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Repeated Riemann integrals

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Let $f(x, y)$ be a bounded real function on the unit square. Consider the Riemann integrals (if they exist)

$$g(x) = (R)\int_0^1 f(x,y)\,dy \quad \text{and} \quad h(y) = (R)\int_0^1 f(x,y)\,dx, \tag{1}$$

$$A = (R)\int_0^1 g(x)\,dx \quad \text{and} \quad B = (R)\int_0^1 h(y)\,dy. \tag{2}$$

Young in 1910, reference (1), published the result given in Theorem 2 below, and in the same year Lichtenstein (2) used this result to show (Theorem 1 below), that if the integrals (1) exist then the integrals (2) also exist and $A = B$. Gillespie (3) pointed out in 1919 a flaw in Young's proof, and showed that assuming the existence of the integrals (2) as well as (1) is sufficient for showing that $A = B$.

This paper supplies a proof for Young's result (Theorem 2) as well as an independent proof of Lichtenstein's theorem (Theorem 1).

A Riemann dissection D of the unit interval is a finite family of points

$$0 = a_0 < a_1 < \ldots < a_m = 1$$

with an associated point b_j in each closed sub-interval $[a_{j-1}, a_j]$ $(j = 1, 2, \ldots, m)$. With D and any function F is associated the Riemann sum

$$R(D, F) = \sum_1^m (a_j - a_{j-1})\,F(b_j).$$

A sequence of dissections converges if the lengths of the longest sub-interval tends to zero. For any bounded function, there is a convergent sequence of Riemann dissections for which the Riemann sums converge to the lower integral and another for which they converge to the upper integral. A function is Riemann integrable if and only if for any convergent sequence of dissections the Riemann sums converge.

Theorem 1. *If the integrals* (1) *both exist then the integrals* (2) *both exist and* $A = B$.

Proof. Let A_1 and A_2 be respectively the upper and lower integrals of $g(x)$. Take a convergent sequence of Riemann dissections D_n such that $R(D_n, g) \to A_1$, then

$$R(D_n, g) = \int_0^1 R(D_n, f(x,y))\,dy, \tag{3}$$

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where the sum operation on the right-hand side is performed with the variable x. Similarly there is D_n' such that $R(D_n', g) \to A_2$. Now consider

$$R(D_n, g) - R(D_n', g) = \int_0^1 [R(D_n, f) - R(D_n', f)]\, dy. \tag{4}$$

Because f is integrable, the integrand on the right-hand side tends to zero for each y, and therefore by Arzela's theorem on bounded convergence, the right-hand side tends to zero. The left-hand side of (4) tends to $A_1 - A_2$ and therefore $A_1 = A_2$ and g is Riemann integrable. Similar reasoning shows that h is also Riemann integrable.

The second part of this theorem, that $A = B$, has already been proved by Lichtenstein and by Gillespie, but it may also be seen by applying Arzela's theorem to equation (3) above. The left-hand side tends to A and the right to B, so that $A = B$.

Lichtenstein's proof(2) of Theorem 1 above depends on the following theorem of Young(1) involving a mixture of Riemann and Lebesgue integrals.

THEOREM 2. *Let $f(x, y)$ be bounded on the unit square, let $g(x) = (R)\int_0^1 f(x, y)\, dy$ exist for each x and let $h(y) = (L)\int_0^1 f(x, y)\, dx$ exist for each y. Then $(R)\int_0^1 h(y)\, dy$ also exists.*

Proof. Take any convergent sequence D_n of Riemann dissections of the unit interval, then

$$R(D_n, h) = (L)\int_0^1 R(D_n, f(x, y))\, dx,$$

where the Riemann sum on the right is taken over the variable y. In this equation, the integrand converges pointwise to $g(x)$ which is bounded and L-measurable, so that by Lebesgue's theorem on bounded or dominated convergence

$$\lim R(D_n, h) \text{ exists and equals } (L)\int_0^1 g(x)\, dx.$$

For the function $h(y)$, for any convergent sequence of dissections, the Riemann sums converge. Therefore $h(y)$ is Riemann integrable.

Young's proof of the above result in (1), pages 127 and 128, applies Fubini's theorem to the set $S = \{(x, y); f(x, u)$ is a discontinuous function of u at $u = y\}$. Gillespie(3) pointed out the weakness that Young gave no reason why S should be plane L-measurable. The following example shows that S may not be.

Sierpiński(4) has shown the existence of a subset E of the unit square not plane L-measurable but meeting any straight line in at most two points. Let $f(x, y)$ be its characteristic function. Then the set S (above) is E and is not L-measurable.

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