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Homeomorphically Irreducible Spanning Trees in Locally Connected Graphs

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This paper is dedicated to the memory of Professor Richard H. Schelp

A spanning tree T of a graph G is called a *homeomorphically irreducible spanning tree* (HIST) if T does not contain vertices of degree 2. A graph G is called *locally connected* if, for every vertex $v \in V(G)$, the subgraph induced by the neighbourhood of v is connected. In this paper, we prove that every connected and locally connected graph with more than 3 vertices contains a HIST. Consequently, we confirm the following conjecture due to Archdeacon: every graph that triangulates some surface has a HIST, which was proposed as a question by Albertson, Berman, Hutchinson and Thomassen.

1. Introduction

In this paper, we will consider simple graphs only and generally follow Bondy and Murty [2] for the notation and terminology. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. A tree is called *homeomorphically irreducible* if it does not contain vertices of degree 2. A *homeomorphically irreducible spanning tree* of a graph G is called a *HIST* of G. Albertson, Berman, Hutchinson and Thomassen [1] obtained various sufficient conditions for a graph to contain a HIST. They also showed that *it is NP-complete to decide whether a graph G contains a HIST*. Hill [5] conjectured that every *triangulation of the plane other than K_3 contains a HIST*. Malkevitch [7] conjectured that the same result is held for near-triangulations of the plane (2-connected plane graphs such that all, but at most one, faces are triangles). Albertson, Berman, Hutchinson and Thomassen [1] confirmed the conjecture. Furthermore, they asked *whether every graph that triangulates some surface has a HIST*. In [3], Archdeacon conjectured that the answer to the above question is positive. Davidow, Hutchinson and Huneke [4] showed that every triangulation of the torus contains a HIST.

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A graph G is *locally connected* if, for every vertex $v \in V(G)$, the subgraph induced by the neighbourhood $N(v)$ is connected. Ringel [8] showed that every triangulation (including orientable and non-orientable embeddings) is a connected and locally connected graph. In this paper, we prove the following much more general result, which, consequently, answers the above question asked by Albertson, Berman, Hutchinson and Thomassen positively.

Theorem 1.1. *Every connected and locally connected graph with order at least four contains a HIST.*

Corollary 1.2. *Let* Π *be a surface* (*orientable or non-orientable*). *Then every triangulation of* Π *with at least four vertices contains a HIST.*

Let G be a graph. Write $v \in G$ if $v \in V(G)$ and similarly $e \in G$ if $e \in E(G)$. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. For each vertex $v \in G$ and vertex set $S \subseteq V$, let $N(v)$ and $N[v]$ denote the *open and closed neighbourhoods* of v, respectively, and $N_S(v) := N(v) \cap S$.

2. Proof of Theorem 1.1

Let k be a positive integer. A graph G is called a *k-tree* if there is an ordering $v_1 \prec v_2 \prec \cdots \prec v_n$ of $V(G)$ such that (i) $G[\{v_1, v_2, \ldots, v_k\}]$ is a complete graph and (ii), for each $i > k$, $N(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}$ induces a clique of order k. Clearly, 1-trees are the same as trees. Hwang, Richards and Winter [6] proved that 2-trees are *maximal series-parallel graphs*. As shown in Lemmas 2.2 and 2.3, we observe that *every 2-tree with more than three vertices contains a HIST*. However, not every connected and locally connected graph contains a 2-tree as a spanning subgraph. Let $W_n := K_1 + C_n$ be a wheel of order $n+1$ and let G_n be obtained from W_n by adding n new vertices such that each is adjacent to a distinct pair of two consecutive vertices on the cycle C_n. It is not difficult to verify that G_n does not contain a spanning 2-tree. G_4 is depicted in Figure 1.

Let G be a graph, and let v be a vertex not in $V(G)$. We write $H = G \oplus v$ if there exist two distinct vertices $u_1, u_2 \in G$ such that $V(H) = V(G) \cup \{v\}$ and $E(H) = E(G) \cup \{u_1v, u_2v, u_1u_2\}$. Note that the edge u_1u_2 may already exist in G. We let $P(v) := \{u_1, u_2\}$ and call u_1 and u_2 the *parents* of v.

Definition. A graph T of order $n \geqslant 3$ is called a *weak 2-tree* (W2-tree) if there is an ordering $\prec: v_1 \prec v_2 \prec \cdots \prec v_n$ of vertices of T and a sequence of graphs $G_3 \subset G_4 \subset \cdots \subset G_n = T$ such that the following properties hold:

(1) $G_3 = T[\{v_1, v_2, v_3\}] \cong K_3$, and
(2) for each $i = 3, 4, \ldots, n-1$, $G_{i+1} \cong G_i \oplus v_{i+1}$.

In addition, we call the ordering $\prec$ a *W2-tree ordering* of T.

Clearly, every 2-tree is a W2-tree. However, the converse is not true; for example, the above graphs G_n are W2-trees but not 2-trees.

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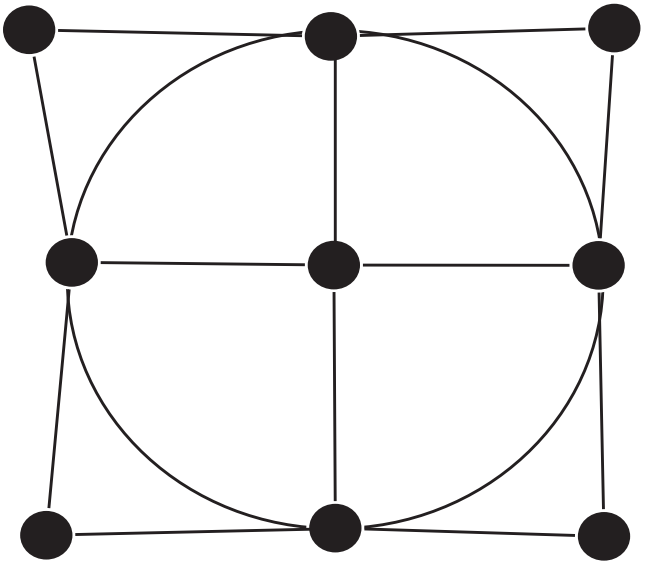


Figure 1. G_4, a locally connected graph without a spanning 2-tree.

Given a W2-tree with a W2-tree ordering $\prec$, if we shift a degree 2 vertex to the end and keep the remaining ordering unchanged, we obtain another W2-tree ordering. So, the following result holds.

Lemma 2.1. *Let G be a W2-tree with $n \geqslant 4$ vertices. Let $w \in G$ be a degree 2 vertex and $N(w) = \{u, v\}$. Then either $G - w$ or $G - w - uv$ is a W2-tree.*

Lemma 2.2. *Let T be W2-tree with $n \geqslant 4$ vertices. Then, there exist two vertices u and v such that $T = (T' \oplus u) \oplus v$ and $N[u] \cap N(v) \neq \emptyset$, where T' is a W2-tree, K_3, or K_2. In this case, $\{u, v\}$ is called a* removable pair *of T.*

Proof. We prove Lemma 2.2 by applying induction on $n = |V(G)|$. Since K_4^- (K_4 minus an edge) is the unique W2-tree with 4 vertices, Lemma 2.2 holds for $n = 4$.

Suppose $n \geqslant 5$ and that Lemma 2.2 holds for all W2-trees with less than n vertices. Let T be a W2-tree with n vertices and let w be the last vertex in a W2-ordering of T. Moreover, we assume that $T = T' \oplus w$, where T' is a W2 tree-tree with $n - 1$ vertices. Suppose that $\{u, v\}$ is a removable pair of T' and $T' = (T^* \oplus u) \oplus v$, where T^* is a W2-tree, K_3, or K_2. We complete the proof by considering the following five cases regarding $N(w) \cap \{u, v\}$.

- If $N(w) \cap \{u, v\} = \emptyset$, then $T = [(T^* \oplus w) \oplus u] \oplus v$, so $\{u, v\}$ is a removable pair of T.
- If $N(w) \cap \{u, v\} = \{v\}$, then $T = [(T^* \oplus u) \oplus v] \oplus w$, so $\{v, w\}$ is a removable pair of T.
- If $N(w) \cap \{u, v\} = \{u\}$ and $uv \notin E(T')$, then $T = [(T^* \oplus v) \oplus u] \oplus w$, so $\{u, w\}$ is a removable pair of T.

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- If $N(w) \cap \{u,v\} = \{u\}$ and $uv \in E(T')$, then $T = [(T^* \oplus u) \oplus v] \oplus w$, so $\{v,w\}$ is a removable pair of T.
- If $N(w) = \{u,v\}$, then $T = [(T^* \oplus u) \oplus v] \oplus w$, so $\{v,w\}$ is a removable pair of T. □

Lemma 2.3. *A W2-tree with at least* 4 *vertices contains a HIST.*

Proof. Let G be a W2-tree with $n \geqslant 4$ vertices. We proceed by induction on n. If $n = 4$, then $G = K_4^-$, which contains a spanning star. If $n = 5$, by case analysis, we can show that G contains a spanning star, so a HIST.

Assume $n \geqslant 6$ and let G be a W2-tree with n vertices. By Lemma 2.2, let $\{u,v\}$ be a removable pair of G and assume $G = (G' \oplus u) \oplus v$, where G' is a W2-tree with $n-2$ vertices. By the induction hypothesis, G' contains a HIST, say T'. Since $\{u,v\}$ is a removable pair of G, $N[u] \cap N(v) \neq \emptyset$. If $uv \notin E(G)$, then $N(u) \cap N(v) \neq \emptyset$; if $uv \in E(G)$, by the definition of $\oplus$, the other neighbour of v is adjacent to u. In either case, there exists a vertex $w \in N(u) \cap N(v)$. Then, $T := T' \cup \{wu, wv\}$ is a HIST of G. □

Lemma 2.4. *Every connected and locally connected graph with at least three vertices contains a spanning W2-tree.*

Proof. Let G be a connected and locally connected graph of order $n \geqslant 3$. Since every triangle is a W2-tree, G contains W2-trees as subgraphs. Let $T \subseteq G$ be a W2-tree such that $|V(T)|$ is maximum. We claim that $V(T) = V(G)$. Otherwise, $W := V(G) - V(T) \neq \emptyset$. Since G is connected, there is a vertex $v \in V(T)$ such that $N_W(v) \neq \emptyset$, where $N_W(v)$ is the set of neighbours of v in W. Since T is a W2-tree, $N(v) \cap V(T) \supseteq N_T(v) \neq \emptyset$. Since $G[N(v)]$ is connected, there is an edge $uw \in E(G)$ with $u \in N_T(v)$ and $w \in N_W(v)$. Then, $T \oplus w$ is a W2-tree containing more vertices than T, where $P(w) = \{u,v\}$. Since $uv, wv, uw \in E(G)$ and $T \subseteq G$, we have $T \oplus w \subseteq G$, which contradicts the maximality of $|V(T)|$. □

So, the proof of Theorem 1.1 is completed.

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