106.37 An unusual identity for odd-powers

Introduction and main results

We begin our mathematical journey with investigation of the pattern in terms of finite differences Δ of cubes n^3. Consider the table of finite differences Δ of the polynomial n^3.

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

TABLE 1: Table of finite differences Δ of n^3

It is easy to observe that finite differences Δ of polynomial n^3 may be expressed according to the pattern

$$\Delta(0^3) = 1 + 6 \times 0$$

$$\Delta(1^3) = 1 + 6 \times 0 + 6 \times 1$$

$$\Delta(2^3) = 1 + 6 \times 0 + 6 \times 1 + 6 \times 2$$

$$\Delta(3^3) = 1 + 6 \times 0 + 6 \times 1 + 6 \times 2 + 6 \times 3$$

$$\vdots$$

$$\Delta(n^3) = 1 + 6 \times 0 + 6 \times 1 + 6 \times 2 + 6 \times 3 + \ \ldots \ + 6 \times n$$

Furthermore, the polynomial n^3 turns into

$$n^3 = [1 + 6 \times 0] + [1 + 6 \times 0 + 6 \times 1] + [1 + 6 \times 0 + 6 \times 1 + 6 \times 2] + \ \ldots$$

$$+ [1 + 6 \times 0 + 6 \times 1 + 6 \times 2 + \ \ldots \ + 6 \times (n-1)]$$

Rearranging the above equation we get

$$n^3 = n + (n-0) \times 6 \times 0 + (n-1) \times 6 \times 1 + (n-2) \times 6 \times 2 + \ \ldots \ 1 \times 6 \times (n-1)$$

Therefore, we can consider n^3 as

$$n^3 = \sum_{k=1}^{n} (6k(n-k) + 1). \tag{1}$$

Assume that (1) has an implicit form such as

$$n^3 = \sum_{k=1}^{n} \left(A_{1,1}k^1(n-k)^1 + A_{1,0}k^0(n-k)^0\right), \tag{2}$$

https://doi.org/10.1017/mag.2022.129 Published online by Cambridge University Press

where $A_{1,1} = 6$ and $A_{1,0} = 1$, respectively. The main problem we meet is to answer the question: Could the pattern (2) be generalised for all positive odd powers? Let us consider this conjecture

Conjecture 1: For every $n \geqslant 1$, $n, m \in \mathbb{N}$, there are coefficients $A_{m,0}, A_{m,1}, \ldots, A_{m,m}$ such that

$$n^{2m+1} = \sum_{k=1}^{n} \left(A_{m,0}k^0(n-k)^0 + A_{m,1}k^1(n-1)^1 + \ \ldots \ + A_{m,m}k^m(n-k)^m\right).$$

Consider the case $m = 1$. We assume that following identity holds

$$n^3 = \sum_{k=1}^{n} \left(A_{1,1}k^1(n-k)^1 + A_{1,0}k^0(n-k)^0\right).$$

We evaluate the coefficients $A_{1,0}, A_{1,1}$ as follows

$$n^3 = \sum_{k=1}^{n} \left(A_{1,1}kn - A_{1,1}k^2 + A_{1,0}\right)$$

$$n^3 = A_{1,1}n\sum_{k=1}^{n} k - A_{1,1}\sum_{k=1}^{n} k^2 + A_{1,0}n. \tag{3}$$

Next, consider Faulhaber's formula, see [1]

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p} \binom{p+1}{j} B_j\, n^{p+1-j},$$

where the B_j are Bernoulli numbers, see [2], and $\binom{n}{k}$ are Binomial coefficients, see [3]. Faulhaber's formula allows us to compute the closed forms of the sums $\sum k, \sum k^2$

$$\sum_{k=1}^{n} k = \frac{n^2+n}{2}, \ \sum_{k=1}^{n} k^2 = \frac{2n^3+3n^2+n}{6}. \tag{4}$$

By (4) we collapse the sums $\sum k, \sum k^2$ in (3)

$$n^3 = A_{1,1}n\frac{n^2+n}{2} - A_{1,1}\frac{2n^3+3n^2+n}{6} + A_{1,0}n.$$

Simplifying this equation we get

$$n^3 = A_{1,1}\frac{n^3-n}{6} + A_{1,0}n.$$

Multiplying by 6 gives

$$6n^3 = A_{1,1}\left(n^3-n\right) + 6A_{1,0}n.$$

Rearranging, we get

$$A_{1,1}n^3 - A_{1,1}n + 6A_{1,0}n - 6n^3 = 0.$$

https://doi.org/10.1017/mag.2022.129 Published online by Cambridge University Press

Hence

$$n^3(A_{1,1} - 6) + n(6A_{1,0} - A_{1,1}) = 0.$$

To determine the coefficients $A_{1,0}$, $A_{1,1}$ we construct the following system of linear equations

$$\begin{cases} A_{1,1} - 6 = 0, \\ 6A_{1,0} - A_{1,1} = 0. \end{cases} \tag{5}$$

Solving the system (5) we conclude that $A_{1,1} = 6$ and $A_{1,0} = 1$. Therefore,

$$n^3 = \sum_{k=1}^{n} \left(6k(n-k) + 1\right).$$

Consider the case $m = 2$. We assume that following identity holds

$$\begin{aligned} n^5 &= \sum_{k=1}^{n} \left(A_{2,2}k^2(n-k)^2 + A_{2,1}k(n-k) + A_{2,0}\right) \\ &= A_{2,2}\sum_{k=1}^{n} k^2(n-k)^2 + A_{2,1}\sum_{k=1}^{n} k(n-k) + A_{2,0}n. \end{aligned} \tag{6}$$

By means of the Binomial theorem, see [3], that is $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, we expand the term $(n-k)^2$ in (6)

$$\begin{aligned} n^5 &= A_{2,2}\sum_{k=1}^{n} k^2\left(n^2 - 2nk + k^2\right) + A_{2,1}\sum_{k=1}^{n} k(n-k) + A_{2,0}n \\ &= A_{2,2}\sum_{k=1}^{n} \left(k^2n^2 - 2nk^3 + k^4\right) + A_{2,1}\sum_{k=1}^{n} \left(kn - k^2\right) + A_{2,0}n. \end{aligned} \tag{7}$$

Now we are able to arrange the sums in (7) in the following way

$$\begin{aligned} n^5 &= A_{2,2}n^2\sum_{k=1}^{n} k^2 - A_{2,2}2n\sum_{k=1}^{n} k^3 + A_{2,2}\sum_{k=1}^{n} k^4 \\ &\quad + A_{2,1}n\sum_{k=1}^{n} k - A_{2,1}\sum_{k=1}^{n} k^2 + A_{2,0}n. \end{aligned}$$

Rearranging yields

$$\begin{aligned} n^5 &= A_{2,2}\left(n^2\sum_{k=1}^{n} k^2 - 2n\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^4\right) \\ &\quad + A_{2,1}\left(n\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2\right) + A_{2,0}n. \end{aligned} \tag{8}$$

Just as before, the idea is quite simple, we have to replace the sums $\sum k$, $\sum k^2$, $\sum k^3$, $\sum k^4$ in (8) by their closed forms. The sums $\sum k$, $\sum k^2$ are

https://doi.org/10.1017/mag.2022.129 Published online by Cambridge University Press

calculated in (4), the others, $\sum k^3$, $\sum k^4$, are

$$\sum_{k=1}^{n} k^3 = \frac{n^4 + 2n^3 + n^2}{4}, \sum_{k=1}^{n} k^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}. \quad (9)$$

By (4) and (9) we collapse the sums $\sum k$, $\sum k^2$, $\sum k^3$, $\sum k^4$ in (8)

$$n^5 = A_{2,2}\left(n^2\frac{2n^3+3n^2+n}{6} - 2n\frac{n^4+2n^3+n^2}{4} + \frac{6n^5+15n^4+10n^3-n}{30}\right)$$
$$+ A_{2,1}\left(n\frac{n^2+n}{2} - \frac{2n^3+3n^2+n}{6}\right) + A_{2,0}n.$$

Simplifying the above expression we get

$$n^5 = A_{2,2}\frac{n^5 - n}{30} + A_{2,1}\frac{n^3 - n}{6} + A_{2,0}n.$$

Therefore,

$$\frac{A_{2,2}n^5 - A_{2,2}n}{30} + \frac{A_{2,1}n^3 - A_{2,1}n}{6} + A_{2,0}n - n^5 = 0.$$

Multiplying by 30 gives

$$A_{2,2}n^5 - A_{2,2}n + 5A_{2,1}n^3 - 5A_{2,1}n + 30A_{2,0}n - 30n^5 = 0.$$

Rearranging we get

$$n^5(A_{2,2} - 30) + 5A_{2,1}n^3 - n(A_{2,2} - 30A_{2,0} + 5A_{2,1}) = 0.$$

To determine the coefficients $A_{2,0}$, $A_{2,1}$, $A_{2,2}$ we construct the following system of linear equations

$$\begin{cases} A_{2,2} - 30 = 0 \\ A_{2,1} = 0 \\ A_{2,2} - 30A_{2,0} + 5A_{2,1} = 0 \end{cases} \quad (10)$$

Solving the system (10) we conclude that $A_{2,2} = 30$, $A_{2,1} = 0$, $A_{2,0} = 1$. Finally, we get another polynomial identity

$$n^5 = \sum_{k=1}^{n} \left(30k^2(n-k)^2 + 1\right).$$

This suggests the following

Theorem 2: For every $n \geqslant 1$, $n, m \in \mathbb{N}$, there are $A_{m,0}, A_{m,1},\ldots,A_{m,m}$, such that

$$n^{2m+1} = \sum_{k=1}^{n}\sum_{r=0}^{m} A_{m,r}k^r(n-k)^r,$$

where $A_{m,r}$ is a real coefficient.

https://doi.org/10.1017/mag.2022.129 Published online by Cambridge University Press

Therefore Conjecture 1 is true. We have the following identities for positive odd powers

$$n^3 = \sum_{k=1}^{n} \left(6k\,(n-k)+1\right)$$

$$n^5 = \sum_{k=1}^{n} \left(30k^2\,(n-k)^2+1\right)$$

$$n^7 = \sum_{k=1}^{n} \left(140k^3\,(n-k)^3-14k\,(n-k)+1\right)$$

$$n^9 = \sum_{k=1}^{n} \left(630k^4\,(n-k)^4-120k\,(n-k)+1\right)$$

$$n^{11} = \sum_{k=1}^{n} \left(2772k^5\,(n-k)^5+660k^2\,(n-k)^2-1386k\,(n-k)+1\right)$$

$$n^{15} = \sum_{k=1}^{n} \left(51480k^7(n-k)^7-60060k^3(n-k)^3+491400k^2(n-k)^2 - 450054k\,(n-k)+1\right).$$

Readers may be familiar with the recursive formula to calculate the coefficients $A_{m,r}$ in [4, equation (2.1)]. Theorem 2 could be verified using the Wolfram Mathematica scripts at github.com/kolosovpetro/mathematica-scripts.

References

1. Alan F. Beardon, Sums of powers of integers, *Amer. Math. Monthly*, 103(3) (1996) pp. 201-213.
2. Harry Bateman, *Higher transcendental functions [volumes i-iii]*, volume 1. McGraw-Hill Book Company (1953).
3. Milton Abramowitz, Irene A. Stegun, et al. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, volume 55. National bureau of standards, Washington, DC (1972).
4. Petro Kolosov, On the link between Binomial Theorem and Discrete Convolution of Polynomials. arXiv preprint arXiv:1603.02468, 2016. https://arxiv.org/abs/1603.02468.

10.1017/mag.2022.129 © The Authors, 2022
Published by Cambridge University Press on behalf of The Mathematical Association

PETRO KOLOSOV
Jawornicka 14, Poznan, Poland, 60–161.
Department of Computer Science, Wyzsza Szkola Bankowa, Poznan, Poland
e-mail: *kolosovp94@gmail.com*

https://doi.org/10.1017/mag.2022.129 Published online by Cambridge University Press