle(\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2)\cdot\boldsymbol{n}, \phi_1\rangle_{\Gamma_z} = 0$ for $\phi_1 \in L^2(\Gamma_z)$, and $\langle(\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2 - \boldsymbol{\psi}_3)\cdot\boldsymbol{n}, \phi_2\rangle_{H^{-1/2},H^{1/2}} = 0$ for $\phi_2 \in H^{1/2}(\Gamma_{as})\}$, with $\mathcal{V}_a(Y) = \{v \in H(\operatorname{div}, Y_a), v\cdot\boldsymbol{n} \in L^2(\Gamma_a)\}$ and $\mathcal{V}_{as}(Y) = \{v \in H(\operatorname{div}, Y_{as}), v\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_{aw}\}$. Then scaling argument will imply inequality (4.4) for $\boldsymbol{\psi} \in \tilde{\mathcal{V}}^\varepsilon$. Suppose it is not true that there exists a constant $\tilde C$ such that for $\hat{\boldsymbol{\psi}} \in \mathcal{V}(Y)$

$$\begin{aligned}\|\hat{\boldsymbol{\psi}}_1\|_{L^2(Y_z)} + \|\nabla\hat{\boldsymbol{\psi}}_1\|_{L^2(Y_z)} \le \tilde C\Big(\|\,\mathrm{S}\,\hat{\boldsymbol{\psi}}_1\|_{L^2(Y_z)} + \|\hat{\boldsymbol{\psi}}_1\times\boldsymbol{n}\|_{L^2(\Gamma_{zs})} + \|\hat{\boldsymbol{\psi}}_2\|_{L^2(Y_a)} + \|\hat{\boldsymbol{\psi}}_3\|_{L^2(Y_{as})}\\ +\|\operatorname{div}\hat{\boldsymbol{\psi}}_1\|_{L^2(Y_z)} + \|\operatorname{div}\hat{\boldsymbol{\psi}}_2\|_{L^2(Y_a)} + \|\operatorname{div}\hat{\boldsymbol{\psi}}_3\|_{L^2(Y_{as})}\Big)\,.\end{aligned} \tag{4.5}$$

Then there exists a sequence $\{\hat{\boldsymbol{\psi}}^m\} \subset \mathcal{V}(Y)$ such that

$$\|\hat{\boldsymbol{\psi}}_1^m\|_{L^2(Y_z)} + \|\nabla\hat{\boldsymbol{\psi}}_1^m\|_{L^2(Y_z)} = 1 \tag{4.6}$$

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and

$$\begin{aligned}\|\,\mathrm{S}\,\hat{\boldsymbol{\psi}}_1^{\,m}\|_{L^2(Y_z)} + \|\hat{\boldsymbol{\psi}}_1^{\,m}\times\boldsymbol{n}\|_{L^2(\Gamma_{zs})} + \|\hat{\boldsymbol{\psi}}_2^{\,m}\|_{L^2(Y_a)} + \|\hat{\boldsymbol{\psi}}_3^{\,m}\|_{L^2(Y_{as})}\\ +\|\,\mathrm{div}\,\hat{\boldsymbol{\psi}}_1^{\,m}\|_{L^2(Y_z)} + \|\,\mathrm{div}\,\hat{\boldsymbol{\psi}}_2^{\,m}\|_{L^2(Y_a)} + \|\,\mathrm{div}\,\hat{\boldsymbol{\psi}}_3^{\,m}\|_{L^2(Y_{as})} \le \frac{1}{m}\,.\end{aligned} \tag{4.7}$$

The last inequality implies that

$$\hat{\boldsymbol{\psi}}_2^{\,m}\to 0 \quad \text{in } H(\mathrm{div},Y_a) \qquad \text{and} \qquad \hat{\boldsymbol{\psi}}_3^{\,m}\to 0 \quad \text{in } H(\mathrm{div},Y_{as})\,,$$

whereas, due to (4.6), there exists $\hat{\boldsymbol{\psi}}_1\in H^1(Y_z)$ such that

$$\hat{\boldsymbol{\psi}}_1^{\,m}\to\hat{\boldsymbol{\psi}}_1 \quad \text{weakly in } H^1(Y_z)\,.$$

We shall denote by $\hat{\boldsymbol{\psi}}$ the extension of $\hat{\boldsymbol{\psi}}_1$ by zero into Y. Since $\hat{\boldsymbol{\psi}}^{\,m}=\hat{\boldsymbol{\psi}}_1^{\,m}\chi_{Y_z}+\hat{\boldsymbol{\psi}}_2^{\,m}\chi_{Y_a}+\hat{\boldsymbol{\psi}}_3^{\,m}\chi_{Y_{as}}$ is bounded in $H(\mathrm{div},Y)$, it converges weakly and we conclude that

$$\hat{\boldsymbol{\psi}}^{\,m}\to\hat{\boldsymbol{\psi}} \quad \text{weakly in } H(\mathrm{div},Y)\,. \tag{4.8}$$

Using the estimate for the norm in the space $H^{-1/2}(\Gamma_{zs})$, i.e.

$$\|\hat{\boldsymbol{\psi}}_2^{\,m}\cdot\boldsymbol{n}\|_{H^{-1/2}(\Gamma_{zs})}\le C\|\hat{\boldsymbol{\psi}}_2^{\,m}\|_{H(\mathrm{div},Y_a)}\to 0\,, \tag{4.9}$$

and the boundary condition $\hat{\boldsymbol{\psi}}_1^{\,m}\cdot\boldsymbol{n}=\hat{\boldsymbol{\psi}}_2^{\,m}\cdot\boldsymbol{n}$ a.e. on Γ_z, we obtain $\hat{\boldsymbol{\psi}}_1\cdot\boldsymbol{n}=\hat{\boldsymbol{\psi}}_2\cdot\boldsymbol{n}=0$ a.e. on Γ_z. Additionally we have that

$$\|\hat{\boldsymbol{\psi}}_1^{\,m}\times\boldsymbol{n}\|_{L^2(\Gamma_{zs})}\to 0 \quad \text{as } m\to\infty, \quad \text{and therefore} \quad \hat{\boldsymbol{\psi}}_1\times\boldsymbol{n}=0 \text{ a.e. on } \Gamma_{zs}\,.$$

Now the classical Korn inequality in Y_z with $\hat{\boldsymbol{\psi}}_1=0$ on $\Gamma_z\subset\partial Y_z$ can be applied and we obtain

$$\|\hat{\boldsymbol{\psi}}_1\|_{L^2(Y_z)}+\|\nabla\hat{\boldsymbol{\psi}}_1\|_{L^2(Y_z)}\le\hat{C}\|\,\mathrm{S}_y\,\hat{\boldsymbol{\psi}}_1\|_{L^2(Y_z)}\,. \tag{4.10}$$

Considering (4.6) and (4.7), we obtain that the left-hand side in (4.10) is equal to one, whereas the right-hand side is zero. This yields the contradiction to the assumption that there no such constant $\tilde{C}$ for which (4.5) hold true.
Due to the geometrical assumption on Ω we can write $\Omega=\cup_{j=1}^{J}\varepsilon(Y+k^j)$ with some $J\in\mathbb{N}$ and $k^j\in\mathbb{Z}^3$. We consider now $\boldsymbol{\psi}\in\mathcal{V}^\varepsilon$ and for $y\in Y$ define $\hat{\boldsymbol{\psi}}^j(y)=\boldsymbol{\psi}(\varepsilon y+\varepsilon k^j)$, and obtain $\hat{\boldsymbol{\psi}}^{\,j}\in\mathcal{V}(Y)$. Applying (4.5) for each $Y_j=(Y+k^j)$ we obtain estimate for $\hat{\boldsymbol{\psi}}^{\,j}$

$$\begin{aligned}\|\hat{\boldsymbol{\psi}}_1^{\,j}\|^2_{L^2(Y_z^j)}+\|\nabla\hat{\boldsymbol{\psi}}_1^{\,j}\|^2_{L^2(Y_z^j)}\le\tilde{C}\big(\|\,\mathrm{S}\,\hat{\boldsymbol{\psi}}_1^{\,j}\|^2_{L^2(Y_z^j)}+\|\hat{\boldsymbol{\psi}}_1^{\,j}\times\boldsymbol{n}\|^2_{L^2(\Gamma_{zs}^j)}+\|\hat{\boldsymbol{\psi}}_2^{\,j}\|^2_{L^2(Y_a^j)}+\|\hat{\boldsymbol{\psi}}_3^{\,j}\|^2_{L^2(Y_{as}^j)}\\ +\|\,\mathrm{div}\,\hat{\boldsymbol{\psi}}_1^{\,j}\|^2_{L^2(Y_z^j)}+\|\,\mathrm{div}\,\hat{\boldsymbol{\psi}}_2^{\,j}\|^2_{L^2(Y_a^j)}+\|\,\mathrm{div}\,\hat{\boldsymbol{\psi}}_3^{\,j}\|^2_{L^2(Y_{as}^j)}\big)\,.\end{aligned} \tag{4.11}$$

Summation over $j=1,\dots,J$ and change of variables $x=\varepsilon(y+k^j)$ for $y\in Y$ in (4.11) yield (4.4). □

For the proof of existence and uniqueness of $\boldsymbol{v}^\varepsilon\in L^2(0,T;\mathcal{V}^\varepsilon)$ and $p^\varepsilon\in L^2(0,T;\mathcal{P}^\varepsilon)$ we shall define two bilinear forms $a^\varepsilon(\cdot,\cdot):L^2(0,T;\mathcal{V}^\varepsilon)\times L^2(0,T;\mathcal{V}^\varepsilon)\to\mathbb{R}$ and $b(\cdot,\cdot):L^2(0,T;\mathcal{V}^\varepsilon)\times L^2(0,T;\mathcal{P}^\varepsilon)\to\mathbb{R}$:

$$\begin{aligned}a^\varepsilon(\varphi,\psi)&=\varepsilon^2\langle 2\eta\,\mathrm{S}\,\varphi_1,\mathrm{S}\,\psi_1\rangle_{\Omega_z^\varepsilon,T}+\langle K_{a,\varepsilon}^{-1}\varphi_2,\psi_2\rangle_{\Omega_a^\varepsilon,T}+\langle K_{sp,\varepsilon}^{-1}\varphi_3,\psi_3\rangle_{\Omega_{as}^\varepsilon,T}+\varepsilon\langle\kappa^\varepsilon\,\varphi_2\cdot\boldsymbol{n},\psi_2\cdot\boldsymbol{n}\rangle_{\Gamma_{zs}^\varepsilon,T}\,,\\ b(\psi,q)&=\langle\mathrm{div}\,\psi_1,q_1\rangle_{\Omega_z^\varepsilon,T}+\langle\mathrm{div}\,\psi_2,q_2\rangle_{\Omega_a^\varepsilon,T}+\langle\mathrm{div}\,\psi_3,q_3\rangle_{\Omega_{as}^\varepsilon,T}\,,\end{aligned}$$

for φ, $\psi\in L^2(0,T;\mathcal{V}^\varepsilon)$ and $q\in L^2(0,T;\mathcal{P}^\varepsilon)$.

For $c_s^\varepsilon,c_a^\varepsilon\in L^2((0,T)\times\Gamma_{zs}^\varepsilon)$ we define a linear form $f^\varepsilon(\cdot):L^2(0,T;\mathcal{V}^\varepsilon)\to\mathbb{R}$

$$f^\varepsilon(\psi)=\varepsilon\langle\delta^\varepsilon(c_s^\varepsilon-c_a^\varepsilon),\psi_2\cdot\boldsymbol{n}\rangle_{\Gamma_{zs}^\varepsilon,T} \quad \text{for } \psi\in L^2(0,T;\mathcal{V}^\varepsilon)\,.$$

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Theorem 4.3. *For $c_a^\varepsilon, c_s^\varepsilon \in L^2((0,T)\times\Gamma_{zs}^\varepsilon)$ and K_a, K_{sp}, v_D satisfying Assumption 1 and for any $\varepsilon>0$ there exists a unique solution $(\boldsymbol{v}^\varepsilon - \bar{V}_D^\varepsilon) \in L^2(0,T;\mathcal{V}^\varepsilon)$, $p^\varepsilon \in L^2(0,T;\mathcal{P}^\varepsilon)$ of (3.1) with transmission conditions (3.8) and boundary conditions (3.10) satisfying*

$$\begin{aligned}&\varepsilon||\nabla \boldsymbol{v}_z^\varepsilon||_{L^2((0,T)\times\Omega_z^\varepsilon)} + ||\boldsymbol{v}_s^\varepsilon||_{L^2((0,T)\times\Omega_s^\varepsilon)} + ||\boldsymbol{v}_a^\varepsilon||_{L^2((0,T)\times\Omega_a^\varepsilon)} + \varepsilon^{\frac12}||\boldsymbol{v}_a^\varepsilon\cdot\boldsymbol{n}||_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} \\ &+||p_s^\varepsilon||_{L^2((0,T)\times\Omega_s^\varepsilon)} + ||p_a^\varepsilon||_{L^2((0,T)\times\Omega_a^\varepsilon)} \le C\varepsilon^{\frac12}(\|c_a^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} + \|c_s^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)}) + C\|V_D\|_{H^1(\Omega)}\,,\end{aligned} \tag{4.12}$$

and there exist extensions P_s^ε of p_s^ε from Ω_s^ε into Ω and P_a^ε of p_a^ε from Ω_a^ε into Ω, satisfying

$$||P_s^\varepsilon||_{L^2(0,T;L^2(\Omega)/\mathbb{R})} + ||P_a^\varepsilon||_{L^2(0,T;L^2(\Omega)/\mathbb{R})} \le C\varepsilon^{\frac12}(\|c_a^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} + \|c_s^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)}) + C\|V_D\|_{H^1(\Omega)}\,, \tag{4.13}$$

where a universal constant C is independent of ε and $\Omega_T = \Omega\times(0,T)$.
Additionally for $c_{a,i}^\varepsilon, c_{s,i}^\varepsilon \in L^2((0,T)\times\Gamma_{zs}^\varepsilon)$, with $i=1,2$, holds

$$\begin{aligned}&||\boldsymbol{v}_{s,1}^\varepsilon - \boldsymbol{v}_{s,2}^\varepsilon||_{L^2((0,T)\times\Omega_s^\varepsilon)} + ||\boldsymbol{v}_{a,1}^\varepsilon - \boldsymbol{v}_{a,2}^\varepsilon||_{L^2((0,T)\times\Omega_a^\varepsilon)} \\ &\qquad\le C(\varepsilon^{\frac12}\|c_{a,1}^\varepsilon - c_{a,2}^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} + \varepsilon^{\frac12}\|c_{s,1}^\varepsilon - c_{s,2}^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)})\,.\end{aligned} \tag{4.14}$$

Proof. We can reformulate Stokes-Darcy problem (4.1) as

$$\begin{cases} a^\varepsilon(\boldsymbol{v}^\varepsilon - \bar{V}_D^\varepsilon, \psi) + b(\psi, p^\varepsilon) = -a^\varepsilon(\bar{V}_D^\varepsilon,\psi) - f^\varepsilon(\psi) & \text{for } \psi\in L^2(0,T;\mathcal{V}^\varepsilon)\,,\\ b(\boldsymbol{v}^\varepsilon, q) = 0 & \text{for } q\in L^2(0,T;\mathcal{P}^\varepsilon)\,,\end{cases} \tag{4.15}$$

and apply the abstract theory of mixed problems, [17], to show the existence of a unique solution of (4.15). Considering $\psi\in L^2(0,T;\mathcal{V}_d^\varepsilon)$, using $\psi_1\times\boldsymbol{n}=0$ on $(0,T)\times\Gamma_{zs}^\varepsilon$ and applying inequality (4.4), we obtain

$$\begin{aligned}a^\varepsilon(\psi,\psi) &\ge C\big(\varepsilon^2\|\operatorname{S}\psi_1\|^2_{L^2((0,T)\times\Omega_z^\varepsilon)} + \|\psi_2\|^2_{L^2((0,T)\times\Omega_a^\varepsilon)} + \|\psi_3\|^2_{L^2((0,T)\times\Omega_{as}^\varepsilon)} + \varepsilon\|\psi_2\cdot\boldsymbol{n}\|^2_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)}\big)\\ &\ge C\|\psi\|^2_{L^2(0,T;\mathcal{V}^\varepsilon)}\,,\end{aligned}$$

and conclude that $a^\varepsilon(\cdot,\cdot)$ is $L^2(0,T;\mathcal{V}_d^\varepsilon)$-elliptic.
The bilinear forms $a^\varepsilon(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous with constants independent of ε, i.e. for $\psi,\varphi\in L^2(0,T;\mathcal{V}^\varepsilon)$ and $q\in L^2(0,T;\mathcal{P}^\varepsilon)$, applying Hölder's inequality, we have

$$\begin{aligned}|a^\varepsilon(\psi,\varphi)| &\le C\big(\varepsilon^2\|\operatorname{S}\psi_1\|_{L^2((0,T)\times\Omega_z^\varepsilon)}\|\operatorname{S}\varphi_1\|_{L^2((0,T)\times\Omega_z^\varepsilon)} + \|\psi_2\|_{L^2((0,T)\times\Omega_a^\varepsilon)}\|\varphi_2\|_{L^2((0,T)\times\Omega_a^\varepsilon)}\\ &\quad + \|\psi_3\|_{L^2((0,T)\times\Omega_{as}^\varepsilon)}\|\varphi_3\|_{L^2((0,T)\times\Omega_{as}^\varepsilon)} + \varepsilon\|\psi_2\cdot\boldsymbol{n}\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)}\|\varphi_2\cdot\boldsymbol{n}\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)}\big)\\ &\le C\|\psi\|_{L^2(0,T;\mathcal{V}^\varepsilon)}\|\varphi\|_{L^2(0,T;\mathcal{V}^\varepsilon)}\,,\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}|b(\psi,q)| &\le (\|\operatorname{div}\psi_1\|_{L^2((0,T)\times\Omega_z^\varepsilon)} + \|\operatorname{div}\psi_2\|_{L^2((0,T)\times\Omega_a^\varepsilon)} + \|\operatorname{div}\psi_3\|_{L^2((0,T)\times\Omega_{as}^\varepsilon)})\times\\ &\quad\times(\|q_1\|_{L^2((0,T)\times\Omega_z^\varepsilon)} + \|q_2\|_{L^2((0,T)\times\Omega_a^\varepsilon)} + \|q_3\|_{L^2((0,T)\times\Omega_{as}^\varepsilon)}) \le \|\psi\|_{L^2(0,T;\mathcal{V}^\varepsilon)}\|q\|_{L^2(0,T;\mathcal{P}^\varepsilon)}\,.\end{aligned}$$

Now we shall prove that $b(\cdot,\cdot)$ satisfies the inf-sup condition. For any $q\in L^2(0,T;\mathcal{P}^\varepsilon)\setminus\{0\}$ we shall construct $\psi\in L^2(0,T;\mathcal{V}^\varepsilon)\setminus\{0\}$ such that

$$b(\psi,q)\ge C\|\psi\|_{L^2(0,T;\mathcal{V}^\varepsilon)}\|q\|_{L^2(0,T;\mathcal{P}^\varepsilon)}\,.$$

For given $q=(q_1,q_2,q_3)\in L^2(0,T;\mathcal{P}^\varepsilon)$ we define $\tilde{q}_1\in L^2\big((0,T)\times\Omega\big)$ and $\tilde{q}_2\in L^2\big((0,T)\times\Omega_s^\varepsilon\big)$ as

$$\tilde{q}_1 = q_2 \ \text{ in } (0,T)\times\Omega_a^\varepsilon\,,\ \ \tilde{q}_1 = q_1 + \frac{1}{|\Omega_z^\varepsilon|}\langle q_3,1\rangle_{\Omega_{as}^\varepsilon} \ \text{ in } (0,T)\times\Omega_z^\varepsilon\,,$$

$$\tilde{q}_2 = q_3 \ \text{ in } (0,T)\times\Omega_{as}^\varepsilon\,,\ \ \tilde{q}_2 = -\frac{1}{|\Omega_z^\varepsilon|}\langle q_3,1\rangle_{\Omega_{as}^\varepsilon} \qquad \text{in } (0,T)\times\Omega_z^\varepsilon\,.$$

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Then, due to $\langle \tilde{q}_1, 1\rangle_\Omega = 0$ and $\langle \tilde{q}_2, 1\rangle_{\Omega_s^\varepsilon} = 0$, there exist $\tilde{\psi}_1 \in L^2(0,T;H_0^1(\Omega))$ and $\tilde{\psi}_2 \in L^2(0,T;H^1(\Omega_s^\varepsilon))$, see [17, Corollary 2.4, Lemma 2.2] or [39, Lemma 2.4], such that

$$\begin{aligned} \operatorname{div}\tilde{\psi}_1 &= \tilde{q}_1 \quad \text{in } \Omega\times(0,T)\,, & \tilde{\psi}_1 &= 0 \quad \text{on } \partial\Omega\times(0,T)\,,\\ \operatorname{div}\tilde{\psi}_2 &= \tilde{q}_2 \quad \text{in } \Omega_s^\varepsilon\times(0,T)\,, & \tilde{\psi}_2\cdot\boldsymbol{n} &= 0 \quad \text{on } \partial\Omega_s^\varepsilon\times(0,T)\,,\\ & & \tilde{\psi}_2\times\boldsymbol{n} &= -\tilde{\psi}_1\times\boldsymbol{n} \quad \text{on } \Gamma_{zs}^\varepsilon\times(0,T)\,, \end{aligned}$$

and satisfy estimates

$$\begin{aligned} \|\tilde{\psi}_1\|_{L^2(0,T;H^1(\Omega))} &\le C\|\tilde{q}_1\|_{L^2((0,T)\times\Omega)} \le C(\|q_1\|_{L^2((0,T)\times\Omega_z^\varepsilon)} + \|q_2\|_{L^2((0,T)\times\Omega_a^\varepsilon)} + \|q_3\|_{L^2((0,T)\times\Omega_{as}^\varepsilon)})\,,\\ \|\tilde{\psi}_2\|_{L^2(0,T;H^1(\Omega_s^\varepsilon))} &\le C(\varepsilon)\|\tilde{q}_2\|_{L^2((0,T)\times\Omega_s^\varepsilon)} \le C(\varepsilon)\|q_3\|_{L^2((0,T)\times\Omega_{as}^\varepsilon)}\,. \end{aligned}$$

Notice that $\tilde{\psi}_1 \in L^2(0,T;H_0^1(\Omega))$ ensures $\tilde{\psi}_1\times\boldsymbol{n} \in L^2(0,T;H^{1/2}(\Gamma_{zs}^\varepsilon))$ and, therefore, also the existence of $\tilde{\psi}_2$ with $\tilde{\psi}_2\times\boldsymbol{n} = -\tilde{\psi}_1\times\boldsymbol{n}$ on $\Gamma_{zs}^\varepsilon\times(0,T)$, whereas $t\in(0,T)$ plays the role of a parameter.

We define $\psi = (\psi_1,\psi_2,\psi_3)$ by $\psi_1 = (\tilde{\psi}_1+\tilde{\psi}_2)|_{\Omega_z^\varepsilon}$, $\psi_2 = \tilde{\psi}_1|_{\Omega_a^\varepsilon}$, $\psi_3 = \tilde{\psi}_2|_{\Omega_{as}^\varepsilon}$. The definitions of $\tilde{\psi}_1$ and $\tilde{\psi}_2$ imply that $\psi_2\cdot\boldsymbol{n} = 0$ on $\partial\Omega\times(0,T)$ and $\psi_3\cdot\boldsymbol{n} = 0$ on $(\partial\Omega_s^\varepsilon\cap\partial\Omega)\times(0,T)$. The properties $\tilde{\psi}_1 \in L^2(0,T;H^1(\Omega))$ and $\tilde{\psi}_2\cdot\boldsymbol{n} = 0$ on $\partial\Omega_s^\varepsilon\times(0,T)$ and $\tilde{\psi}_2\times\boldsymbol{n} = -\tilde{\psi}_1\times\boldsymbol{n}$ on $\Gamma_{zs}^\varepsilon\times(0,T)$ ensure $\psi_1\cdot\boldsymbol{n} = \psi_2\cdot\boldsymbol{n}$ on $\Gamma_z^\varepsilon\times(0,T)$ and $\psi_1\times\boldsymbol{n} = 0$ on $\Gamma_{zs}^\varepsilon\times(0,T)$, as well as $\psi_1\cdot\boldsymbol{n} = (\psi_2+\psi_3)\cdot\boldsymbol{n}$ on $\Gamma_{as}^\varepsilon\times(0,T)$ and $\psi_3\cdot\boldsymbol{n} = 0$ on $\Gamma_{aw}^\varepsilon\times(0,T)$. The regularity of $\tilde{\psi}_1$ and $\tilde{\psi}_2$ implies also that $\psi_1 \in L^2(0,T;H^1(\Omega_z^\varepsilon))$, $\psi_2 \in L^2(0,T;H(\operatorname{div},\Omega_a^\varepsilon))$ and $\psi_3 \in L^2(0,T;H(\operatorname{div},\Omega_{as}^\varepsilon))$. Thus $\psi = (\psi_1,\psi_2,\psi_2) \in W^\varepsilon\cap L^2(0,T;\mathcal{V}^\varepsilon)$ with $W^\varepsilon = L^2(0,T;H^1(\Omega_z^\varepsilon))\times L^2(0,T;H^1(\Omega_a^\varepsilon))\times L^2(0,T;H^1(\Omega_{as}^\varepsilon))$ and $W^\varepsilon\cap L^2(0,T;\mathcal{V}^\varepsilon)$ is continuously embedded in $L^2(0,T;\mathcal{V}^\varepsilon)$ and for $\psi\in W^\varepsilon\cap L^2(0,T;\mathcal{V}^\varepsilon)$ we have

$$\|\psi\|_{L^2(0,T;\mathcal{V}^\varepsilon)} \le C\|\psi\|_{W^\varepsilon} \le C(\varepsilon)(\|\tilde{q}_1\|_{L^2((0,T)\times\Omega)} + \|\tilde{q}_2\|_{L^2((0,T)\times\Omega_s^\varepsilon)}) \le C(\varepsilon)\|q\|_{L^2(0,T;\mathcal{P}^\varepsilon)}\,.$$

Here we used that $q_1 = (\tilde{q}_1+\tilde{q}_2)|_{\Omega_z^\varepsilon}$, $q_2 = \tilde{q}_1|_{\Omega_a^\varepsilon}$ and $q_3 = \tilde{q}_2|_{\Omega_{as}^\varepsilon}$. Thus, the definition of $b(\cdot,\cdot)$ yields

$$\begin{aligned} b(\psi,q) &= \langle \tilde{q}_1+\tilde{q}_2, \operatorname{div}(\tilde{\psi}_1+\tilde{\psi}_2)\rangle_{\Omega_z^\varepsilon,T} + \langle \tilde{q}_1, \operatorname{div}\tilde{\psi}_1\rangle_{\Omega_a^\varepsilon,T} + \langle \tilde{q}_2, \operatorname{div}\tilde{\psi}_2\rangle_{\Omega_{as}^\varepsilon,T}\\ &= \|q_1\|^2_{L^2((0,T)\times\Omega_z^\varepsilon)} + \|q_2\|^2_{L^2((0,T)\times\Omega_a^\varepsilon)} + \|q_3\|^2_{L^2((0,T)\times\Omega_{as}^\varepsilon)} \ge C_1(\varepsilon)(\|\tilde{\psi}_1\|_{L^2(0,T;H^1(\Omega))}\\ &+\|\tilde{\psi}_2\|_{L^2((0,T)\times\Omega_s^\varepsilon)} + \|\operatorname{div}\tilde{\psi}_2\|_{L^2((0,T)\times\Omega_s^\varepsilon)} + \varepsilon\|\nabla\tilde{\psi}_2\|_{L^2((0,T)\times\Omega_s^\varepsilon)})(\|q_1\|_{L^2((0,T)\times\Omega_z^\varepsilon)}\\ &+\|q_2\|_{L^2((0,T)\times\Omega_a^\varepsilon)} + \|q_3\|_{L^2((0,T)\times\Omega_{as}^\varepsilon)}) \ge C_2(\varepsilon)\|\psi\|_{L^2(0,T;\mathcal{V}^\varepsilon)}\|q\|_{L^2(0,T;\mathcal{P}^\varepsilon)}\,, \end{aligned}$$

and $b(\cdot,\cdot)$ satisfies the inf-sup condition

$$\inf_{q\in L^2(0,T;\mathcal{P}^\varepsilon),q\neq 0}\ \sup_{\psi\in L^2(0,T;\mathcal{V}^\varepsilon),\psi\neq 0} \frac{b(\psi,q)}{\|\psi\|_{L^2(0,T;\mathcal{V}^\varepsilon)}\|q\|_{L^2(0,T;\mathcal{P}^\varepsilon)}} \ge C > 0\,. \tag{4.17}$$

The regularity of c_s^ε, c_a^ε and Hölder's inequality imply the boundedness in $L^2(0,T;\mathcal{V}^\varepsilon)$ of $f^\varepsilon(\cdot)$

$$\begin{aligned} |f^\varepsilon(\psi)| &\le C\varepsilon(\|c_a^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} + \|c_s^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)})\|\psi_2\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)}\\ &\le C\varepsilon^{1/2}(\|c_a^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} + \|c_s^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)})\|\psi\|_{L^2(0,T;\mathcal{V}^\varepsilon)}\,. \end{aligned}$$

The uniform boundedness of V_D^ε in $L^2(\Omega_a^\varepsilon)$ and estimate (4.16) ensure the boundedness of the linear form $a^\varepsilon(\bar{V}_D^\varepsilon,\cdot)$. Combining all estimates and applying Corollary I.4.1 in [17] imply the existence of a unique solution $(u^\varepsilon,p^\varepsilon)\in L^2(0,T;\mathcal{V}^\varepsilon)\times L^2(0,T;\mathcal{P}^\varepsilon)$ of (4.1) for given $c_a^\varepsilon, c_s^\varepsilon \in L^2((0,T)\times\Gamma_{zs}^\varepsilon)$.

For the proof of a priori estimates (4.12) we start with the estimates for $\boldsymbol{v}_z^\varepsilon$, $\boldsymbol{v}_a^\varepsilon$ and $\boldsymbol{v}_{sp}^\varepsilon$. We consider $\psi_1 = \boldsymbol{v}_z^\varepsilon$ in $\Omega_z^\varepsilon\times(0,T)$ and $\psi_2 = \boldsymbol{v}_a^\varepsilon - V_D^\varepsilon$ in $\Omega_a^\varepsilon\times(0,T)$, and $\psi_3 = \boldsymbol{v}_{sp}^\varepsilon$ in $\Omega_{as}^\varepsilon\times(0,T)$ as test functions in (4.1) and, using the divergence-free property, obtain

$$\begin{aligned} \varepsilon^2\|\mathrm{S}\,\boldsymbol{v}_z^\varepsilon\|^2_{L^2((0,T)\times\Omega_z^\varepsilon)} + \|\boldsymbol{v}_{sp}^\varepsilon\|^2_{L^2((0,T)\times\Omega_{as}^\varepsilon)} + \|\boldsymbol{v}_a^\varepsilon\|^2_{L^2((0,T)\times\Omega_a^\varepsilon)} + \varepsilon\|\boldsymbol{v}_a^\varepsilon\cdot\boldsymbol{n}\|^2_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)}\\ \le C\varepsilon(\|c_s^\varepsilon\|^2_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} + \|c_a^\varepsilon\|^2_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)}) + C\|V_D^\varepsilon\|^2_{L^2(\Omega_a^\varepsilon)}\,. \end{aligned} \tag{4.18}$$

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This together with $\|V_D^\varepsilon\|_{L^2(\Omega_a^\varepsilon)} \leq C\|V_D\|_{H^1(\Omega)}$, inequality (4.4) and $\boldsymbol{v}_z^\varepsilon \times \boldsymbol{n} = 0$ on $\Gamma_{zs}^\varepsilon \times (0,T)$ imply the estimates for $\boldsymbol{v}_i^\varepsilon$, with $i = z, a$, or sp, stated in (4.12).
For $c_{a,i}^\varepsilon$ and $c_{s,i}^\varepsilon$ and corresponding $\boldsymbol{v}_i^\varepsilon$ with $i = 1, 2$, the linearity of the problem, the coercivity of $a^\varepsilon(\cdot,\cdot)$ and the boundedness of $f^\varepsilon(\cdot)$ give

$$\|\boldsymbol{v}_1^\varepsilon - \boldsymbol{v}_2^\varepsilon\|^2_{L^2(0,T;\mathcal{V}_d^\varepsilon)} \leq C\varepsilon(\|c_{a,1}^\varepsilon - c_{a,2}^\varepsilon\|^2_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} + \|c_{s,1}^\varepsilon - c_{s,2}^\varepsilon\|^2_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)}) \,. \tag{4.19}$$

This will ensure the uniqueness of a weak solution of the coupled problem (3.1)-(3.10).

By Lemma 4 in [38] there exists a restriction operator $R^\varepsilon_{\Omega_a^\varepsilon} : H_0^1(\Omega) \to H_0^1(\Omega_a^\varepsilon)$

$$\begin{cases} \psi \in H_0^1(\Omega_a^\varepsilon) & \Longrightarrow \ R^\varepsilon_{\Omega_a^\varepsilon}\psi = \psi \,, \\ \operatorname{div}\psi = 0 & \Longrightarrow \ \operatorname{div} R^\varepsilon_{\Omega_a^\varepsilon}\psi = 0 \,, \\ \|R^\varepsilon_{\Omega_a^\varepsilon}\psi\|_{L^2(\Omega_a^\varepsilon)} & \leq \ C(\|\psi\|_{L^2(\Omega)} + \varepsilon\|\nabla\psi\|_{L^2(\Omega)}) \,, \\ \varepsilon\|\nabla R^\varepsilon_{\Omega_a^\varepsilon}\psi\|_{L^2(\Omega_a^\varepsilon)} & \leq \ C(\|\psi\|_{L^2(\Omega)} + \varepsilon\|\nabla\psi\|_{L^2(\Omega)}) \,. \end{cases} \tag{4.20}$$

As in [3, Theorem 2.3], to define $R^\varepsilon_{\Omega_s^\varepsilon} : H_0^1(\Omega) \to V(\Omega_s^\varepsilon)$, with $V(\Omega_s^\varepsilon) = \{v \in H_0^1(\Omega_s^\varepsilon) : v \times \boldsymbol{n} = 0 \text{ on } \Gamma_{as}^\varepsilon\}$, such that

$$\begin{cases} \psi \in V(\Omega_s^\varepsilon) & \Longrightarrow \ R^\varepsilon_{\Omega_s^\varepsilon}\psi = \psi \,, \\ \operatorname{div}\psi = 0 & \Longrightarrow \ \operatorname{div} R^\varepsilon_{\Omega_s^\varepsilon}\psi = 0 \,, \\ \|R^\varepsilon_{\Omega_s^\varepsilon}\psi\|_{L^2(\Omega_s^\varepsilon)} & \leq \ C(\|\psi\|_{L^2(\Omega)} + \varepsilon\|\nabla\psi\|_{L^2(\Omega)}) \,, \\ \varepsilon\|\nabla R^\varepsilon_{\Omega_s^\varepsilon}\psi\|_{L^2(\Omega_s^\varepsilon)} & \leq \ C(\|\psi\|_{L^2(\Omega)} + \varepsilon\|\nabla\psi\|_{L^2(\Omega)}) \,, \end{cases} \tag{4.21}$$

we consider for $u \in H^1(Y)$, as in [3, Lemma 3.4], a linear operator $Q : H^1(Y) \to H^1(Y)$, such that $Qu = 0$ in Y_a, and a modified problem

$$\begin{cases} (w,q) \in H^1(Y_s) \times L^2(Y_s)/\mathbb{R} \,, & \\ -\eta\,\Delta w + \nabla q = \Delta u & \text{in } Y_z \cup Y_{as} \,, \\ \operatorname{div} w = \operatorname{div} u + \frac{1}{|Y_s|}\int_{Y_{aw}} \operatorname{div} u \, dy & \text{in } Y_s \,, \\ w \times \boldsymbol{n} = 0 \,, \quad [w \cdot \boldsymbol{n}] = 0 \,, \quad [-2\eta\,(\mathrm{S}\, w\, \boldsymbol{n}) \cdot \boldsymbol{n} + q] = 0 & \text{on } \Gamma_{as} \,, \\ w = Qu + \left[\frac{\zeta_i}{\langle\zeta_i,1\rangle_{\partial Y_i}}\int_{\partial Y_i}(u - Qu)\cdot e_i dy\right] e_i & \text{on } \partial Y_{s,i} \cap \partial Y \,, \\ w = 0 & \text{on } \partial Y_s \setminus \partial Y \,, \end{cases} \tag{4.22}$$

where ∂Y_i, with $i = -3,-2,-1,1,2,3$, are the six faces of the cube Y, such that ∂Y_k, ∂Y_{-k} are orthogonal to the unit vector e_k, with $k = 1,2,3$. Here $\zeta_i \in C^\infty(\overline{Y})$, with $i = -3,-2,-1,1,2,3$, satisfy $\zeta_i \geq 0$, $\zeta_i \not\equiv 0$ on ∂Y_i, $\zeta_i \equiv 0$ in $\overline{Y}_a$ and on ∂Y_j for $j \neq i$, and $\zeta_k|_{\partial Y_k} \equiv \zeta_{-k}|_{\partial Y_{-k}}$ for $k = 1,2,3$. The existence of such ζ_i is ensured by the geometrical structure of Y_s. Notice that the construction of the boundary conditions on ∂Y_s ensures that $\int_{Y_s} \operatorname{div} w \, dy = \int_{\partial Y_s} w \cdot \boldsymbol{n} \, dy$. The existence of a unique solution of (4.22) can be shown by applying the abstract theory of mixed problems, similar as for (4.15). Then R_{Y_s}, given by $R_{Y_s}u = w$, belongs to $\mathcal{L}(H^1(Y); H^1(Y_s))$ and in each cell $\varepsilon Y^k \subset \Omega$, with $k \in \mathbb{Z}^3$, we define $R^\varepsilon_{\Omega_s^\varepsilon}u = (R_{Y_s}u(\varepsilon y))(x/\varepsilon)$ for $y \in Y^k$ and $x \in \varepsilon Y_s^k$, respectively.

Equations (3.1) imply that $\nabla p_s \in L^2(0,T;H^{-1}(\Omega_s^\varepsilon))$ and $\nabla p_a \in L^2(0,T;H^{-1}(\Omega_a^\varepsilon))$. We define the extensions of ∇p_s^ε and ∇p_a^ε into Ω using the duality argument and consider $\mathcal{F}_s^\varepsilon \in L^2(0,T;H^{-1}(\Omega))$ and $\mathcal{F}_a^\varepsilon \in L^2(0,T;H^{-1}(\Omega))$ given by

$$\begin{aligned} \langle \mathcal{F}_s^\varepsilon, \psi\rangle_{H^{-1},H^1} &= \langle \nabla p_s^\varepsilon, R^\varepsilon_{\Omega_s^\varepsilon}\psi\rangle_{L^2(0,T;H^{-1}(\Omega_s^\varepsilon)),L^2(0,T;H_0^1(\Omega_s^\varepsilon))} \quad \text{for } \psi \in L^2(0,T;H_0^1(\Omega)) \,, \\ \langle \mathcal{F}_a^\varepsilon, \psi\rangle_{H^{-1},H^1} &= \langle \nabla p_a^\varepsilon, R^\varepsilon_{\Omega_a^\varepsilon}\psi\rangle_{L^2(0,T;H^{-1}(\Omega_a^\varepsilon)),L^2(0,T;H_0^1(\Omega_a^\varepsilon))} \quad \text{for } \psi \in L^2(0,T;H_0^1(\Omega)) \,, \end{aligned}$$

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where $\langle\cdot,\cdot\rangle_{H^{-1},H^1} := \langle\cdot,\cdot\rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H^1_0(\Omega))}$. Applying the estimates for $R^\varepsilon_{\Omega^\varepsilon_s}(\psi)$ and $R^\varepsilon_{\Omega^\varepsilon_a}(\psi)$ in (4.20) and (4.21), we obtain from (3.1) for $\psi \in L^2(0,T;H^1_0(\Omega))$ that

$$\begin{aligned}\langle \mathcal{F}^\varepsilon_s,\psi\rangle_{H^{-1},H^1} &= \langle \varepsilon^2\eta\nabla \boldsymbol{v}^\varepsilon_z, \nabla(R^\varepsilon_{\Omega^\varepsilon_s}\psi)\rangle_{\Omega^\varepsilon_z,T} + \langle K^{-1}_{sp}\boldsymbol{v}^\varepsilon_{sp}, R^\varepsilon_{\Omega^\varepsilon_s}\psi\rangle_{\Omega^\varepsilon_{as},T}\\ &\le C\left(\varepsilon\|\nabla \boldsymbol{v}^\varepsilon_z\|_{L^2((0,T)\times\Omega^\varepsilon_z)} + \|\boldsymbol{v}^\varepsilon_{sp}\|_{L^2((0,T)\times\Omega^\varepsilon_{as})}\right)\left(\|\psi\|_{L^2(\Omega_T)} + \varepsilon\|\nabla\psi\|_{L^2(\Omega_T)}\right)\ ,\end{aligned} \tag{4.23}$$

where $\Omega_T = (0,T)\times\Omega$, and

$$\langle \mathcal{F}^\varepsilon_a,\psi\rangle_{H^{-1},H^1} = \langle K^{-1}_a\boldsymbol{v}^\varepsilon_a, R^\varepsilon_{\Omega^\varepsilon_a}\psi\rangle_{\Omega^\varepsilon_a,T} \le C\|\boldsymbol{v}^\varepsilon_a\|_{L^2((0,T)\times\Omega^\varepsilon_a)}\left(\|\psi\|_{L^2((\Omega_T)} + \varepsilon\|\nabla\psi\|_{L^2(\Omega_T)}\right)\ . \tag{4.24}$$

Thus for each $\varepsilon>0$ we have that $\mathcal{F}^\varepsilon_a$ and $\mathcal{F}^\varepsilon_s$ are bounded functionals on $L^2(0,T;H^1_0(\Omega))$.

For $\psi\in L^2(0,T;H^1_0(\Omega^\varepsilon_l))$ with $\psi\times\boldsymbol{n}=0$ on $\Gamma^\varepsilon_{as}\times(0,T)$, where $l=a,s$, the properties of $R^\varepsilon_{\Omega^\varepsilon_l}$ imply $R^\varepsilon_{\Omega^\varepsilon_l}(\psi)=\psi$ and $\mathcal{F}^\varepsilon_l|_{\Omega^\varepsilon_l} = \nabla p^\varepsilon_l$. We have also $\operatorname{div} R^\varepsilon_{\Omega^\varepsilon_l}(\psi)=0$, for $l=a,s$, provided $\operatorname{div}\psi=0$, and the orthogonality property ensures that $\mathcal{F}^\varepsilon_s$ and $\mathcal{F}^\varepsilon_a$ are the gradients with respect to x of functions in $L^2((0,T)\times\Omega)$. It means that $\mathcal{F}^\varepsilon_s$ and $\mathcal{F}^\varepsilon_a$ are continuations of ∇p^ε_a and ∇p^ε_s to Ω, respectively, and

$$\mathcal{F}^\varepsilon_s = \nabla P^\varepsilon_s\ ,\quad \mathcal{F}^\varepsilon_a = \nabla P^\varepsilon_a \qquad \text{with} \quad P^\varepsilon_s, P^\varepsilon_a \in L^2((0,T)\times\Omega)/\mathbb{R}\ .$$

We have also an explicit formula for the extension P^ε_l, with $l=s$ or a, see [3,19], for $t\in(0,T)$

$$P^\varepsilon_l(t,x) = \begin{cases} p^\varepsilon_l(t,x) & \text{in } \Omega^\varepsilon_l\ ,\\ \frac{1}{|\varepsilon Y^k_l|}\int_{\varepsilon Y^k_l} p^\varepsilon_l(t,x)dx & \text{in } \varepsilon Y^k\setminus\varepsilon\overline{Y^k_l} \text{ for } k\in\mathbb{Z}^3 \text{ such that } \varepsilon Y^k\cap\Omega\neq\emptyset\ .\end{cases}$$

Applying now estimates (4.23), (4.24) together with (4.18) and (4.4), and using the estimate of L^2-norm by H^{-1}-norm, see [17,39], give

$$\begin{aligned}||P^\varepsilon_s||_{L^2(0,T;L^2(\Omega)/\mathbb{R})} &\le C_1||P^\varepsilon_s||_{L^2(0,T;H^{-1}(\Omega))} \le C_2(\varepsilon\|\nabla\boldsymbol{v}^\varepsilon_z\|_{L^2((0,T)\times\Omega^\varepsilon_z)} + \|\boldsymbol{v}^\varepsilon_{sp}\|_{L^2((0,T)\times\Omega^\varepsilon_{as})})\\ &\le C_3\varepsilon^{1/2}(\|c^\varepsilon_s\|_{L^2((0,T)\times\Gamma^\varepsilon_{zs})} + \|c^\varepsilon_a\|_{L^2((0,T)\times\Gamma^\varepsilon_{zs})}) + C_4\|V^\varepsilon_D\|_{L^2(\Omega^\varepsilon_a)}\ ,\\ ||P^\varepsilon_a||_{L^2(0,T;L^2(\Omega)/\mathbb{R})} &\le C_1||P^\varepsilon_a||_{L^2(0,T;H^{-1}(\Omega))} \le C_2\|\boldsymbol{v}^\varepsilon_a\|_{L^2((0,T)\times\Omega^\varepsilon_a)}\\ &\le C_3\varepsilon^{1/2}(\|c^\varepsilon_s\|_{L^2((0,T)\times\Gamma^\varepsilon_{zs})} + \|c^\varepsilon_a\|_{L^2((0,T)\times\Gamma^\varepsilon_{zs})}) + C_4\|V^\varepsilon_D\|_{L^2(\Omega^\varepsilon_a)}\ .\end{aligned}$$

The last estimates together with $\|V^\varepsilon_D\|_{L^2(\Omega^\varepsilon_a)}\le C\|V_D\|_{H^1(\Omega)}$ and the definition of P^ε_s and P^ε_a ensure (4.13) and L^2-estimates for p^ε_s and p^ε_a in (4.12). □

4.2. Existence and estimates for c^ε_s, c^ε_a, $\vartheta^\varepsilon_{f,a}$, $\vartheta^\varepsilon_{b,a}$, $\vartheta^\varepsilon_{f,s}$ and $\vartheta^\varepsilon_{b,s}$

Using classical results [1,8] we can extend the domain of definition of solute concentrations c^ε_l from a connected domain Ω^ε_l to Ω, where $l=a,s$.

Lemma 4.4. *1. For $c_l\in W^{1,p}(Y_l)$, with $1<p<\infty$, there exists an extension $\tilde{c}_l$ to Y such that*

$$\|\tilde{c}_l\|_{L^p(Y)}\le\Xi\|c_l\|_{L^p(Y_l)} \quad \text{and} \quad \|\nabla\tilde{c}_l\|_{L^p(Y)}\le\Xi\|\nabla c_l\|_{L^p(Y_l)}\ .$$

2. There exists an extension $\tilde{c}^\varepsilon_l$ of c^ε_l from $W^{1,p}(\Omega^\varepsilon_l)$ into $W^{1,p}(\Omega)$ such that

$$\|\tilde{c}^\varepsilon_l\|_{L^p(\Omega)}\le\Xi\|c^\varepsilon_l\|_{L^p(\Omega^\varepsilon_l)} \quad \text{and} \quad \|\nabla\tilde{c}^\varepsilon_l\|_{L^p(\Omega)}\le\Xi\|\nabla c^\varepsilon_l\|_{L^p(\Omega^\varepsilon_l)}\ ,$$

where the constant Ξ depends on Y and Y_l only.

Due to the geometric assumptions on Ω^ε_a, holes in the domain do not touch each other, have smooth boundary and do not touch the boundary $\partial\Omega$, i.e. $\Gamma^\varepsilon_{zs}\cap\partial\Omega=\emptyset$. Therefore, classical extension results [1,8] apply to c^ε_a. Due to the structural assumptions, Ω^ε_s is a connected domain in $\mathbb{R}^3$ with Lipschitz-continous boundary $\partial\Omega^\varepsilon_s$. The geometrical assumptions on Ω^ε_s ensure also that it is sufficient to extend

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c_s^ε by reflection in tangential directions near the boundary $\partial\Omega_s^\varepsilon \cap \partial\Omega$. Therefore the extension results apply also for c_s^ε, see [1, 33], and the extension operator is defined globally in Ω.

For $c_l^\varepsilon \in H^1(0,T;H^1(\Omega_l^\varepsilon))$ we define $\hat{c}_l^\varepsilon(\cdot,t) := \tilde{c}_l^\varepsilon(\cdot,t)$ a.e. in time. Since the extension operator is linear and bounded and Ω_l^ε does not depend on t, we obtain $\hat{c}_l^\varepsilon \in H^1(0,T;H^1(\Omega))$ and

$$\|\partial_t \hat{c}_l^\varepsilon\|_{L^2(\Omega_T)} \leq \Xi \|\partial_t c_l^\varepsilon\|_{L^2((0,T)\times\Omega_l^\varepsilon)}, \quad \|\partial_t \nabla \hat{c}_l^\varepsilon\|_{L^2(\Omega_T)} \leq \Xi \|\partial_t \nabla c_l^\varepsilon\|_{L^2((0,T)\times\Omega_l^\varepsilon)} ,$$

where $l = a, s$. In the sequel, we shall identify c_l^ε with its extension $\hat{c}_l^\varepsilon$.

Theorem 4.5. *Under Assumption 1 there exists a nonnegative unique weak solution* $(c_a^\varepsilon, c_s^\varepsilon, \vartheta_{f,a}^\varepsilon, \vartheta_{b,a}^\varepsilon, \vartheta_{f,s}^\varepsilon, \vartheta_{b,s}^\varepsilon)$ *of* (3.2) *and* (3.5) *with boundary conditions* (3.3), (3.6), *and* (3.9) *and initial conditions* (3.7) *such that* $c_l^\varepsilon \in H^1(0,T;H^1(\Omega_l^\varepsilon)) \cap L^\infty((0,T)\times\Omega_l^\varepsilon)$, $\vartheta_{j,l}^\varepsilon \in W^{1,\infty}(0,T;L^\infty(\Gamma_{zs}^\varepsilon))$, *with* $l = a, s$, $j = f, b$, *and satisfies the estimates*

$$\begin{aligned} &\|c_l^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_l^\varepsilon))} + \|\nabla c_l^\varepsilon\|_{L^2((0,T)\times\Omega_l^\varepsilon)} + \varepsilon^{\frac{1}{2}} \|c_l^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} \leq C , \\ &\|c_l^\varepsilon\|_{L^\infty((0,T)\times\Omega)} \leq C , \qquad \|\vartheta_{j,l}^\varepsilon\|_{L^\infty((0,T)\times\Gamma_{zs}^\varepsilon)} \leq C , \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} &\|\partial_t c_l^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_l^\varepsilon))} + \|\partial_t \nabla c_l^\varepsilon\|_{L^2((0,T)\times\Omega_l^\varepsilon)} + \varepsilon^{\frac{1}{2}} \|\partial_t c_l^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} + \varepsilon^{\frac{1}{2}} \|\partial_t \vartheta_{j,l}^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} \leq C , \\ &\|\partial_t \boldsymbol{v}_a^\varepsilon\|_{L^2((0,T)\times\Omega_a^\varepsilon)} + \|\partial_t \boldsymbol{v}_s^\varepsilon\|_{L^2((0,T)\times\Omega_s^\varepsilon)} + \varepsilon \|\partial_t \nabla \boldsymbol{v}_z^\varepsilon\|_{L^2((0,T)\times\Omega_z^\varepsilon)} + \varepsilon^{\frac{1}{2}} \|\partial_t \boldsymbol{v}_a^\varepsilon \cdot \boldsymbol{n}\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} \leq C , \end{aligned} \tag{4.26}$$

with $j = f, b$ *and* $l = a, s$, *and the constant* C *depends on* M *and is independent of* ε.

Proof. The existence of a solution will be proven by showing the existence of a fix point of the operator $\mathcal{B}$ defined on $\big(C([0,T];H^\varsigma(\Omega_a^\varepsilon)) \cap L^\infty((0,T)\times\Omega_a^\varepsilon)\big) \times \big(C([0,T];H^\varsigma(\Omega_s^\varepsilon)) \cap L^\infty((0,T)\times\Omega_s^\varepsilon)\big)$, with $1/2 < \varsigma < 1$, by $(c_a^{\varepsilon,n}, c_s^{\varepsilon,n}) = \mathcal{B}(c_a^{\varepsilon,n-1}, c_s^{\varepsilon,n-1})$ given as solutions of (4.2) and (4.3) with $(\vartheta_{f,l}^{\varepsilon,n}, \vartheta_{b,l}^{\varepsilon,n})$ solving

$$\begin{cases} \partial_t \vartheta_{f,l}^{\varepsilon,n} &= R_l^\varepsilon(t,x,\vartheta_{f,l}^{\varepsilon,n}) - \alpha_l^\varepsilon(t,x) c_l^{\varepsilon,n-1} \vartheta_{f,l}^{\varepsilon,n} + \beta_l^\varepsilon(t,x) \vartheta_{b,l}^{\varepsilon,n} - \gamma_{f,l}^\varepsilon(t,x) \vartheta_{f,l}^{\varepsilon,n} & \text{on } \Gamma_{zs}^\varepsilon \times (0,T) , \\ \partial_t \vartheta_{b,l}^{\varepsilon,n} &= \alpha_l^\varepsilon(t,x) c_l^{\varepsilon,n-1} \vartheta_{f,l}^{\varepsilon,n} - \beta_l^\varepsilon(t,x) \vartheta_{b,l}^{\varepsilon,n} - \gamma_{b,l}^\varepsilon(t,x) \vartheta_{b,l}^{\varepsilon,n} & \text{on } \Gamma_{zs}^\varepsilon \times (0,T) , \end{cases} \tag{4.27}$$

where $l = a, s$. For a given nonnegative $c_l^{n-1,\varepsilon} \in C([0,T];H^\varsigma(\Omega_l^\varepsilon)) \cap L^\infty((0,T)\times\Omega_l^\varepsilon)$, with $l = a, s$, due to the Lipschitz-continuity of the right-hand side of system (4.27), there exists a unique solution $\vartheta_{j,l}^{\varepsilon,n} \in C^1([0,T];L^2(\Gamma_{zs}^\varepsilon))$ of (4.27), with $j = f, b$ and $l = a, s$. Using that initial values are nonnegative, function $R_l^\varepsilon(t,x,\xi)$ is nonnegative for nonnegative ξ, and $c_l^{\varepsilon,n-1}(t,x) \geq 0$ a.e. on $[0,T]\times\Gamma_{zs}^\varepsilon$, we obtain $\vartheta_{j,l}^{n,\varepsilon}(t,x) \geq 0$ a.e on $[0,T]\times\Gamma_{zs}^\varepsilon$ for $j = f, b$ and $l = a, s$. Adding equations in (4.27) yields

$$\partial_t(\vartheta_{f,l}^{\varepsilon,n} + \vartheta_{b,l}^{\varepsilon,n}) = R_l^\varepsilon(t,x,\vartheta_{f,l}^{\varepsilon,n}) - \gamma_{f,l}^\varepsilon(t,x)\vartheta_{f,l}^{\varepsilon,n} - \gamma_{b,l}^\varepsilon(t,x)\vartheta_{b,l}^{\varepsilon,n} , \quad \text{for } t \in [0,T] \text{ and a.a } x \in \Gamma_{zs}^\varepsilon , \tag{4.28}$$

where $l = a, s$. Considering the Lipschitz continuity of R_l^ε, the nonnegativity of $\vartheta_{j,l}^{\varepsilon,n}$, the boundedness of initial conditions and applying Gronwall's inequality, we obtain the boundedness of $\vartheta_{j,l}^{\varepsilon,n}$ a.e. on $[0,T]\times\Gamma_{zs}^\varepsilon$, with $j = f, b$ and $l = a, s$. The boundedness of $c_l^{\varepsilon,n-1}$ and $\vartheta_{j,l}^{\varepsilon,n}$ implies also the boundedness of $\partial_t\vartheta_{j,l}^{\varepsilon,n}$ a.e. on $[0,T]\times\Gamma_{zs}^\varepsilon$, whereas $j = f, b$ and $l = a, s$.

Using Galerkin's method and a priori estimates similar to those shown below, we obtain the existence of a weak nonnegative solution $c_l^{\varepsilon,n} \in H^1(0,T;H^1(\Omega_l^\varepsilon)) \cap L^\infty((0,T)\times\Omega_l^\varepsilon)$, for $l = s, a$, see [22]. The embedding $H^1(0,T;H^1(\Omega_l^\varepsilon)) \subset C([0,T];H^\varsigma(\Omega_l^\varepsilon))$, for $1/2 < \varsigma < 1$, is compact and, by virtue of the Schauder theorem, there exists a fix point of $\mathcal{B}$, a solution $(c_l^\varepsilon, \vartheta_{j,l}^\varepsilon)$ of the microscopic problem, where $j = f, b$ and $l = a, s$. In addition, we obtain boundedness of $c_l^\varepsilon(t,x)$ a.e. in $[0,T]\times\Omega_l^\varepsilon$ and on $[0,T]\times\Gamma_{zs}^\varepsilon$, and $c_l^\varepsilon(t,x) \geq 0$ a.e. in $[0,T]\times\Omega_l^\varepsilon$ and on $[0,T]\times\Gamma_{zs}^\varepsilon$, together with $\vartheta_{j,l}^\varepsilon(t,x) \geq 0$ a.e. on $[0,T]\times\Gamma_{zs}^\varepsilon$, where $j = f, b$ and $l = a, s$. This ensures also the boundedness of $\partial_t\vartheta_{j,l}^\varepsilon$ a.e. on $[0,T]\times\Gamma_{zs}^\varepsilon$, as well as by applying (4.28) the uniform in ε boundedness of $\vartheta_{j,l}^\varepsilon$ a.e. on $[0,T]\times\Gamma_{zs}^\varepsilon$, and thus the last estimate in (4.25).

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Now we shall prove the non-negativity of $c_l^{\varepsilon,n}$, and thus of c_l^ε, and the a priori estimates for $c_l^{\varepsilon,n}$, and therefore for c_l^ε, uniformly in ε, where $l = a, s$. To show the non-negativity of $c_l^{\varepsilon,n}$, with $l = a, s$ we take $c_{l,-}^\varepsilon = \min\{0, c_l^{\varepsilon,n}\}$ as test functions in (4.2) and (4.3), respectively. Considering the assumptions on the coefficients and functions F_l^ε, and the nonnegativity of $\vartheta_{j,l}^{\varepsilon,n}$, with $l = a, s$ and $j = f, b$, terms on the right-hand side in (4.2) and (4.3) can be estimated by $C_1\|c_{l,-}^\varepsilon\|_{L^2((0,\tau)\times\Omega_l^\varepsilon)}$ for $\tau \in (0,T]$. Using Gagliardo-Nirenberg-inequality, i.e. for $\phi \in W^{1,q}(\Omega)$

$$\|\phi\|_{L^p(\Omega)} \le C_G\big(\|\phi\|_{L^r(\Omega)}^\lambda \|\nabla\phi\|_{L^q(\Omega)}^{(1-\lambda)} + \|\phi\|_{L^1(\Omega)}\big), \tag{4.29}$$

where $1 \le r < p$ and $1 \le q \le p$, with $\lambda \in [0,1]$ satisfies the relation $1/p = \lambda(1/r - 1/n) + (1-\lambda)1/q$ and $p \in [q, qn/(n-q)]$ for $q < n$, $p \in [q,\infty)$ for $q = n$, $p \in [q,\infty]$ for $q > n$, and considering extension of $c_{l,-}^\varepsilon$, defined in Lemma 4.4, we can estimate the advective term by

$$\begin{aligned}
|\langle H_M(\boldsymbol{v}_l^\varepsilon)c_{l,-}^\varepsilon, \nabla c_{l,-}^\varepsilon\rangle_{\Omega_l^\varepsilon,\tau}| &\le \frac{1}{d_l}\|H_M(\boldsymbol{v}_l^\varepsilon)\|^2_{L^\infty(0,\tau;L^4(\Omega_l^\varepsilon))}\|c_{l,-}^\varepsilon\|^2_{L^2(0,\tau;L^4(\Omega_l^\varepsilon))} + \frac{d_l}{4}\|\nabla c_{l,-}^\varepsilon\|^2_{L^2((0,\tau)\times\Omega_l^\varepsilon)}\\
&\le C\Big(\|H_M(\boldsymbol{v}_l^\varepsilon)\|^8_{L^\infty(0,\tau;L^4(\Omega_l^\varepsilon))} + \|H_M(\boldsymbol{v}_l^\varepsilon)\|^2_{L^\infty(0,\tau;L^4(\Omega_l^\varepsilon))}\Big)\|c_{l,-}^\varepsilon\|^2_{L^2((0,\tau)\times\Omega_l^\varepsilon)} + \frac{d_l}{2}\|\nabla c_{l,-}^\varepsilon\|^2_{L^2((0,\tau)\times\Omega_l^\varepsilon)}
\end{aligned}$$

for $\tau \in (0,T]$. Considering uniform boundedness of $\|H_M(\boldsymbol{v}_l^\varepsilon)\|_{L^\infty(0,T;L^4(\Omega_l^\varepsilon))} \le C_v M$, the nonegativity of initial condition c_l^0 and applying Gronwall's inequality we obtain

$$\|c_{a,-}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_a^\varepsilon))} + \|c_{s,-}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_s^\varepsilon))} \le 0\,,$$

which implies the nonnegativity of $c_l^{\varepsilon,n}$ a.e. in $(0,T)\times\Omega_l^\varepsilon$, where $l = a, s$. For $c_l^{\varepsilon,n} \in H^1(0,T;H^1(\Omega_l^\varepsilon))$ we obtain also that $c_l^{\varepsilon,n} \ge 0$ a.e. on $[0,T]\times\Gamma_{zs}^\varepsilon$. Considering weak convergence of $\{c_l^{\varepsilon,n}\}$ in $H^1(0,T;H^1(\Omega_l^\varepsilon))$, as $n\to\infty$, we conclude that $c_l^\varepsilon(t,x) \ge 0$ a.e. in $[0,T]\times\Omega_l^\varepsilon$ and a.e. on $[0,T]\times\Gamma_{zs}^\varepsilon$.

Taking $\varphi_1 = c_a^\varepsilon$ and $\varphi_2 = c_s^\varepsilon$ as test functions in (4.2) and (4.3), using the nonnegativity of $\vartheta_{f,l}^\varepsilon$ and of coefficients α_l^ε, with $l = a, s$, assumptions on F_l, the first estimate in

$$\begin{aligned}
\varepsilon\|c_l^\varepsilon\|^2_{L^2(\Gamma_{zs}^\varepsilon)} &\le C\big(\|c_l^\varepsilon\|^2_{L^2(\Omega_l^\varepsilon)} + \varepsilon^2\|\nabla c_l^\varepsilon\|^2_{L^2(\Omega_l^\varepsilon)}\big)\,,\\
\varepsilon\|c_l^\varepsilon\|_{L^1(\Gamma_{zs}^\varepsilon)} &\le C\big(\|c_l^\varepsilon\|_{L^1(\Omega_l^\varepsilon)} + \varepsilon\|\nabla c_l^\varepsilon\|_{L^1(\Omega_l^\varepsilon)}\big)\,,
\end{aligned} \tag{4.30}$$

see [27], considering that $\varepsilon|\Gamma_{zs}^\varepsilon| \le C$ independently of ε, and applying Hölder's inequality, we obtain

$$\begin{aligned}
\|c_l^\varepsilon(\tau)\|^2_{L^2(\Omega_l^\varepsilon)} + (d_l - \zeta\varepsilon^2)\|\nabla c_l^\varepsilon\|^2_{L^2((0,\tau)\Omega_l^\varepsilon)} \le{}& \frac{1}{d_l}\|H_M(\boldsymbol{v}_l^\varepsilon)\|^2_{L^\infty(0,\tau;L^4(\Omega_l^\varepsilon))}\|c_l^\varepsilon\|^2_{L^2(0,\tau;L^4(\Omega_l^\varepsilon))}\\
&+ C_1\|c_l^\varepsilon\|^2_{L^2((0,\tau)\times\Omega_l^\varepsilon)} + \|c_l^0\|^2_{L^2(\Omega_l^\varepsilon)} + C_2\|\vartheta_{b,l-1}^\varepsilon\|^2_{L^\infty((0,\tau)\times\Gamma_{zs}^\varepsilon)}
\end{aligned}$$

for $\tau \in (0,T]$, with $l = a, s$, whereas $a - 1 := s$ and $s - 1 := a$. Using the extension of c_l^ε from Ω_l^ε into Ω, given by Lemma 4.4, and applying inequality (4.29) in the first term on the right-hand side imply

$$\begin{aligned}
\|c_l^\varepsilon(\tau)\|^2_{L^2(\Omega)} + (d_l/4 - \zeta\varepsilon^2)\|\nabla c_l^\varepsilon\|^2_{L^2((0,\tau)\times\Omega)} \le{}& \|c_l^0\|^2_{L^2(\Omega_l^\varepsilon)} + C_1\|\vartheta_{b,l-1}^\varepsilon\|^2_{L^\infty((0,T)\times\Gamma_{zs}^\varepsilon)}\\
&+C_2\big(\|H_M(\boldsymbol{v}_l^\varepsilon)\|^8_{L^\infty(0,T;L^4(\Omega_l^\varepsilon))} + \|H_M(\boldsymbol{v}_l^\varepsilon)\|^2_{L^\infty(0,T;L^4(\Omega_l^\varepsilon))} + 1\big)\|c_l^\varepsilon\|^2_{L^2((0,\tau)\times\Omega_l^\varepsilon)}\,.
\end{aligned} \tag{4.31}$$

Then, for all $\varepsilon \le \varepsilon_0$, with some $\varepsilon_0 > 0$, and ζ such that $d_l/4 - \zeta\varepsilon_0^2 \ge d_0 > 0$, considering the regularity assumption on initial data, the boundedness of $\vartheta_{b,l}^\varepsilon$, where $l = a, s$, applying Gronwall's Lemma in inequality (4.31), and using the first inequality in (4.30), we obtain the first estimate in (4.25).

To show L^∞-estimates for the extension of c_l^ε into Ω, given by Lemma 4.4, we shall apply Theorem II.6.1 from [22] stating that inequality $\|(c_l^\varepsilon - S)_+\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(c_l^\varepsilon - S)_+\|_{L^2(\Omega_T)} \le \zeta S\|\Omega_{l,S}(t)\|_{L^{\tilde q}(0,T)}^{\tilde r}$, for appropriate $\tilde r, \tilde q$, a positive constant ζ and $\Omega_{l,S}(t) = \{x \in \Omega : c_l^\varepsilon(t,x) > S\}$ for a.a. $t \in (0,T)$, ensures the corresponding estimate for the L^∞-norm of c_l^ε.

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We take $(c_a^\varepsilon - S)_+$ and $(c_s^\varepsilon - S)_+$ as test functions in (4.2) and (4.3). Considering the nonnegativity of $\alpha_l^\varepsilon(t,x)$, $c_l^\varepsilon(t,x)$ and $\vartheta_{f,l}^\varepsilon(t,x)$ for $(t,x) \in (0,T) \times \Gamma_{zs}^\varepsilon$, and inequalities (4.30), the boundary integrals can be estimated by

$$\varepsilon\langle \beta_{l-1}^\varepsilon(t,x)\vartheta_{b,l-1}^\varepsilon - \alpha_l^\varepsilon(t,x)c_l^\varepsilon\vartheta_{f,l}^\varepsilon, (c_l^\varepsilon - S)_+\rangle_{\Gamma_{zs}^\varepsilon} \leq \sup_{\Gamma_{zs}^\varepsilon}(\beta_{l-1}^\varepsilon(t,x)\vartheta_{b,l-1}^\varepsilon)\varepsilon\|(c_l^\varepsilon - S)_+\|_{L^1(\Gamma_{zs}^\varepsilon)}$$
$$\leq 1/2\left(\Upsilon_l\|(c_l^\varepsilon - S)_+\|^2_{L^2(\Omega_l^\varepsilon)} + d_l\|\nabla(c_l^\varepsilon - S)_+\|^2_{L^2(\Omega_l^\varepsilon)} + C_{Y_l}^2\Upsilon_l(1+\Upsilon_l\varepsilon^2/d_l)|\Omega_{l,S}^\varepsilon(t)|\right), \quad (4.32)$$

where $l = a, s$, with $a - 1 = s$, $s - 1 = a$, $\Omega_{l,S}^\varepsilon(t) = \{x \in \Omega_l^\varepsilon : c_l^\varepsilon(t,x) > S\}$ for a.a. $t \in (0,T)$ and $\Upsilon_l = \sup_{(0,T)\times\Gamma_{zs}^\varepsilon}(\beta_{l-1}^\varepsilon(t,x)\vartheta_{b,l-1}^\varepsilon)$. The Lipschitz continuity of F_l, with $l = a, s$, ensures

$$\langle F_l^\varepsilon(t,x,c_l^\varepsilon), (c_l^\varepsilon - S)_+\rangle_{\Omega_l^\varepsilon} \leq 3C_{F_l}/2\|(c_l^\varepsilon - S)_+\|^2_{L^2(\Omega_l^\varepsilon)} + (S^2/C_{F_l} + C_{F_l})|\Omega_{l,S}^\varepsilon(t)|\,. \quad (4.33)$$

As next we shall estimate $|\langle H_M(\boldsymbol{v}_l^\varepsilon)c_l^\varepsilon, \nabla(c_l^\varepsilon - S)_+\rangle_{\Omega_l^\varepsilon,T}|$ with $l = a, s$. Using Hölder's inequality yields

$$|\langle H_M(\boldsymbol{v}_l^\varepsilon)c_l^\varepsilon, \nabla(c_l^\varepsilon - S)_+\rangle_{\Omega_l^\varepsilon,T}| \leq \frac{d_l}{4}\|\nabla(c_l^\varepsilon - S)_+\|^2_{L^2((0,T)\times\Omega_l^\varepsilon)}$$
$$+\frac{2}{d_l}\|H_M(\boldsymbol{v}_l^\varepsilon)\|^2_{L^{10}(0,T;L^4(\Omega_l^\varepsilon))}\Big(\|(c_l^\varepsilon - S)_+\|^2_{L^{5/2}(0,T;L^4(\Omega_l^\varepsilon))} + S^2\Big(\int_0^T |\Omega_{l,S}^\varepsilon(t)|^{\frac{5}{8}}dt\Big)^{\frac{4}{5}}\Big)\,.$$

Choosing $\varsigma = 1/30$ we can estimate

$$\|(c_l^\varepsilon - S)_+\|^2_{L^{5/2}(0,T;L^4(\Omega_l^\varepsilon))} \leq \|(c_l^\varepsilon - S)_+\|^2_{L^{5(1+\varsigma)/2}(0,T;L^{4(1+\varsigma)}(\Omega_l^\varepsilon))}\Big(\int_0^T |\Omega_{l,S}^\varepsilon(t)|^{\frac{5}{8}}dt\Big)^{\frac{4}{5}\frac{\varsigma}{\varsigma+1}}\,.$$

Using imbedding result, see [22, Chapter II, Eq. (3.4)], and extension of c_l^ε into Ω, we obtain

$$\|(c_l^\varepsilon - S)_+\|^2_{L^{5(1+\varsigma)/2}(0,T;L^{4(1+\varsigma)}(\Omega_l^\varepsilon))} \leq \|(c_l^\varepsilon - S)_+\|^2_{L^{5(1+\varsigma)/2}(0,T;L^{4(1+\varsigma)}(\Omega))}$$
$$\leq C_\Omega\left(\|(c_l^\varepsilon - S)_+\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(c_l^\varepsilon - S)_+\|^2_{L^2(0,T;L^2(\Omega))}\right)\,.$$

Considering $\|H_M(\boldsymbol{v}_l^\varepsilon)\|^2_{L^{10}(0,T;L^4(\Omega_l^\varepsilon))} \leq T^{\frac{1}{5}}\|H_M(\boldsymbol{v}_l^\varepsilon)\|^2_{L^\infty(0,T;L^4(\Omega_l^\varepsilon))} \leq T^{\frac{1}{5}}C_vM^2$ we conclude

$$|\langle H_M(\boldsymbol{v}_l^\varepsilon)c_l^\varepsilon, \nabla(c_l^\varepsilon - S)_+\rangle_{\Omega_l^\varepsilon,T}| \leq \frac{d_l}{4}\|\nabla(c_l^\varepsilon - S)_+\|^2_{L^2((0,T)\times\Omega_l^\varepsilon)} + \frac{2}{d_l}T^{\frac{1}{5}}C_\Omega C_vM^2 \times \quad (4.34)$$
$$\Big[S^2\Big(\int_0^T |\Omega_{l,S}^\varepsilon|^{\frac{5}{8}}dt\Big)^{\frac{4}{5}} + \Big(\int_0^T |\Omega_{l,S}^\varepsilon|^{\frac{5}{8}}dt\Big)^{\frac{4}{5}\frac{\varsigma}{\varsigma+1}}\Big(\|(c_l^\varepsilon - S)_+\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(c_l^\varepsilon - S)_+\|^2_{L^2(\Omega_T)}\Big)\Big].$$

We shall define

$$S = \max_{l=a,s}\max\{1, \|c_l^0\|_{L^\infty(\Omega)}, \Xi\big(\|c_l^0\|_{L^1(\Omega)} + \Upsilon_l|\Gamma||\Omega| + C_{F_l}|\Omega|\big)e^{C_{F_l}T}, C_{F_l}, \Upsilon_l\},$$

where $\Xi = \Xi(Y, Y_l)$ is the constant from Lemma 4.4. The choice of S ensures that $|\Omega_{l,S}^\varepsilon(t)| \leq 1$ and $|\Omega_{l,S}(t)| \leq 1$. Choosing

$$\tilde{T} = \min_{l=a,s}\min\{[d_l^2/(16\Xi C_\Omega C_vM^2|\Omega|^{\frac{\varsigma}{2\varsigma+2}})]^{\frac{5(\varsigma+1)}{5\varsigma+1}}, [d_l/(8\Xi C_\Omega C_vM^2|\Omega|^{\frac{\varsigma}{2\varsigma+2}})]^{\frac{5(\varsigma+1)}{5\varsigma+1}}, [4\Upsilon_l+12C_{F_l}]^{-1}, |\Omega|^{-\frac{5}{8}}\}\,,$$

combining estimates (4.32)-(4.34) and using the fact that $|\Omega_{l,S}^\varepsilon(t)| \leq 1$ yield

$$\|(c_l^\varepsilon - S)_+\|^2_{L^\infty(0,\tilde{T};L^2(\Omega))} + \|\nabla(c_l^\varepsilon - S)_+\|^2_{L^2((0,\tilde{T})\times\Omega)}$$
$$\leq S^2 8\Xi/\min\{1, d_l\}\left(2\tilde{T}^{\frac{1}{5}}C_vC_\Omega M^2/d_l + 1/C_{F_l} + 1 + C_{Y_l}^2(1+\Upsilon_l\varepsilon^2/d_l)/2\right)\Big(\int_0^{\tilde{T}} |\Omega_{l,S}^\varepsilon(t)|^{\frac{5}{8}}dt\Big)^{\frac{4}{5}}\,.$$

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Choosing $q = 4(1+\varsigma)$ together with $r = 5(1+\varsigma)/2$ and using $|\Omega^\varepsilon_{l,S}(t)| \leq |\Omega_{l,S}(t)| \leq 1$, we obtain

$$\|(c^\varepsilon_l - S)_+\|_{L^\infty(0,\tilde{T};L^2(\Omega))} + \|\nabla(c^\varepsilon_l - S)_+\|_{L^2((0,\tilde{T})\times\Omega)} \leq S\Theta\Big(\int_0^{\tilde{T}} |\Omega_{l,S}(t)|^{\frac{r}{q}}\,dt\Big)^{\frac{2(1+\varsigma)}{r}},$$

where $\Theta = 8\Xi/\min\{1,d_l\}\left(2\tilde{T}^{\frac{1}{5}}C_vC_\Omega M^2/d_l + 1/C_{F_l} + 1 + C^2_{Y_l}(1+\Upsilon_l\varepsilon^2/d_l)/2\right)$. Applying Theorem II.6.1 in [22] yields

$$\|c^\varepsilon_l\|_{L^\infty((0,\tilde{T})\times\Omega)} \leq 2S[1 + 2^{\frac{2\varsigma+1}{\varsigma^2}}(C_\Omega\Theta)^{1+\frac{1}{\varsigma}}\tilde{T}^{1+\frac{1}{\varsigma}}|\Omega|^{\frac{1}{4}}]. \tag{4.35}$$

Considering $T_c = \min\limits_{l=a,s}\min\{\tilde{T}, \frac{d_l^5}{(2C_\Omega C_v M^2)^5}, |\Omega|^{-\frac{\varsigma}{4(1+\varsigma)}}[\frac{8C_\Omega\Xi}{\min\{1,d_l\}}[2 + \frac{1}{C_{F_l}} + \frac{C^2_{Y_l}}{2}(1+\Upsilon_l\frac{\varepsilon^2}{d_l})]2^{\frac{2+\varsigma}{\varsigma+\varsigma^2}}]^{-1}\}$ implies

$$\|c^\varepsilon_l\|_{L^\infty((0,\tau)\times\Omega)} \leq 4S \quad \text{for all } \tau \leq T_c.$$

The iteration over time-intervals will then ensure the boundedness of c^ε_l in $(0,T)\times\Omega$, and thus also the boundedness of c^ε_l in $(0,T)\times\Omega^\varepsilon_l$, for $l = a,s$, and the second estimate in (4.25).

Using the estimates for c^ε_l and $\vartheta^\varepsilon_{j,l}$, with $j = f,b$ and $l = a,s$, and considering $\partial_t\vartheta^\varepsilon_{j,l}$ as test functions in (3.5), we conclude

$$\varepsilon^{1/2}\|\partial_t\vartheta^\varepsilon_{f,l}\|_{L^2((0,T)\times\Gamma^\varepsilon_{zs})} + \varepsilon^{1/2}\|\partial_t\vartheta^\varepsilon_{b,l}\|_{L^2((0,T)\times\Gamma^\varepsilon_{zs})} \leq C_1(1+\varepsilon^{1/2}\|c^\varepsilon_l\|_{L^2((0,T)\times\Gamma^\varepsilon_{zs})}) \leq C_2\,. \tag{4.36}$$

Differentiating equations (3.1) and boundary conditions (3.8), (3.10) with respect to t, considering $(\partial_t\boldsymbol{v}^\varepsilon_z, \partial_t\boldsymbol{v}^\varepsilon_a, \partial_t\boldsymbol{v}^\varepsilon_{sp})$ as a test function, and applying Korn inequality (4.4) we obtain for $\tau \in (0,T]$

$$\begin{aligned}\varepsilon^2\|\nabla\partial_t\boldsymbol{v}^\varepsilon_z\|^2_{L^2((0,\tau)\times\Omega^\varepsilon_z)} + \|\partial_t\boldsymbol{v}^\varepsilon_s\|^2_{L^2((0,\tau)\times\Omega^\varepsilon_s)} + \|\partial_t\boldsymbol{v}^\varepsilon_a\|^2_{L^2((0,\tau)\times\Omega^\varepsilon_a)} + \varepsilon\,\|\partial_t\boldsymbol{v}^\varepsilon_a\cdot\boldsymbol{n}\|^2_{L^2((0,\tau)\times\Gamma^\varepsilon_{zs})}\\ \leq C\varepsilon\big(\|\partial_t c^\varepsilon_s\|^2_{L^2((0,\tau)\times\Gamma^\varepsilon_{zs})} + \|\partial_t c^\varepsilon_a\|^2_{L^2((0,\tau)\times\Gamma^\varepsilon_{zs})}\big)\,.\end{aligned} \tag{4.37}$$

Now we differentiate with respect to t equations (3.2) and use $\partial_t c^\varepsilon_a$ and $\partial_t c^\varepsilon_s$ as test functions. Estimates (4.25) together with inequalities (4.29) and (4.30), estimates in Lemma 4.4, and Assumption 1 give

$$\begin{aligned}&\|\partial_t c^\varepsilon_l(\tau)\|^2_{L^2(\Omega^\varepsilon_l)} + \int_0^\tau \|\partial_t\nabla c^\varepsilon_l\|^2_{L^2(\Omega^\varepsilon_l)}dt \leq C\|c^\varepsilon_l\|^2_{L^\infty((0,T)\times\Omega^\varepsilon_l)}\int_0^\tau \|\partial_t\boldsymbol{v}^\varepsilon_l\|^2_{L^2(\Omega^\varepsilon_l)}dt + \|\partial_t c^\varepsilon_l(0)\|^2_{L^2(\Omega^\varepsilon_l)}\\ &+C(1+\|H_M(\boldsymbol{v}^\varepsilon_l)\|^2_{L^\infty(0,\tau;L^4(\Omega^\varepsilon_l))})\int_0^\tau\Big[\|\partial_t c^\varepsilon_l\|^2_{L^2(\Omega^\varepsilon_l)} + \varepsilon\|\partial_t\vartheta^\varepsilon_{b,l-1}\|^2_{L^2(\Gamma^\varepsilon_{zs})} + \varepsilon\|\partial_t\vartheta^\varepsilon_{f,l}\|^2_{L^2(\Gamma^\varepsilon_{zs})}\Big]\,dt\,,\end{aligned}$$

where $l = a,s$ with $a-1 := s$ and $s-1 := a$. The regularity assumption on c^0_l ensures that $\|\partial_t c^\varepsilon_l(0)\|_{L^2(\Omega^\varepsilon_l)} \leq C\|c^0_l\|_{H^2(\Omega^\varepsilon_l)}$, with $l = a,s$. Combining the last inequality together with (4.36) and (4.37), using the boundedness of $H_M(\boldsymbol{v}^\varepsilon_l)$ and inequality (4.30), choosing ε sufficient small, and applying Gronwall's inequality imply (4.26).

Uniqueness. Suppose there are two solutions of the problem. We denote $c^\varepsilon_l = c^\varepsilon_{l,1} - c^\varepsilon_{l,2}$ and $\vartheta^\varepsilon_{j,l} = \vartheta^\varepsilon_{j,l,1} - \vartheta^\varepsilon_{j,l,2}$, with $j = f,b$ and $l = a,s$, and choose $\varphi_1 = c^\varepsilon_a$ in (4.2) and $\varphi_2 = c^\varepsilon_s$ in (4.3)

$$\begin{aligned}&\|c^\varepsilon_l(\tau)\|^2_{L^2(\Omega^\varepsilon_l)} + 2(d_l-\zeta_1)\int_0^\tau\|\nabla c^\varepsilon_l\|_{L^2(\Omega^\varepsilon_l)}dt \leq C_{\zeta_1}\|c^\varepsilon_{l,1}\|^2_{L^\infty((0,T)\times\Omega^\varepsilon_l)}\int_0^\tau\|\boldsymbol{v}^\varepsilon_{l,1} - \boldsymbol{v}^\varepsilon_{l,2}\|^2_{L^2(\Omega^\varepsilon_l)}dt\\ &+C_{\zeta_1}(1+\|H_M(\boldsymbol{v}^\varepsilon_{l,2})\|^2_{L^\infty(0,\tau;L^4(\Omega^\varepsilon_l))})\int_0^\tau\|c^\varepsilon_l\|^2_{L^2(\Omega^\varepsilon_l)}dt + C_{\zeta_2}\|c^\varepsilon_{l,1}\|^2_{L^\infty((0,T)\times\Gamma^\varepsilon_{zs})}\int_0^\tau\varepsilon\|\vartheta^\varepsilon_{f,l}\|^2_{L^2(\Gamma^\varepsilon_{zs})}dt\\ &+C_{\zeta_2}\int_0^\tau\varepsilon\|\vartheta^\varepsilon_{b,l-1}\|^2_{L^2(\Gamma^\varepsilon_{zs})}dt + \zeta_2\int_0^\tau\varepsilon\|c^\varepsilon_l\|^2_{L^2(\Gamma^\varepsilon_{zs})}dt\,,\end{aligned}$$

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for any $\tau \in (0,T]$, where $a-1:=s$ and $s-1:=a$. For $\vartheta^\varepsilon_{j,l}$, with $j=f,b$ and $l=a,s$, we obtain

$$\partial_t\|\vartheta^\varepsilon_{f,l}\|^2_{L^2(\Gamma^\varepsilon_{zs})} \le C\left[(1+\|c^\varepsilon_{l,1}\|_{L^\infty})\|\vartheta^\varepsilon_{f,l}\|^2_{L^2(\Gamma^\varepsilon_{zs})}+\|\vartheta^\varepsilon_{f,l,2}\|^2_{L^\infty}\|c^\varepsilon_l\|^2_{L^2(\Gamma^\varepsilon_{zs})}+\|\vartheta^\varepsilon_{b,l}\|^2_{L^2(\Gamma^\varepsilon_{zs})}\right],$$
$$\partial_t\|\vartheta^\varepsilon_{b,l}\|^2_{L^2(\Gamma^\varepsilon_{zs})} \le C\left[\|c^\varepsilon_{l,1}\|^2_{L^\infty}\|\vartheta^\varepsilon_{f,l}\|^2_{L^2(\Gamma^\varepsilon_{zs})}+\|\vartheta^\varepsilon_{f,l,2}\|^2_{L^\infty}\|c^\varepsilon_l\|^2_{L^2(\Gamma^\varepsilon_{zs})}+\|\vartheta^\varepsilon_{b,l}\|^2_{L^2(\Gamma^\varepsilon_{zs})}\right].$$

Combining those estimates, using (4.14) and (4.30), boundedness of c^ε_l and $H_M(\boldsymbol{v}^\varepsilon_l)$, and applying Gronwall's inequality yield

$$\varepsilon\|\vartheta^\varepsilon_{j,l}\|^2_{L^\infty(0,T;L^2(\Gamma^\varepsilon_{zs}))}+\|c^\varepsilon_l\|^2_{L^\infty(0,T;L^2(\Omega^\varepsilon_l))}+\|\boldsymbol{v}^\varepsilon_{l,1}-\boldsymbol{v}^\varepsilon_{l,2}\|^2_{L^2((0,T)\times\Omega^\varepsilon_l)}\le 0\,.$$

Thus, $c^\varepsilon_{l,1}=c^\varepsilon_{l,2}$, $\boldsymbol{v}^\varepsilon_{l,1}=\boldsymbol{v}^\varepsilon_{l,2}$ a.e. in $(0,T)\times\Omega^\varepsilon_l$, and $\vartheta^\varepsilon_{j,l,1}=\vartheta^\varepsilon_{j,l,2}$ a.e. on $[0,T]\times\Gamma^\varepsilon_{zs}$, with $l=a,s$ and $j=f,b$. □

5. Derivation of macroscopic equations

We denote by $\tilde{\boldsymbol{v}}^\varepsilon_i$, for $i=z,a,sp$ the extension by zero from $(0,T)\times\Omega^\varepsilon_i$ into $\Omega_T=(0,T)\times\Omega$, where $\Omega^\varepsilon_{sp}:=\Omega^\varepsilon_{as}$, and define $\boldsymbol{v}^\varepsilon=\tilde{\boldsymbol{v}}^\varepsilon_z+\tilde{\boldsymbol{v}}^\varepsilon_a+\tilde{\boldsymbol{v}}^\varepsilon_{sp}$ in Ω_T.

Lemma 5.1. *Under Assumption 1 there exist* $\boldsymbol{v}_z\in L^2(\Omega_T;H^1_{per}(Y_z)/\mathbb{R})$, $\boldsymbol{v}_a\in L^2(\Omega_T\times Y_a)$, $\boldsymbol{v}_{sp}\in L^2(\Omega_T\times Y_{as})$ *and* $p_a,p_s\in L^2(\Omega_T\times Y)$, *with* $\langle p_a(t,x,y),1\rangle_{\Omega\times Y_a}+\langle p_s(t,x,y),1\rangle_{\Omega\times Y_s}=0$ *for a.a.* $t\in(0,T)$, *such that, up to a subsequence,*

$$\begin{cases}\boldsymbol{v}^\varepsilon_z\to\boldsymbol{v}_z,\quad \boldsymbol{v}^\varepsilon_a\to\boldsymbol{v}_a,\quad \boldsymbol{v}^\varepsilon_{sp}\to\boldsymbol{v}_{sp} & \text{two-scale},\\ \boldsymbol{v}^\varepsilon_s\to\tilde{\boldsymbol{v}}_z+\tilde{\boldsymbol{v}}_{sp} & \text{two-scale in } \Omega_T\times Y_s,\\ \boldsymbol{v}^\varepsilon\to\overline{\boldsymbol{v}}=\tilde{\boldsymbol{v}}_z+\tilde{\boldsymbol{v}}_a+\tilde{\boldsymbol{v}}_{sp} & \text{two-scale in } \Omega_T\times Y,\\ \varepsilon\nabla\boldsymbol{v}^\varepsilon_z\to\nabla_y\boldsymbol{v}_z & \text{two-scale in } \Omega_T\times Y_z,\\ P^\varepsilon_a\to p_a,\ P^\varepsilon_s\to p_s & \text{two-scale in } \Omega_T\times Y,\\ p^\varepsilon_a\to p_a|_{\Omega_T\times Y_a}\ \text{two-scale in } \Omega_T\times Y_a, & p^\varepsilon_s\to p_s|_{\Omega_T\times Y_s}\ \text{two-scale in } \Omega_T\times Y_s,\end{cases}\tag{5.1}$$

and

$$\begin{cases}P^\varepsilon_a\to\frac{1}{|Y|}\int_Y p_a\,dy, \quad P^\varepsilon_s\to\frac{1}{|Y|}\int_Y p_s\,dy & \text{weakly in } L^2(\Omega_T),\\ \boldsymbol{v}^\varepsilon\to\boldsymbol{v}=\frac{1}{|Y|}\int_{Y_z}\boldsymbol{v}_z\,dy+\frac{1}{|Y|}\int_{Y_a}\boldsymbol{v}_a\,dy+\frac{1}{|Y|}\int_{Y_{as}}\boldsymbol{v}_{sp}\,dy & \text{weakly in } L^2(\Omega_T),\\ \boldsymbol{v}^\varepsilon_a\cdot\boldsymbol{n}\to\boldsymbol{v}_a\cdot\boldsymbol{n} & \text{two-scale on } \Omega_T\times\Gamma_{zs}.\end{cases}\tag{5.2}$$

Proof. The convergences in (5.1) follows directly from estimates (4.12), (4.13) and (4.25) together with Lemma 4.1 in [4] and the definition and compactness theorems for two-scale convergence, see [2, 29] or Definition 5.5 and Theorem 5.7 in Appendix. Since $\tilde{\boldsymbol{v}}^\varepsilon_i$, for $i=z,a,sp$, are zero in $\Omega\setminus\Omega^\varepsilon_i$, also the two-scale limits are equal to zero in $Y\setminus Y_i$, respectively. Applying two-scale limit in equality $\langle p^\varepsilon_a,1\rangle_{\Omega^\varepsilon_a}+\langle p^\varepsilon_s,1\rangle_{\Omega^\varepsilon_s}=0$ from the definition of the space $\mathcal{P}^\varepsilon$ we obtain the stated relation for mean values of p_a and p_s. A priori estimates (4.12), (4.13) and the relation between two-scale and weak limits, [2,4,29], ensure the first two convergences in (5.2). The uniform in ε boundedness of $\varepsilon^{1/2}\|\boldsymbol{v}^\varepsilon_a\cdot\boldsymbol{n}\|_{L^2((0,T)\times\Gamma^\varepsilon_{zs})}$ implies that there exists $w\in L^2(\Omega_T\times\Gamma_{zs})$ such that $\boldsymbol{v}^\varepsilon_a\cdot\boldsymbol{n}\to w$ two-scale, see [28] or Definition 5.8 and Theorem 5.9 in Appendix. Then $\operatorname{div}\boldsymbol{v}^\varepsilon_a=0$ in $(0,T)\times\Omega^\varepsilon_a$ and two-scale convergence of $\boldsymbol{v}^\varepsilon_a$ to $\boldsymbol{v}_a$ in $\Omega_T\times Y_a$ ensure $w=\boldsymbol{v}_a\cdot\boldsymbol{n}$ a.e. on $\Omega_T\times\Gamma_{zs}$. □

We consider the extension of c^ε_l from Ω^ε_l into Ω, for $l=a,s$, as in Lemma 4.4, and identify c^ε_l with its extension. We denote by $\mathcal{T}^\varepsilon_{\Gamma_{zs}}:\Gamma^\varepsilon_{zs}\to\Omega\times\Gamma_{zs}$ the boundary unfolding operator, see [10] or Definition 5.10 in Appendix. Here we shall use a shorted notation $\mathcal{T}^\varepsilon_\Gamma(\psi):=\mathcal{T}^\varepsilon_{\Gamma_{zs}}(\psi)$.

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Lemma 5.2. *Under Assumption 1 there exist functions* $c_l \in H^1(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T)$, $c_l^1 \in L^2(\Omega_T; H^1_{per}(Y_l)/\mathbb{R})$ *and* $\vartheta_{j,l} \in L^\infty(\Omega_T \times \Gamma_{zs}) \cap H^1(0,T;L^2(\Omega \times \Gamma_{zs}))$, *with* $j = f,b$ *and* $l = a,s$, *such that, up to a subsequence,*

$$\begin{aligned} c_l^\varepsilon \to c_l \text{ weakly in } H^1(0,T;H^1(\Omega)) , &\quad \text{weakly-}* \text{ in } L^\infty(\Omega_T) , \\ c_l^\varepsilon \to c_l \text{ strongly in } L^2(0,T;H^\varsigma(\Omega)) , \ \varsigma<1 , &\quad \varepsilon^{1/2}\|c_l^\varepsilon - c_l\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} \to 0 , \\ c_l^\varepsilon \to c_l , \ \partial_t c_l^\varepsilon \to \partial_t c_l , \ \nabla c_l^\varepsilon \to \nabla c_l + \nabla_y c_l^1 &\quad \text{two-scale} , \\ \vartheta_{j,l}^\varepsilon \to \vartheta_{j,l} , \ \partial_t \vartheta_{j,l}^\varepsilon \to \partial_t \vartheta_{j,l} &\quad \text{two-scale} , \\ \mathcal{T}_\Gamma^\varepsilon c_l^\varepsilon \to c_l , \ \mathcal{T}_\Gamma^\varepsilon \vartheta_{j,l}^\varepsilon \to \vartheta_{j,l} &\quad \text{strongly in } L^2(\Omega_T \times \Gamma_{zs}) . \end{aligned} \tag{5.3}$$

Proof. The first two convergences are a direct consequence of the a priori estimates for c_l^ε in (4.25)-(4.26) and properties of the extension from Ω_l^ε into Ω, with $l = a,s$.

The compact embedding $H^1(0,T;H^1(\Omega)) \subset L^2(0,T;H^\varsigma(\Omega))$ for $\varsigma<1$ ensures the strong convergence of c_a^ε and c_s^ε. Then, the strong convergence in $L^2(0,T;H^\varsigma(\Omega))$, with $1/2<\varsigma<1$, and the estimate

$$\varepsilon^{1/2}\|c_l^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} \le C\|c_l^\varepsilon\|_{L^2(0,T;H^\varsigma(\Omega_l^\varepsilon))} , \qquad \text{with } l = a,s ,$$

see [27] for the proof, give the fourth convergence in (5.3). A priori estimates (4.25), (4.26) and compactness theorems for two-scale convergence, see [2, 28, 29] or Theorem 5.7 and 5.9 in Appendix, imply two-scale convergences stated in (5.3). Theorem 5.9 in Appendix ensures also that $\vartheta_{j,l} \in L^\infty(\Omega_T \times \Gamma_{zs})$.

The assumed structure of the initial data $\vartheta_{j,l}^{0,\varepsilon}$, i.e. $\mathcal{T}_\Gamma^\varepsilon(\vartheta_{j,l}^{0,\varepsilon})(x,y) = \vartheta_{2j,l}^0(y)\mathcal{T}_\Gamma^\varepsilon(\vartheta_{1j,l}^0(x))$ and the strong convergence in $L^2(\Omega\times\Gamma_{zs})$ of $\mathcal{T}_\Gamma^\varepsilon(\psi)$ for $\psi \in L^2(\Omega)$, see [10], yield $\mathcal{T}_\Gamma^\varepsilon(\vartheta_{j,l}^{0,\varepsilon}) \to \vartheta_{j,l}^0$ strongly in $L^2(\Omega\times\Gamma_{zs})$, with $j = f,b$ and $l = a,s$. The properties of the unfolding operator, see [10], give

$$\|\mathcal{T}_\Gamma^\varepsilon(c_l^\varepsilon) - \mathcal{T}_\Gamma^\varepsilon(c_l)\|_{L^2(\Omega_T\times\Gamma_{zs})} \le C\varepsilon^{1/2}\|c_l^\varepsilon - c_l\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)} \quad \text{for } l = a,s ,$$

and $\mathcal{T}_\Gamma^\varepsilon(c_l) \to c_l$ strongly in $L^2(\Omega_T\times\Gamma_{zs})$. Then, together with the fourth convergence in (5.3) we obtain that $\mathcal{T}_\Gamma^\varepsilon(c_l^\varepsilon) \to c_l$ strongly in $L^2(\Omega_T\times\Gamma_{zs})$, where $l = a,s$. Applying now the unfolding operator to the equations in (3.5), using the convergence of c_a^ε, c_s^ε, and the equivalence between the two-scale convergence of a sequence and the weak convergence of the corresponding unfolded sequence, see [27] or Lemma 5.11 in Appendix, we can show, in the same manner as in [27], that $\{\mathcal{T}_\Gamma^\varepsilon(\vartheta_{j,l}^\varepsilon)\}$, with $j = f,b$ and $l = a,s$, are Cauchy sequences in $L^2(\Omega_T\times\Gamma_{zs})$ and conclude the last strong convergences stated in (5.3). □

Theorem 5.3. *For the sequence of solutions* $(\boldsymbol{v}_z^\varepsilon, \boldsymbol{v}_a^\varepsilon, \boldsymbol{v}_{sp}^\varepsilon)$ *and* $(p_z^\varepsilon, p_a^\varepsilon, p_{sp}^\varepsilon)$ *of microscopic problem* (3.1), (3.8), (3.10), *we have* $\boldsymbol{v}^\varepsilon \to \boldsymbol{v}$ *weakly in* $L^2(0,T;H(\mathrm{div},\Omega))$, *with* $\boldsymbol{v}^\varepsilon = \tilde{\boldsymbol{v}}_z^\varepsilon + \tilde{\boldsymbol{v}}_{sp}^\varepsilon + \tilde{\boldsymbol{v}}_a^\varepsilon$, *extensions* $P_s^\varepsilon \to p$ *and* $P_a^\varepsilon \to p$ *weakly in* $L^2(0,T;L^2(\Omega)/\mathbb{R})$ *as* $\varepsilon \to 0$, *and* $(\boldsymbol{v},p) \in L^2(0,T;H(\mathrm{div},\Omega)) \times L^2(0,T;L^2_0(\Omega))$, *where* $L^2_0(\Omega) = \{\phi \in L^2(\Omega) : \langle\phi,1\rangle_\Omega = 0\}$, *is the unique solution of the Darcy problem*

$$\begin{cases} \boldsymbol{v} + \boldsymbol{K}\nabla p = \boldsymbol{M}(c_s - c_a) & \text{in } (0,T)\times\Omega , \\ \mathrm{div}\,\boldsymbol{v} = 0 & \text{in } (0,T)\times\Omega , \\ \boldsymbol{v}\cdot\boldsymbol{n} = v_D & \text{on } (0,T)\times\partial\Omega , \end{cases} \tag{5.4}$$

where the tensor $\boldsymbol{K}$ *and vector* $\boldsymbol{M}$ *are defined in* (5.13), *and* c_s, c_a *are solutions of the macroscopic equations* (5.14)-(5.16).

Proof. The weak convergences of $\boldsymbol{v}^\varepsilon$ in $L^2(\Omega_T)$ and of P_l^ε in $L^2(0,T;L^2(\Omega)/\mathbb{R})$, with $l = a,s$, follows from Lemma 5.1. Using $\mathrm{div}\,\boldsymbol{v}^\varepsilon \in L^2(\Omega_T)$ and $\mathrm{div}\,\boldsymbol{v}^\varepsilon = 0$ in Ω_T, and applying the weak convergence of $\boldsymbol{v}^\varepsilon$ we obtain for $\psi \in C_0^\infty(\Omega_T)$

$$0 = \lim_{\varepsilon\to0}\langle\mathrm{div}\,\boldsymbol{v}^\varepsilon,\psi\rangle_{\Omega,T} = \lim_{\varepsilon\to0}\langle\boldsymbol{v}^\varepsilon,\nabla\psi\rangle_{\Omega,T} = \langle\boldsymbol{v},\nabla\psi\rangle_{\Omega,T} = \langle\mathrm{div}\,\boldsymbol{v},\psi\rangle_{\Omega,T} . \tag{5.5}$$

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For $\psi \in C^\infty(\Omega_T)$ we have

$$0 = \lim_{\varepsilon\to 0}\langle \operatorname{div} \boldsymbol{v}^\varepsilon, \psi\rangle_{\Omega,T} = \lim_{\varepsilon\to 0}\left(\langle \boldsymbol{v}^\varepsilon\cdot\boldsymbol{n},\psi\rangle_{\partial\Omega,T} - \langle \boldsymbol{v}^\varepsilon, \nabla\psi\rangle_{\Omega,T}\right) = \langle v_D,\psi\rangle_{\partial\Omega,T} - \langle \boldsymbol{v},\nabla\psi\rangle_{\Omega,T}\ , \quad (5.6)$$

and, together with $\operatorname{div}\boldsymbol{v}=0$ in Ω_T, given by (5.5), conclude $\boldsymbol{v}\cdot\boldsymbol{n} = v_D$ on $(0,T)\times\partial\Omega$.

Considering $\psi_1,\psi_2 \in C_0^\infty(\Omega_T; C^\infty_{per}(Y))$ with $\psi_1(t,x,y)=0$ in $Y\setminus Y_s$, such that $\psi_1\times\boldsymbol{n}=0$ on Γ_{as}, and $\psi_2(t,x,y)=0$ in $Y\setminus Y_a$, for $(t,x)\in\Omega_T$, taking $\psi^\varepsilon(t,x)=\varepsilon(\psi_1(t,x,x/\varepsilon),\psi_2(t,x,x/\varepsilon),\psi_1(t,x,x/\varepsilon))$ as a test function in (4.1) and applying two-scale convergences of $\boldsymbol{v}_j^\varepsilon$, with $j=z,a,sp$ and of p_l^ε, with $l=a,s$, stated in Lemma 5.1, yield

$$\langle p_s, \operatorname{div}_y\psi_1\rangle_{\Omega\times Y_s,T} + \langle p_a, \operatorname{div}_y\psi_2\rangle_{\Omega\times Y_a,T} = 0\ .$$

This implies that $p_l\in L^2(\Omega_T;H^1(Y_l))$ with $\nabla_y p_l = 0$ a.e. in $\Omega_T\times Y_l$, where $l=a,s$. Thus p_l is independent of y in $\Omega_T\times Y_l$ and $p_l = p_l(t,x)$ a.e. in $\Omega_T\times Y_l$, where $l=a,s$. Taking now $\psi^\varepsilon(t,x) = \varepsilon(\psi_1(t,x,x/\varepsilon),\psi_1(t,x,x/\varepsilon),0)$ with $\psi_1\in C_0^\infty(\Omega_T;C^\infty_{per}(Y))$ and $\psi_1(t,x,y)=0$ in $\overline{Y}_{as}$ and $\psi_1\times\boldsymbol{n}=0$ on Γ_z as a test function in (4.1) and letting $\varepsilon\to 0$ give

$$\langle p_a(t,x)-p_s(t,x), \psi_1(t,x,y)\cdot\boldsymbol{n}\rangle_{\Omega\times\Gamma_z,T} = 0\ .$$

Thus $p_a(t,x)=p_s(t,x)$ a.e. in $\Omega_T\times\Gamma_z$ and $p_a(t,x)=p_s(t,x)=p(t,x)$ a.e. in Ω_T. From the relation $\langle p_a,1\rangle_{\Omega\times Y_a}+\langle p_s,1\rangle_{\Omega\times Y_s}=0$ in Lemma 5.1 we obtain that $\langle p(t,x),1\rangle_\Omega = 0$ for a.a. $t\in(0,T)$.

Now we consider $\psi^\varepsilon(t,x) = (\psi_1(t,x,x/\varepsilon)+\psi_2(t,x,x/\varepsilon),\psi_1(t,x,x/\varepsilon),\psi_2(t,x,x/\varepsilon))$, where $\psi_1,\psi_2\in C_0^\infty(\Omega_T, C^\infty_{per}(Y))$ with $\psi_2\cdot\boldsymbol{n}=0$ on $\Gamma_z\cup\Gamma_{aw}$ and $\psi_2\times\boldsymbol{n}=-\psi_1\times\boldsymbol{n}$ on Γ_{zs} and $\operatorname{div}_y\psi_1=0$ in $\Omega_T\times Y$, $\operatorname{div}_y\psi_2=0$ in $\Omega_T\times Y_s$ as a test function in (4.1). The two-scale convergences in (5.1) and (5.2), and the convergence of c_a^ε and c_s^ε on $(0,T)\times\Gamma_{zs}^\varepsilon$ stated in (5.3) imply

$$\begin{aligned}\langle 2\eta\,\mathrm{S}_y\boldsymbol{v}_z, \mathrm{S}_y(\psi_1+\psi_2)\rangle_{Y_z\times\Omega_T} - \langle p, \operatorname{div}_x(\psi_1+\psi_2)\rangle_{Y_z\times\Omega_T} + \langle K_a^{-1}(y)\boldsymbol{v}_a,\psi_1\rangle_{Y_a\times\Omega_T}\\ -\langle p,\operatorname{div}_x\psi_1\rangle_{Y_a\times\Omega_T} + \langle K_{sp}^{-1}(y)\boldsymbol{v}_{sp},\psi_2\rangle_{Y_{as}\times\Omega_T} - \langle p,\operatorname{div}_x\psi_2\rangle_{Y_{as}\times\Omega_T}\\ +\langle\delta(y)(c_s-c_a)+\kappa(y)\boldsymbol{v}_a\cdot\boldsymbol{n},\psi_1\cdot\boldsymbol{n}\rangle_{\Gamma_{zs}\times\Omega_T}=0\ ,\end{aligned} \quad (5.7)$$

where $\mathrm{S}_y\boldsymbol{v} = 1/2(\nabla_y\boldsymbol{v}+\nabla_y\boldsymbol{v}^T)$. Choosing $\psi_1=0$, restricting ψ_2 to Y_z, i.e. $\psi_2\in C_0^\infty(\Omega_T,C_0^\infty(Y_z))$, and applying the integration by parts imply

$$\langle -2\eta\operatorname{div}_y(\mathrm{S}_y\boldsymbol{v}_z)+\nabla_x p,\psi_2\rangle_{\Omega_T\times Y_z}=0\ .$$

Since $\operatorname{div}_y\psi_2(t,x,y)=0$ in $\Omega_T\times Y_z$, there exists $p_{1,z}\in L^2(\Omega_T,L^2_{\mathrm{per}}(Y_z)/\mathbb{R})$, see [17,19], such that

$$-2\eta\operatorname{div}_y(\mathrm{S}_y\boldsymbol{v}_z)+\nabla_x p+\nabla_y p_{1,z}=0 \quad \text{in } \Omega_T\times Y_z\ . \quad (5.8)$$

Similarly we obtain the existence of $p_{1,a}\in L^2(\Omega_T,L^2_{\mathrm{per}}(Y_a)/\mathbb{R})$, $p_{1,sp}\in L^2(\Omega_T,L^2_{\mathrm{per}}(Y_{as})/\mathbb{R})$ and

$$\begin{aligned}K_a^{-1}\boldsymbol{v}_a+\nabla_x p+\nabla_y p_{1,a}=0 \quad &\text{in } \Omega_T\times Y_a\ ,\\ K_{sp}^{-1}\boldsymbol{v}_{sp}+\nabla_x p+\nabla_y p_{1,sp}=0 \quad &\text{in } \Omega_T\times Y_{as}\ .\end{aligned} \quad (5.9)$$

Considering $\operatorname{div}\boldsymbol{v}_j^\varepsilon\in L^2((0,T)\times\Omega_j^\varepsilon)$ and $\operatorname{div}\boldsymbol{v}_j^\varepsilon=0$ in $(0,T)\times\Omega_j^\varepsilon$, with $j=z,a,sp$ and $\Omega_{sp}^\varepsilon:=\Omega_{as}^\varepsilon$, we obtain for $\psi\in C_0^\infty((0,T)\times\Omega;C_0^\infty(Y_j))$, where $Y_{sp}:=Y_{as}$,

$$0=\lim_{\varepsilon\to0}\langle\operatorname{div}\boldsymbol{v}_j^\varepsilon,\psi(t,x,x/\varepsilon)\rangle_{\Omega_j^\varepsilon,T} = \lim_{\varepsilon\to0}\langle\boldsymbol{v}_j^\varepsilon,\nabla_x\psi(t,x,x/\varepsilon)+\varepsilon^{-1}\nabla_y\psi(t,x,x/\varepsilon)\rangle_{\Omega_j^\varepsilon,T}\ .$$

The two-scale convergence of $\boldsymbol{v}_j^\varepsilon$ ensures $\lim_{\varepsilon\to0}\langle\boldsymbol{v}_j^\varepsilon(t,x),\nabla_y\psi(t,x,x/\varepsilon)\rangle_{\Omega_j^\varepsilon,T}=0$ and then implies that $\operatorname{div}_y\boldsymbol{v}_z=0$ in $\Omega_T\times Y_z$, $\operatorname{div}_y\boldsymbol{v}_{sp}=0$ in $\Omega_T\times Y_{as}$, and $\operatorname{div}_y\boldsymbol{v}_a=0$ in $\Omega_T\times Y_a$. Similarly, using $\operatorname{div}\boldsymbol{v}^\varepsilon\in L^2(\Omega_T)$ and $\operatorname{div}\boldsymbol{v}^\varepsilon=0$ in Ω_T, the two-scale convergence of $\boldsymbol{v}^\varepsilon$ ensures $\operatorname{div}_y\overline{\boldsymbol{v}}=0$ in $\Omega_T\times Y$.

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Considering the two-scale convergence of $\boldsymbol{v}_z^\varepsilon$, $\boldsymbol{v}_a^\varepsilon$, $\boldsymbol{v}_{sp}^\varepsilon$, and using the calculations similar to (5.5)-(5.6) we can conclude $\boldsymbol{v}_z \cdot \boldsymbol{n} = (\boldsymbol{v}_a + \boldsymbol{v}_{sp}) \cdot \boldsymbol{n}$ on $\Omega_T \times \Gamma_{as}$ and $\boldsymbol{v}_z \cdot \boldsymbol{n} = \boldsymbol{v}_a \cdot \boldsymbol{n}$ on $\Omega_T \times \Gamma_z$.

Inequality (4.30), applied to $\boldsymbol{v}_z^\varepsilon$, with Γ_{zs}^ε and Ω_z^ε, and the estimates for $\|\boldsymbol{v}_z^\varepsilon\|_{L^2((0,T)\times\Omega_z^\varepsilon)}$ and $\varepsilon\|\nabla \boldsymbol{v}_z^\varepsilon\|_{L^2((0,T)\times\Omega_z^\varepsilon)}$ in (4.12) ensure the uniform in ε boundedness of $\varepsilon^{1/2}\|\boldsymbol{v}_z^\varepsilon\|_{L^2((0,T)\times\Gamma_{zs}^\varepsilon)}$. Then the compactness theorem for the two-scale convergence on oscillating boundaries, see [28] or Theorem 5.9 in Appendix, together with the two-scale convergence of $\varepsilon\nabla \boldsymbol{v}_z^\varepsilon$ and $\operatorname{div} \boldsymbol{v}_z^\varepsilon = 0$, as well as $\operatorname{div}_y \boldsymbol{v}_z = 0$, implies the two-scale convergence of $\boldsymbol{v}_z^\varepsilon$ to $\boldsymbol{v}_z$ on $\Omega_T \times \Gamma_{zs}$ and ensures $\boldsymbol{v}_z \times \boldsymbol{n} = 0$ on $\Omega_T \times \Gamma_{zs}$.

Applying in (5.7) integration by parts and accounting equations (5.8) and (5.9) yield

$$\begin{aligned}\langle 2\eta(\mathrm{S}_y \boldsymbol{v}_z \boldsymbol{n}) \cdot \boldsymbol{n} - p_{1,z}, \psi_1 \cdot \boldsymbol{n}\rangle_{\Omega_T\times\Gamma_z} + \langle 2\eta(\mathrm{S}_y \boldsymbol{v}_z \boldsymbol{n}) \cdot \boldsymbol{n} - p_{1,z}, (\psi_1 + \psi_2) \cdot \boldsymbol{n}\rangle_{\Omega_T\times\Gamma_{as}} \\ + \langle p_{1,a} + \delta(y)(c_s - c_a) + \kappa(y)\boldsymbol{v}_a \cdot \boldsymbol{n}, \psi_1 \cdot \boldsymbol{n}\rangle_{\Omega_T\times\Gamma_{zs}} + \langle p_{1,sp}, \psi_2 \cdot \boldsymbol{n}\rangle_{\Omega_T\times\Gamma_{as}} = 0\end{aligned}$$

for $\psi_1, \psi_2 \in C_0^\infty(\Omega_T, C_{per}^\infty(Y))$ with $\psi_2 \cdot \boldsymbol{n} = 0$ on $\Gamma_z \cup \Gamma_{aw}$ and $\psi_2 \times \boldsymbol{n} = -\psi_1 \times \boldsymbol{n}$ on Γ_{zs}, $\operatorname{div}_y \psi_1 = 0$ in $\Omega_T \times Y$, $\operatorname{div}_y \psi_2 = 0$ in $\Omega_T \times Y_s$. Thus, we conclude that $\boldsymbol{v}_z \in L^2(\Omega_T; H^1_{\mathrm{per}}(Y_z)/\mathbb{R})$, $\boldsymbol{v}_{sp} \in L^2(\Omega_T \times Y_{as})$, $\boldsymbol{v}_a \in L^2(\Omega_T \times Y_a)$, and $p \in L^2(0,T; L_0^2(\Omega))$, $p_{1,z} \in L^2(\Omega_T; L^2_{\mathrm{per}}(Y_z)/\mathbb{R})$, $p_{1,sp} \in L^2(\Omega_T; L^2_{\mathrm{per}}(Y_{as})/\mathbb{R})$, and $p_{1,a} \in L^2(\Omega_T; L^2_{\mathrm{per}}(Y_a)/\mathbb{R})$, satisfy equations

$$\begin{cases} -\eta \Delta_y \boldsymbol{v}_z + \nabla_x p + \nabla_y p_{1,z} = 0 & \text{in } \Omega_T \times Y_z\,, \\ K_a^{-1}\boldsymbol{v}_a + \nabla_x p + \nabla_y p_{1,a} = 0 & \text{in } \Omega_T \times Y_a\,, \\ K_{sp}^{-1}\boldsymbol{v}_{sp} + \nabla_x p + \nabla_y p_{1,sp} = 0 & \text{in } \Omega_T \times Y_{as}\,, \\ \operatorname{div}_y \boldsymbol{v}_z = 0 \ \text{ in } \Omega_T \times Y_z\,, \quad \operatorname{div}_y \boldsymbol{v}_a = 0 \ \text{ in } \Omega_T \times Y_a\,, \quad \operatorname{div}_y \boldsymbol{v}_{sp} = 0 & \text{in } \Omega_T \times Y_{as}\,, \end{cases} \tag{5.10}$$

with boundary and transmission conditions

$$\begin{cases} -2\eta(\mathrm{S}_y \boldsymbol{v}_z \boldsymbol{n}) \cdot \boldsymbol{n} + p_{1,z} - p_{1,a} = \delta(y)(c_s - c_a) + \kappa(y)\boldsymbol{v}_a \cdot \boldsymbol{n} & \text{on } \Omega_T \times \Gamma_{zs}\,, \\ -2\eta(\mathrm{S}_y \boldsymbol{v}_z \boldsymbol{n}) \cdot \boldsymbol{n} + p_{1,z} = p_{1,sp} & \text{on } \Omega_T \times \Gamma_{as}\,, \\ \boldsymbol{v}_z \cdot \boldsymbol{n} = \boldsymbol{v}_a \cdot \boldsymbol{n} + \boldsymbol{v}_{sp} \cdot \boldsymbol{n}\,,\ \boldsymbol{v}_z \times \boldsymbol{n} = 0 & \text{on } \Omega_T \times \Gamma_{as}\,, \\ \boldsymbol{v}_z \cdot \boldsymbol{n} = \boldsymbol{v}_a \cdot \boldsymbol{n}\,, \quad \boldsymbol{v}_z \times \boldsymbol{n} = 0 & \text{on } \Omega_T \times \Gamma_z\,, \\ \boldsymbol{v}_{sp} \cdot \boldsymbol{n}_{aw} = 0 & \text{on } \Omega_T \times \Gamma_{aw}\,. \end{cases} \tag{5.11}$$

Considering the structure, linearity, and uniqueness of a solution of equations (5.10) and (5.11), the proof of which follows the same lines as for microscopic model (4.1), we can express $\boldsymbol{v}_l$ and $p_{1,l}$, with $l = z, a$ or sp, in the form

$$\begin{cases} \boldsymbol{v}_z = -\sum\limits_{i=1}^3 \partial_{x_i} p\ \boldsymbol{w}_z^i + (c_s - c_a)\boldsymbol{r}_z\,, \quad \boldsymbol{v}_a = -\sum\limits_{i=1}^3 \partial_{x_i} p\ \boldsymbol{w}_a^i + (c_s - c_a)\boldsymbol{r}_a\,, \quad \boldsymbol{v}_{sp} = -\sum\limits_{i=1}^3 \partial_{x_i} p\ \boldsymbol{w}_{sp}^i\,, \\ p_{1,z} = -\sum\limits_{i=1}^3 \partial_{x_i} p\ \pi_z^i + (c_s - c_a)\zeta_z\,, \quad p_{1,a} = -\sum\limits_{i=1}^3 \partial_{x_i} p\ \pi_a^i + (c_s - c_a)\zeta_a\,, \quad p_{1,sp} = -\sum\limits_{i=1}^3 \partial_{x_i} p\ \pi_{sp}^i\,, \end{cases} \tag{5.12}$$

where $\boldsymbol{w}_l^i$, π_l^i, with $l = z, a$ and sp, are solutions of the unit cell problems

$$\begin{cases} -\eta \Delta_y \boldsymbol{w}_z^i + \nabla_y \pi_z^i = e_i & \text{in } Y_z\,, \\ K_a^{-1}\boldsymbol{w}_a^i + \nabla_y \pi_a^i = e_i & \text{in } Y_a\,, \\ K_{sp}^{-1}\boldsymbol{w}_{sp}^i + \nabla_y \pi_{sp}^i = e_i & \text{in } Y_{as}\,, \\ \operatorname{div}_y \boldsymbol{w}_z^i = 0 \quad \text{in } Y_z\,, \quad \operatorname{div}_y \boldsymbol{w}_a^i = 0 \quad \text{in } Y_a\,, \quad \operatorname{div}_y \boldsymbol{w}_{sp}^i = 0 \quad \text{in } Y_{as}\,, \end{cases}$$

with transmission conditions

$$\begin{cases} -2\eta(\mathrm{S}_y \boldsymbol{w}_z^i \boldsymbol{n}) \cdot \boldsymbol{n} + \pi_z^i - \pi_a^i = \kappa(y)\, \boldsymbol{w}_a^i \cdot \boldsymbol{n} & \text{on } \Gamma_{zs}\,, \\ -2\eta(\mathrm{S}_y \boldsymbol{w}_z^i \boldsymbol{n}) \cdot \boldsymbol{n} + \pi_z^i = \pi_{sp}^i & \text{on } \Gamma_{as}\,, \\ \boldsymbol{w}_z^i \cdot \boldsymbol{n} = (\boldsymbol{w}_a^i + \boldsymbol{w}_{sp}^i) \cdot \boldsymbol{n}\,, \quad \boldsymbol{w}_z^i \times \boldsymbol{n} = 0 & \text{on } \Gamma_{as}\,, \\ \boldsymbol{w}_z^i \cdot \boldsymbol{n} = \boldsymbol{w}_a^i \cdot \boldsymbol{n}\,, \quad \boldsymbol{w}_z^i \times \boldsymbol{n} = 0 & \text{on } \Gamma_z\,, \\ \boldsymbol{w}_{sp}^i \cdot \boldsymbol{n}_{aw} = 0 & \text{on } \Gamma_{aw}\,, \end{cases}$$

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and $\boldsymbol{r}_l$, ζ_l, with $l = z$ and a, solve the following problem

$$\begin{cases} -\eta \Delta_y \boldsymbol{r}_z + \nabla_y \zeta_z = 0 & \text{in } Y_z\ , \\ K_a^{-1} \boldsymbol{r}_a + \nabla_y \zeta_a = 0 & \text{in } Y_a\ , \\ -2\eta (\mathrm{S}_y \boldsymbol{r}_z \boldsymbol{n}) \cdot \boldsymbol{n} + \zeta_z - \zeta_a = \delta(y) + \kappa(y)\, \boldsymbol{r}_a \cdot \boldsymbol{n} & \text{on } \Gamma_{zs}\ , \\ \boldsymbol{r}_a \cdot \boldsymbol{n} = \boldsymbol{r}_z \cdot \boldsymbol{n}\ , \qquad \boldsymbol{r}_z \times \boldsymbol{n} = 0 & \text{on } \Gamma_{zs}\ , \\ \mathrm{div}_y\, \boldsymbol{r}_z = 0 \quad \text{in } Y_z\ , \qquad \mathrm{div}_y\, \boldsymbol{r}_a = 0 & \text{in } Y_a\ . \end{cases}$$

Averaging the expressions in (5.12) over Y_z, Y_a and Y_{as}, respectively, using the definition of $\boldsymbol{v}$ and defining the vector $\boldsymbol{M}$ and the permeability tensor $\boldsymbol{K} = (K_{ij})_{1\le i,j\le 3}$ by

$$\boldsymbol{M} = \frac{1}{|Y|}\Big(\int_{Y_z} \boldsymbol{r}_z dy + \int_{Y_a} \boldsymbol{r}_a dy \Big)\ , \quad K_{ij} = \frac{1}{|Y|}\Big(\int_{Y_z} \boldsymbol{w}^i_{z,j} dy + \int_{Y_a} \boldsymbol{w}^i_{a,j} dy + \int_{Y_{as}} \boldsymbol{w}^i_{sp,j} dy \Big)\ , \quad (5.13)$$

we obtain the equation for $\boldsymbol{v}$ stated in (5.4). □

For concentrations of a osmotically active solute and transporters on the cell membrane we obtain the following macroscopic problem

Theorem 5.4. *The sequence of solutions of the microscopic model* (3.2)-(3.7), (3.9) *converges to a unique solution* $c_l \in H^1(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T)$, $\vartheta_{j,l} \in H^1(0,T;L^2(\Omega \times \Gamma_{zs})) \cap L^\infty(\Omega_T \times \Gamma_{zs})$, *with* $l = a, s$, $j = f, b$, *of the initial boundary values problem in* $\Omega_T = (0,T) \times \Omega$

$$\begin{cases} \partial_t c_a - \mathrm{div}(\mathcal{A}_a \nabla c_a - \hat{H}_M(\boldsymbol{v}_a)\, c_a) = \frac{1}{|Y_a|}\langle F_a(t,y,c_a), 1\rangle_{Y_a} + \frac{1}{|Y_a|}\langle \beta_s(t,y)\vartheta_{b,s} - c_a \alpha_a(t,y)\vartheta_{f,a}, 1\rangle_{\Gamma_{zs}}, \\ (\mathcal{A}_a \nabla c_a - \hat{H}_M(\boldsymbol{v}_a)\, c_a) \cdot \boldsymbol{n} = 0 \qquad \text{on } (0,T) \times \partial\Omega, \\ c_a(0,x) = c_a^0(x) \qquad \text{in } \Omega, \end{cases} \quad (5.14)$$

and

$$\begin{cases} \partial_t c_s - \mathrm{div}(\mathcal{A}_s \nabla c_s - \hat{H}_M(\boldsymbol{v}_s)\, c_s) = \frac{1}{|Y_s|}\langle F_s(t,y,c_s), 1\rangle_{Y_s} + \frac{1}{|Y_s|}\langle \beta_a(t,y)\vartheta_{b,a} - c_s \alpha_s(t,y)\vartheta_{f,s}, 1\rangle_{\Gamma_{zs}}, \\ (\mathcal{A}_s \nabla c_s - \hat{H}_M(\boldsymbol{v}_s)\, c_s) \cdot \boldsymbol{n} = 0 \qquad \text{on } (0,T) \times \partial\Omega\ , \\ c_s(0,x) = c_s^0(x) \qquad \text{in } \Omega\ , \end{cases} \quad (5.15)$$

where homogenized diffusion coefficients $\mathcal{A}_a$, $\mathcal{A}_s$ *and velocity fields* $\hat{H}_M(\boldsymbol{v}_a)$, $\hat{H}_M(\boldsymbol{v}_s)$ *are defined by* (5.20) *with* $\boldsymbol{v}_a$ *and* $\boldsymbol{v}_s$ *given as solutions of* (5.10), (5.11), *and transporter concentrations satisfy ordinary differential equations, for* $l = a, s$,

$$\begin{cases} \partial_t \vartheta_{f,l} = R_l(t,y,\vartheta_{f,l}) - \alpha_l(t,y) c_l \vartheta_{f,l} + \beta_l(t,y)\vartheta_{b,l} - \gamma_{f,l}(t,y)\vartheta_{f,l} & \text{on } \Omega_T \times \Gamma_{zs}\ , \\ \partial_t \vartheta_{b,l} = \qquad \alpha_l(t,y) c_l \vartheta_{f,l} - \beta_l(t,y)\vartheta_{b,l} - \gamma_{b,l}(t,y)\vartheta_{b,l} & \text{on } \Omega_T \times \Gamma_{zs}\ , \\ \vartheta_{f,l}(0,x,y) = \vartheta^0_{f,l}(x,y),\ \vartheta_{b,l}(0,x,y) = \vartheta^0_{b,l}(x,y) & \text{on } \Omega \times \Gamma_{zs}\ . \end{cases} \quad (5.16)$$

Proof. To derive macroscopic equations we shall apply two-scale and strong convergences stated in Lemmata 5.1 and 5.2. The Lipschitz continuity of F_l and the strong L^2-convergence of c_l^ε imply $F_l^\varepsilon(t,x,c_l^\varepsilon) \to F_l(t,y,c_l)$ two-scale in $\Omega_T \times Y_l$, with $l = a, s$. Taking $\varphi(t,x) = \psi_1(t,x) + \varepsilon\psi_2(t,x,x/\varepsilon)$ with $\psi_1 \in L^2(0,T;H^1(\Omega))$ and $\psi_2 \in C_0^\infty(\Omega_T; C^\infty_{per}(Y))$ as a test function in (4.2) and (4.3), applying two-scale convergences and strong convergence in $L^2(\Omega_T)$ of c_l^ε, together with two-scale convergences for $\boldsymbol{v}_l^\varepsilon$ and $\vartheta_{j,l}^\varepsilon$, where $j = f, b$ and $l = a, s$, and the linearity of H_M, considering appropriate subsequences and passing to the limit as $\varepsilon \to 0$, imply

$$\begin{aligned} |Y_l|\langle \partial_t c_l, \psi_1\rangle_{\Omega_T} + \langle D_l(t,y)(\nabla c_l + \nabla_y c_l^1) - H_M(\boldsymbol{v}_l(t,x,y)) c_l, \nabla\psi_1 + \nabla_y\psi_2\rangle_{\Omega_T \times Y_l} \\ = \langle \beta_{l-1}(t,y)\vartheta_{b,l-1} - \alpha_l(t,y) c_l\, \vartheta_{f,l}, \psi_1\rangle_{\Omega_T \times \Gamma_{zs}} + \langle F_l(t,y,c_l), \psi_1\rangle_{\Omega_T \times Y_l}\ . \end{aligned} \quad (5.17)$$

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where $l = a, s$, with $a - 1 := s$ and $s - 1 := a$. Choosing now $\psi_1 = 0$ in (5.17) we obtain

$$\langle D_l(t,y)(\nabla c_l + \nabla_y c_l^1) - H_M(\boldsymbol{v}_l)\, c_l, \nabla_y \psi_2 \rangle_{\Omega_T \times Y_l} = 0 \,, \qquad \text{with } \; l = a, s \,,$$

for all $\psi_2 \in C_0^\infty(\Omega_T; C^\infty_{\rm per}(Y))$ and can conclude that c_l^1 depends linearly on ∇c_l and c_l. Thus we can consider an Ansatz

$$c_l^1(t,x,y) = \sum_{i=1}^3 \frac{\partial c_l(t,x)}{\partial x_i} \omega_l^i(y) + c_l(t,x) z_l(t,x,y) \,,$$

where $l = a, s$, and ω_l^i and z_l are solutions of the cell problems

$$\begin{cases} -\operatorname{div}_y(D_l(t,y)\nabla_y \omega_l^i) = \sum_{k=1}^3 \partial_{y_k} D_{l,ki}(t,y) & \text{in } (0,T) \times Y_l \,, \\ -D_l(t,y)\nabla_y \omega_l^i \cdot \boldsymbol{n} = \sum_{k=1}^3 D_{l,ki}(t,y)\boldsymbol{n}_k & \text{in } (0,T) \times (\partial Y_l \setminus \partial Y) \,, \\ \omega_l^i \;\text{ and } D_l(t,y)\nabla_y \omega_l^i \cdot \boldsymbol{n}|_{\partial Y_l \cap \partial Y} & \text{are } Y-\text{periodic} \,, \end{cases} \tag{5.18}$$

and

$$\begin{cases} \operatorname{div}_y \big(D_l(t,y)\nabla_y z_l - H_M(\boldsymbol{v}_l)\big) = 0 & \text{in } \Omega_T \times Y_l, \\ \big(D_l(t,y)\nabla_y z_l - H_M(\boldsymbol{v}_l)\big) \cdot \boldsymbol{n} = 0 & \text{on } \Omega_T \times (\partial Y_l \setminus \partial Y) \,, \\ z_l \text{ and } D_l(t,y)\nabla_y z_l \cdot \boldsymbol{n}|_{\partial Y_l \cap \partial Y} & \text{are } Y-\text{periodic} \,. \end{cases} \tag{5.19}$$

Next, setting $\psi_2 = 0$ in (5.17) yields macroscopic equations (5.14)-(5.15) for c_s and c_a, where homogenized diffusion matrices and macroscopic velocity fields are given by

$$\begin{aligned} \mathcal{A}_{l,ij}(t) &= \frac{1}{|Y_l|} \int_{Y_l} \big(D_{l,ij}(t,y) + \sum_{k=1}^3 D_{l,ik}(t,y)\partial_{y_k}\omega_l^j(y)\big) dy \qquad \text{for } l = a, s \,, \\ \hat{H}_M(\boldsymbol{v}_l)_i(t,x) &= \frac{1}{|Y_l|} \int_{Y_l} H_M(\boldsymbol{v}_l(t,x,y))_i \, dy - \frac{1}{|Y_l|} \int_{Y_l} \sum_{k=1}^3 D_{l,ik}(t,y)\partial_{y_k} z_l(t,x,y) \, dy \,, \end{aligned} \tag{5.20}$$

whereas w_l^j and z_l are solutions of the unit cell problems (5.18) and (5.19).

Applying the boundary unfolding operator $\mathcal{T}_\Gamma^\varepsilon$ to (3.5) and testing with $\psi \in L^2(\Omega_T \times \Gamma_{zs})$ give

$$\begin{aligned} \langle \partial_t \mathcal{T}_\Gamma^\varepsilon(\vartheta_{f,l}^\varepsilon), \psi \rangle_{\Omega \times \Gamma_{zs}, T} &= \langle R_l(t,y,\mathcal{T}_\Gamma^\varepsilon(\vartheta_{f,l}^\varepsilon)), \psi \rangle_{\Omega \times \Gamma_{zs}, T} - \langle \alpha_l(t,y)\mathcal{T}_\Gamma^\varepsilon(\vartheta_{f,l}^\varepsilon)\mathcal{T}_\Gamma^\varepsilon(c_l^\varepsilon), \psi \rangle_{\Omega \times \Gamma_{zs}, T} \\ &\quad + \langle \beta_l(t,y)\mathcal{T}_\Gamma^\varepsilon(\vartheta_{b,l}^\varepsilon) - \gamma_{f,l}(t,y)\mathcal{T}_\Gamma^\varepsilon(\vartheta_{f,l}^\varepsilon), \psi \rangle_{\Omega \times \Gamma_{zs}, T} \,, \\ \langle \partial_t \mathcal{T}_\Gamma^\varepsilon(\vartheta_{b,l}^\varepsilon), \psi \rangle_{\Omega \times \Gamma_{zs}, T} &= \langle \alpha_l(t,y)\mathcal{T}_\Gamma^\varepsilon(\vartheta_{f,l}^\varepsilon)\mathcal{T}_\Gamma^\varepsilon(c_l^\varepsilon) - (\beta_l(t,y) + \gamma_{b,l}(t,y))\mathcal{T}_\Gamma^\varepsilon(\vartheta_{b,l}^\varepsilon), \psi \rangle_{\Omega \times \Gamma_{zs}, T} \,. \end{aligned} \tag{5.21}$$

Considering the strong convergence of $\mathcal{T}_\Gamma^\varepsilon(\vartheta_{f,l}^\varepsilon)$ and $\mathcal{T}_\Gamma^\varepsilon(c_l^\varepsilon)$ stated in (5.3), the equivalence between two-scale convergence and the weak convergence of the unfolded sequence, and Lipschitz continuity of R_l, with $l = a, s$, we can pass in (5.21) to the limit as $\varepsilon \to 0$ and obtain equations (5.16). The assumption on the initial data, the similar arguments as in the proof of Lemma 5.2, the two-scale convergence of c_l^ε, $\vartheta_{j,l}^\varepsilon$ and of their time derivatives ensure that the initial conditions for c_l and $\vartheta_{j,l}$, with $l = a, s$ and $j = f, b$, are satisfied in $L^2(\Omega)$ and in $L^2(\Omega \times \Gamma_{zs})$, respectively. The proof of the uniqueness follows along the same lines as for the microscopic model and implies the convergence of the entire sequence of solutions of the microscopic problems. □

Appendix

We recall here the definition of the two-scale convergence and unfolding operator.

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Definition 5.5. [2, 29] A sequence $\{u^\varepsilon\} \subset L^2((0,T)\times\Omega)$ is said to two-scale converge to a limit $u_0 \in L^2((0,T)\times\Omega\times Y)$ iff for any $\phi \in L^2((0,T)\times\Omega, C_{per}(Y))$ we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega u^\varepsilon(t,x)\phi(t,x,x/\varepsilon)dxdt = \frac{1}{|Y|}\int_0^T\int_\Omega\int_Y u_0(t,x,y)\phi(t,x,y)dydxdt\ .$$

Theorem 5.6. *[2, 29] From each bounded sequence $\{u^\varepsilon\}$ in $L^2((0,T)\times\Omega)$ we can extract a subsequence, which two-scale converges to $u_0 \in L^2((0,T)\times\Omega\times Y)$*

Theorem 5.7. *[2, 29] 1. Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(0,T;H^1(\Omega))$, which converges weakly to $u \in L^2(0,T;H^1(\Omega))$. Then, there exists $u_1 \in L^2(\Omega_T; H^1_{per}(Y)/\mathbb{R})$ such that, up to a subsequence, u^ε two-scale converges to u and ∇u^ε two-scale converges to $\nabla u + \nabla_y u_1$.*

2. Let $\{u^\varepsilon\}$ and $\{\varepsilon\nabla u^\varepsilon\}$ be bounded sequences in $L^2(\Omega_T)$. Then, there exists $u_0 \in L^2(\Omega_T, H^1_{per}(Y)/\mathbb{R})$ such that, up to a subsequence, u^ε and $\varepsilon\nabla u^\varepsilon$ two-scale converge to u_0 and $\nabla_y u_0$, respectively.

Definition 5.8. [28] A sequence $\{w^\varepsilon\} \subset L^2((0,T)\times\Gamma^\varepsilon_{zs})$ is said to two-scale converge to a limit $w \in L^2((0,T)\times\Omega\times\Gamma_{zs})$ iff for every $\psi \in L^2((0,T)\times\Omega; C_{per}(\Gamma_{zs}))$ we have

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\Gamma^\varepsilon_{zs}} w^\varepsilon(t,x)\psi(t,x,x/\varepsilon)d\gamma_x dt = \frac{1}{|Y|}\int_0^T\int_\Omega\int_{\Gamma_{zs}} w(t,x,y)\psi(t,x,y)d\gamma_y dxdt\ .$$

Theorem 5.9. *1.[28] For each sequence $\{w^\varepsilon\} \subset L^2((0,T)\times\Gamma^\varepsilon_{zs})$ with $\varepsilon^{\frac{1}{2}}\|w^\varepsilon\|_{L^2((0,T)\times\Gamma^\varepsilon_{zs})}$ bounded uniformly in ε, there exists a subsequence and $w \in L^2((0,T)\times\Omega\times\Gamma_{zs})$ such that the subsequence two-scale converges to w.*
2.[27] If $\{w^\varepsilon\}$ is bounded in $L^\infty((0,T)\times\Gamma^\varepsilon_{zs})$, then the limit $w \in L^\infty((0,T)\times\Omega\times\Gamma_{zs})$.

Definition 5.10. [10] 1. For any function ϕ Lebesgue-measurable on the perforated domain Ω^ε_l, the unfolding operator $\mathcal{T}^\varepsilon_{Y_l} : \Omega^\varepsilon_l \to \Omega\times Y_l$, $l = a, s$, is defined by

$$\mathcal{T}^\varepsilon_{Y_l}(\phi)(x,y) = \begin{cases}\phi(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y) & \text{a.e. for } y \in Y_l,\ x \in \tilde{\Omega}^\varepsilon_{int}\ ,\\ 0 & \text{a.e. for } y \in Y_l,\ x \in \Omega\setminus\tilde{\Omega}^\varepsilon_{int}\ ,\end{cases}$$

where $k := \left[\frac{x}{\varepsilon}\right]$ denotes the unique integer combination, such that $x - \left[\frac{x}{\varepsilon}\right]$ belongs to Y_l, and $\tilde{\Omega}^\varepsilon_{int} = \mathrm{Int}(\cup_{k\in\mathbb{Z}^3}\{\varepsilon\overline{Y^k}, \varepsilon Y^k \subset \Omega\})$. We note that for $w \in H^1(\Omega)$ it holds that $\mathcal{T}^\varepsilon_{Y_l}(w|_{\Omega^\varepsilon_l}) = \mathcal{T}^\varepsilon_Y(w)|_{\Omega\times Y_l}$.
2. For any function ϕ Lebesgue-measurable on oscillating boundary Γ^ε_l, $l = zs, a, s$, the boundary unfolding operator $\mathcal{T}^\varepsilon_{\Gamma_l} : \Gamma^\varepsilon_l \to \Omega\times\Gamma_l$, is defined by

$$\mathcal{T}^\varepsilon_{\Gamma_l}(\phi)(x,y) = \begin{cases}\phi(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y) & \text{a.e. for } y \in \Gamma_l,\ x \in \tilde{\Omega}^\varepsilon_{int}\ ,\\ 0 & \text{a.e. for } y \in \Gamma_l,\ x \in \Omega\setminus\tilde{\Omega}^\varepsilon_{int}\ .\end{cases}$$

Lemma 5.11. *[27] If $\{\psi^\varepsilon\} \subset L^2((0,T)\times\Gamma^\varepsilon_{zs})$ converges two-scale to $\psi \in L^2((0,T)\times\Omega\times\Gamma_{zs})$ and $\{\mathcal{T}^\varepsilon_{\Gamma_{zs}}(\psi^\varepsilon)\} \subset L^2((0,T)\times\Omega\times\Gamma_{zs})$ converges weakly to ψ^* in $L^2((0,T)\times\Omega\times\Gamma_{zs})$, then $\psi = \psi^*$ a.e. in $(0,T)\times\Omega\times\Gamma_{zs}$.*

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