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Transport scaling and trapping

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It is shown that particle trapping plays an important role in the appearance of anomalous or strange diffusion in turbulent fields, mainly by inducing a non-Gaussian statistics of the stochastic displacements. Trapping processes are responsible for the important deviation observed between the traditional prediction of Bohm scaling for diffusion in low-frequency turbulence and the numerically checked prediction by Isichenko. Also, several theoretical models are proposed or reviewed.

1. Introduction

For the design of future experiments on thermonuclear fusion, scaling of the transport coefficients with respect to the magnetic field strength B appears to be of particular importance. For the diffusion coefficient D, we denote the latter dependence by $D \propto B^{-\gamma}$ in term of the scaling exponent γ. Classical or quasilinear scaling generally implies $\gamma = 2$, which is also known as 'gyro-Bohm' scaling. For several decades, the well-known Bohm scaling $\gamma = 1$ has been predicted to occur for guiding centres in electrostatic turbulence in the limit of 'frozen' or low-frequency turbulence (see Misguich *et al.* 1987, and references therein), which also corresponds to the limit of high-amplitude turbulence.

Isichenko and collaborators (Grusinov *et al.* 1990; Isichenko 1992) have proposed an alternative scaling $\gamma = 0.7$ in this low-frequency domain, based on an analogy with percolation problems in two dimensions. By extensive numerical computations, this prediction has been shown to hold in the low-frequency domain for guiding centres in a k^{-3} turbulent spectrum modelled by a large number of randomly phased waves (Reuss and Misguich 1996; Reuss *et al.* 1998; Weyssow 1997 personal communication).

In Tore Supra experiments various scalings have been observed as functions of the Larmor radius (Zou *et al.* 1997): gyro-Bohm scaling holds for the electrons in the low-confinement regime (L modes), which means $\gamma = 2$, while the scaling for ions rather seems to vary between a Goldston law (Petty *et al.* 1995), with $\gamma = 0.5$, and a gyro-Bohm law, which implies $0.5 < \gamma < 2$. The experimental situation is thus not completely clear, and Isichenko scaling is not excluded.

We show in this paper that the process responsible for the deviation from Bohm scaling and for the reduction of the diffusion coefficient is trajectory trapping. Trajectory trapping also produces large departures from a Gaussian distribution of the displacements. The process of trapping is analysed in various

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contexts related to the anomalous transport induced by electrostatic turbulence or by magnetic line braiding. With this aim, we have used the test-particle approach both in the extended numerical computations and in the theoretical models.

A simple heuristic argument can be used to show that trajectory trapping has the effect of decreasing the diffusion coefficient and changing the scaling parameters to subunitary values $\gamma < 1$. The diffusion coefficients can generally be estimated according to the random-walk expression as the square of the average step l over the average duration τ: $D = l^2/\tau$. The parameters of the stochastic field combine in the Kubo number A, which represents the ratio of the distance covered by the particle in the correlation time τ_c to the correlation length of the stochastic field λ: $A \equiv V\tau_c/\lambda$. Here V is the amplitude of the fluctuating particle velocity. The diffusion coefficient is a function of A. As V is proportional to the inverse of the magnetic field, the scaling exponent of the diffusion in A is just γ.

When A is small, the particles cannot 'see' the structure of the potential – they move only a small distance before the stochastic field is changed. This is a temporal decorrelation, which is characterized by the equality of the time step of the random walk with the correlation time: $\tau = \tau_c$. The space step is the distance covered in τ_c: $l = V\tau_c$. The diffusion coefficient is thus estimated as

$$D = V^2\tau_c = \frac{\lambda^2}{\tau_c}A^2 \quad \text{(the quasilinear regime, } \gamma = 2\text{)}.$$

When A is not small, the diffusion coefficient is much more difficult to estimate. Under these conditions, the particle can explore the structure of the potential so that the space step of the random walk is just the correlation length of the stochastic potential: $l = \lambda$. This is a spatial decorrelation. Considering that particles move freely with the velocity V, the time necessary to cover this distance (the transit time) is $\tau = \lambda/V$, which is smaller than the correlation time τ_c (for large A). Estimation of the diffusion coefficient gives

$$D = V\lambda = \frac{\lambda^2}{\tau_c}A \equiv D_B \quad \text{(Bohm scaling, } \gamma = 1\text{)}.$$

The trajectories are greatly simplified in this estimation: they correspond to a free-motion transit of the particles through the correlated zone. In reality, the motion is much more complicated and various mechanisms of particle trapping occur, leading to a longer average time spent by the particle in the correlated zone. The time step appearing in the estimation of the diffusion coefficient is $\tau = \lambda/V + \tau_t$, where τ_t is the average trapping time. Consequently, the diffusion coefficient is

$$D = \frac{\lambda^2}{\tau_c}\frac{A}{1 + [\tau_t(A)/\tau_c]A}. \tag{1.1}$$

As the trapping time (which is a function of A) may have values between zero (no trapping) and τ_c (total trapping), $D_B \geqslant D \geqslant D_t \equiv \lambda^2/\tau_c$. Thus, in the nonlinear case, the Bohm coefficient represents the upper limit of the diffusion coefficients, which is reached when particle trapping is negligible. The evaluation of $\tau_t(A)$ is a very difficult task, which has not been solved generally.

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This paper is organized as follows. Numerical simulations of particle trajectories in a stochastic potential are presented in Section 2. They clearly show that, during their evolution, the particles are temporarily trapped on almost-closed, small trajectories for rather long durations. The diffusion coefficient is calculated as a function of the Kubo number A, the Lagrangian statistics is determined and the percolation scaling is confirmed at low frequencies. Section 3 presents various theoretical models that attempt to describe the influence of trapping on the effective diffusion coefficient. First, in Section 3.1, the partial trapping in random structures like vortex or magnetic islands is studied. This is an extrinsic trapping, the basic diffusion process and the trapping random structure being independent. Next (Section 3.2), the collisional particle motion in a stochastic magnetic field is presented. It illustrates a special kind of trapping, which is produced by the collisional motion of the particles along braided magnetic lines. Section 3.3 deals with the intrinsic trapping that appears in space-dependent stochastic fields characterized by extended two-point correlations. This is the type of process that is studied numerically in Section 2. Actually, this very difficult problem has no satisfactory solution. A simple one-dimensional model is presented. Other models (based on the continuous-time random-walk method) are briefly reviewed in Section 3.4. The conclusions are summarized in Section 4.

2. Non-Gaussian features of turbulent trajectories

The deviation observed between the predicted Bohm scaling at high amplitude and the more recently described percolation scaling results in a weaker dependence of the diffusion coefficient on the control parameter A (see Section 2.1). In Section 2.2 this slower increase with A is explained by the fact that the observed trajectories present not only long flights contributing to diffusion, but also long trapping events around the extrema of the moving potential. As a result, small displacements are more numerous and not independent: a deviation from a simple Gaussian statistics of the displacements is observed. Moreover, the numerical simulations provide some evidence that the usual Corrsin factorization approximation (McComb 1990) fails in this domain of high values of A.

2.1. Percolation exponent

We have studied numerically the problem of guiding-centre diffusion in a k^{-3} spectrum of electrostatic turbulence, with a simple monochromatic time dependence. The equations of motion for guiding centres in a strong magnetic field $\mathbf{B}$ and a fluctuating electrostatic field $\mathbf{E}(x,y,t) = -\nabla\phi(x,y,t)$ are simply those of a Hamiltonian system with $1\frac{1}{2}$ degrees of freedom, with non-autonomous Hamiltonian $H = -c\phi(x,y,t)/B$. The model of the electrostatic potential is composed of a large number (462396) of randomly phased plane waves, varying with a frequency ω. A similar model has already been used in Reuss and Misguich (1996), but here special attention is paid to the conditions of validity for the Isichenko prediction, and the numerical discretization of the model has been greatly enhanced.

For the domain of low frequencies (almost-frozen turbulence) or high amplitudes, a percolation mechanism has been invoked by Grusinov *et al.* (1990)

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and Isichenko (1992). This yields a new prediction for the diffusion scaling $\gamma_I = \frac{7}{10}$, which is based on the idea that in a frozen potential the dominant contribution to the diffusion process is due to particles making long excursions along the zero-potential curves, and thus 'percolating' among the potential cells (see Balescu 1997*b*, Chap. 13). In this description, sufficiently long contours are considered to be fractal, with a hull dimension $d_h = 1 + 1/\nu_0 = \frac{7}{4}$, where $\nu_0 = \frac{4}{3}$ is the correlation exponent of the 2D percolation problem. A simple physical argument is then used by Isichenko and co-workers to deduce that $\gamma = (\nu_0 + 1)/(\nu_0 + 2) = \frac{7}{10}$.

The domain of large amplitudes in which the Isichenko predictions holds actually involves a maximum value for the amplitude (Grusinov *et al.* 1990): $1 \ll A < A_{\mathrm{Gr}}$, simply because the system size has to be taken larger than the correlation length $\zeta = \lambda_{\max} A^{2/5}$. The condition immediately yields $A < A_{\mathrm{Gr}} \equiv N_0^{5/2}$, where N_0 is the number of $\lambda_{\max}$ in the integration domain.

By considering a larger box ($N_0 = 16$), a better discretization, a wider spectrum, more particles and a new spreading of memory data among 64 processors, we have constructed a new code, RADIGUET 3, for the parallel CRAY T3E computer in order to determine the exponent γ (Reuss *et al.* 1998). In the domain of higher amplitudes $72 < A < 1120$ we have found

$$\gamma = 0.692 \pm 0.010, \tag{2.1}$$

which is fully consistent with the Isichenko prediction $\gamma = \frac{7}{10}$, equivalent to $D \propto \omega^{0.3}(E/B)^{0.7}$, where E is the root-mean-square turbulent electric field.

The most interesting consequence of this result is the reduction of the diffusion coefficient compared with the classical Bohm-like prediction.

2.2. *Non-Gaussian features of turbulent trajectories*

The direct observation of turbulent trajectories allows us to study their statistical properties, in order to possibly take these properties into account in a statistical theory of turbulent transport. The simulations described in Section 2.1 can be used to describe a set of 1024 turbulent trajectories at low and high amplitudes ($A = 4$ and 160). Initial positions are uniformly distributed, and initial velocities are determined univocally by the local values of the field. We have found that the initial distribution of (Eulerian) velocities is of Gaussian shape. Moreover, the probability distributions of Lagrangian velocities $P(v_x, t)$ and $P(v_y, t)$ are also found to remain of Gaussian shape, independently of the time t and of the amplitude A.

The time correlation of the Lagrangian velocities $L_{ij}(t) \equiv \langle v_i(0) v_j(t) \rangle$, averaged over the trajectories, i.e. over the initial conditions, presents a very interesting behaviour. At high amplitude ($A = 160$) it exhibits a rapid decrease, passes through a negative minimum at time τ_m and presents several oscillations with a long negative tail. Similarly to particles-in-cell models for liquids, this indicates a (statistical) inversion of velocities after some time, and hence a statistical trapping phenomenon. We can estimate $\tau_m \approx 0.004 \times$ period. It is simple to give a precise physical interpretation of this result. The equation of motion ((2.21) in Reuss and Misguich 1996) allows us to deduce the average distance covered in one period: we find that a trapping process around a closed potential cell of size $\lambda_{\max}$ can be described by a turnover time $\tau_{t,\lambda_{\max}} \approx \pi/A$ (in units of periods). In the present case, $A = 160$ yields $\tau_{t,\lambda_{\max}} = 0.02$. But a

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typical wavelength $\bar{\lambda}$ can be defined by averaging over the k^{-3} spectrum from $k_{\min} = N_0 2\pi/L$ to $k_{\max} = N 2\pi/L$, with $\mu \equiv N_0/N \ll 1$. From (42) and (43) of Misguich *et al.* (1987), we find

$$\bar{\lambda} = \lambda_{\max}\left[\frac{1-\mu^2}{\ln(1/\mu^2)}\right]^{1/2},$$

which is here $\lambda = 0.4\lambda_{\max}$. For such typical cells, the turnover time is $\tau_{t,\bar{\lambda}} \approx \pi\bar{\lambda}/\lambda_{\max} A = 0.008$. The time τ_m, where the minimum of the Lagrangian correlation is found to occur, can thus be interpreted as one-half of the turnover time $\tau_{t,\bar{\lambda}}$ over a typical potential cell. After that time, the velocity is statistically inverted by the trapping process.

The trapping events essentially occur around the extrema of the potential: small trapping oscillations can be seen for high absolute values of the potential, while long excursions are seen around the small values of the potential. As a consequence, the statistics of the displacements is strongly non-Gaussian at high amplitudes. The direct measurement of the probability density $P(\Delta x, \Delta y, \Delta t)$ of displacements Δx, Δy during the time interval Δt is particularly important, since in all previous theories this function is assumed to be Gaussian. Here we observe that this function exhibits an increasing deviation from a Gaussian shape: a sharp peak rapidly develops, indicating a strong enhancement of small displacements during intervals larger than 0.005 period, and hence a trapping process.

Non-Gaussianity can also be tested by direct measurement of the skewness $S(t) = \langle \Delta x^3(t) \rangle / \langle \Delta x^2(t) \rangle^{3/2}$ and the kurtosis $K(t) = \langle \Delta x^4(t) \rangle / \langle \Delta x^2(t) \rangle^2$. At moderate amplitudes, $S(t)$ remains negligible and $K(t)$ remains of the order of 3, in agreement with a Gaussian distribution, while for $A = 160$ the skewness slowly increases up to unity for $t = 0.05$ period, and kurtosis rapidly increases to around 8 in that time interval, showing an important deviation from a Gaussian process at intermediate times.

Thus, although the Lagrangian velocity distribution remains Gaussian at all time, a strong deviation from Gaussianity is found at high amplitudes in the probability density of displacements $P(\Delta x, \Delta y, \Delta t)$, as should be expected from the direct evidence of the trapping processes in the individual trajectories.

3. Theoretical models for trapping processes

3.1. Diffusion with extrinsic trapping

It is known experimentally that, even under stationary conditions, the particle and energy fluxes at the tokamak edge exhibit fluctuations around average values. One possible explanation relies on the presence of trapping centres in the plasma, which can be magnetic islands or long-lived vortical motions induced by the nonlinear drift instabilities (Spineanu and Vlad 1997). In addition to the fluctuations of the diffusive flux, there also appears a reduction of the effective diffusion coefficient. We note that the effect of trapping is strongly dependent on the particular conditions necessary for the formation of trapping centres. These are extrinsic to the diffusive mechanism of the particles, and the statistics of the random trapping/release events can be prescribed independently.

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In the simplest case the absolute time variation of the particle density in each trapping or release event is a constant, p. The statistics of the events obeys the Poisson law with ν_t the time density and ρ the spatial density of events. The diffusion equation for the particle density $P(x,t)$, modified with an additive noise representing the trapping and release events, is used in the calculation of the generating functional of the correlations. In this simple case the diffusion coefficient D, which is initially prescribed, is not modified by the trapping/release events, but significant fluctuations of the density profile arise:

$$\langle [P(x,t)-\langle P(x,t)\rangle]^2\rangle = \frac{\nu_t \rho p^2}{(2\pi D)^{1/2}} t^{1/2}. \tag{3.1}$$

A more realistic model, in which the trapping is followed by release after a time interval τ short compared with ν_t^{-1}, leads to fluctuations that vanish asymptotically (for $t\to\infty$). The diffusion coefficient is modified by the effect of trapping. In the case of Lagrangian trapping, where the trapping/release events have a distribution specified along the particle trajectory, we obtain $D_{\text{eff}} = D(1-\nu_t \tau_t)$, where τ_t is the average trapping time.

A related problem is random escape from an imperfect trapping structure. We can consider an ensemble of particles that are initially trapped in a potential well. For a particle (acted upon by stochastic kicks), the diffusion is the free state, which can only be attained if the fluctuating barrier of the trap is surpassed. The barrier has an average height W (in energy units), and its fluctuations are given by an Ornstein–Uhlenbeck process characterized by the amplitude K and correlation time τ_c. The mean exit time is $\langle\tau\rangle = \exp(\epsilon W - \epsilon K/4\tau_c)$, where the factor ϵ describes the 'thermal' stochastic kicks. This value can be used as a measure of τ_t defined previously.

3.2. Trapping by collisions

Besides electrostatic turbulence, the fluctuations of the magnetic field observed in magnetically confined plasmas could provide a major contribution to the enhanced particle and energy transport.

The equations for the guiding-centre trajectories in this case are

$$\frac{d}{dt}\mathbf{x}_\perp(t) = \mathbf{b}(\mathbf{x}_\perp, z)\, v_\parallel, \quad \frac{d}{dt} z(t) = v_\parallel, \tag{3.2a, b}$$

where $B\mathbf{b}$ is a small stochastic component of the magnetic field perpendicular to the confining magnetic field $B\mathbf{e}_z$. The amplitude of $\mathbf{b}$ is β, with a parallel correlation length $\lambda_\parallel$. The vector potential $\Psi(\mathbf{x}_\perp, z)\mathbf{e}_z$ can be introduced and the stochastic field can be expressed as $\mathbf{b} = \nabla\Psi \times \mathbf{e}_z$. Equations (3.2) are very similar to the equations of guiding-centre motion in electrostatic turbulence. For particles that move freely along the magnetic lines ($v_\parallel$ = const), the results concerning the mean square deviation (MSD) are similar to those obtained for electrostatic turbulence:

$$\langle \mathbf{x}_\perp^2(t)\rangle = \int_0^t dt_1 \int_0^t dt_2\, v_\parallel^2\, \mathscr{L}(z(t_1)-z(t_2)) \to 2 D_m v_\parallel t, \tag{3.3}$$

where $\mathscr{L}(z_1-z_2) \equiv \langle \mathbf{b}(\mathbf{x}_\perp(z_1), z_1)\cdot \mathbf{b}(\mathbf{z}_\perp(z_2), z_2)\rangle$ is the Lagrangian correlation of

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the stochastic magnetic field on trajectories. Here $\mathscr{D}_m = \int_0^\infty d\zeta\, \mathscr{L}(\zeta)$ is the diffusion coefficient of the magnetic lines. When the velocities $v_\parallel$ are of the order of the plasma thermal velocities, a rather large particle diffusion coefficient $D = D_m v_\parallel$ can be obtained, even for small fluctuations of the magnetic field. But the parallel velocity is a random function of time owing to particle collisions. As was shown by Wang Hai-Da *et al.* (1995), this completely changes the effective process, leading to a subdiffusive evolution of the MSD. The collisions can be modelled by considering the parallel velocity $v_\parallel$ as a Gaussian random variable with zero mean and with a correlation function

$$\langle v_\parallel(t)\, v_\parallel(t')\rangle_{v_\parallel} = \chi_\parallel \nu \exp(-|t-t'|\nu),$$

where $\langle\ \rangle_{v_\parallel}$ is the average over the realizations of the collisional velocity, $\chi_\parallel$ is the collisional diffusivity along magnetic field lines and ν is the collision frequency. The trajectories are randomly reversed, and stick around their initial position $z(0) = 0$ in the sense that the average displacement is zero at any time $\langle z(t)\rangle_{v_\parallel} = 0$ and they come back to the initial position with probability one. A diffusive motion in the z direction is obtained: the probability of the displacements is a Gaussian centred at $z = 0$ and the MSD is

$$\langle z^2(t)\rangle_{v_\parallel} = 2\chi_\parallel\left[t + \frac{\exp(-\nu t)-1}{\nu}\right] \to 2\chi_\parallel t.$$

The important point that we underline here is that the motion in the z direction is now 'confined' owing to collisions. The collisions act in this case as a trapping mechanism: the probability density for the displacements in the z direction shows that the particles are located around $z = 0$ on an interval of size $2\chi_\parallel t$. The effect of this change of the parallel motion on the cross-field MSD is determined by averaging the integrand of (3.3) over the collisional velocity $v_\parallel$. The Lagrangian correlation $L(t) \equiv \langle \mathbf{V}(\tau+t)\cdot\mathbf{V}(\tau)\rangle$ of the cross-field velocity $\mathbf{V} \equiv v_\parallel \mathbf{b}$ is thus obtained (Vlad *et al.* 1996*a*) as

$$L(t) = \beta^2 \frac{\lambda_\parallel \chi_\parallel \nu}{[\lambda_\parallel^2 + \langle z^2(t)\rangle_{v_\parallel}]^{1/2}}\left\{\exp(1-\nu|t|) - \frac{\chi_\parallel[1-\exp(-\nu|t|)]^2}{\nu[\lambda_\parallel^2 + \langle z^2(t)\rangle_{v_\parallel}]}\right\}, \tag{3.4}$$

where, for simplicity, the quasilinear approximation for the correlation of the magnetic field has been considered:

$$\mathscr{L}(\zeta) = \beta^2 \exp\left(-\frac{\zeta^2}{2\lambda_\parallel^2}\right).$$

Equation (3.4) shows that the stochastic parallel velocity $v_\parallel$ has a very strong effect: the shape of the Lagrangian correlation of $\mathbf{V}$ is completely different from that of the correlation of each of the two factors $\mathbf{b}$ and $v_\parallel$. It has a negative tail whose area compensates exactly the area of the positive part: $\int_0^\infty dt\, L(t) = 0$. This is a subdiffusive behaviour: the asymptotic diffusion coefficient is zero, and the MSD grows more slowly than linearly in time: $\langle \mathbf{x}_\perp^2(t)\rangle = 4\pi^{-1/2}\chi_\parallel^{1/2}\mathscr{D}_m t^{1/2}$. Such a process in which the MSD is not linear in time is called 'strange' transport (Balescu 1997*b*; see also Section 3.4).

Thus the collisional trapping of the trajectories along the magnetic lines in the correlated zone of the magnetic field has a very strong effect. It hinders the effective diffusion process, leading to a strange diffusion characterized by an

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increase of the MSD as $t^{1/2}$. This trapping, however, is very unstable: any other contribution to the particle motion produces a release of the trajectories and consequently an effective diffusion. For instance, if the small cross-field collisional diffusion is considered by adding a small stochastic component $\boldsymbol{\eta}_\perp(t) \ll \mathbf{V}$ to the right-hand side of (3.2*a*), the Rechester–Rosenbluth formula for the diffusion coefficient is obtained (Wang Hai-Da *et al.* 1995). Actually, even the small magnetic drifts determined by the curvature of the braided magnetic lines lead to release of the particles and cause diffusion even at very small collision frequencies when $\eta_\perp(t)$ can be neglected (Vlad *et al.* 1996*a*). A similar effect can also be produced by the existence of an average flow of the plasma (parallel or perpendicular to the confining magnetic field) (Vlad *et al.* 1996*b*) or even by a time variation of the stochastic magnetic field (Vlad *et al.* 1996*a*). Usually, the competition between the collisional trapping and a release mechanism produces first an increase in diffusion (which can be rather large), but, at higher amplitudes of the releasing mechanism, the effect is the inverse, leading to a decay of the diffusion coefficient.

3.3. 1D diffusion in intrinsic trapping structures: beyond Corrsin's hypothesis

In the two previous subsections we have presented problems in which there are two independent stochastic processes: one producing diffusion and the other trajectory trapping (extrinsic trapping). Such processes can be studied rather well theoretically. Trajectory trapping also appears when the particles move in a stochastic field that has space-dependent two-point correlations (as in the numerical study in Section 2). Formally, this effect is determined by the Lagrangian nonlinearity consisting in the position dependence of the stochastic field. Both diffusion and trapping are produced by the stochastic field (intrinsic trapping). There is no satisfactory description of this very complicated nonlinear process. Most theoretical treatments lead to Bohm scaling, which was not confirmed by the numerical calculations. Actually, none of these theories take into account trajectory trapping, and so their result $\gamma = 1$ is not surprising. Although the estimation of Isichenko (1992) does not focus on a description of the trapping, this is actually the only model that implicitly takes account of this process.

We present here a one-dimensional model that describes trajectory trapping. It determines a relation between the Eulerian and Lagrangian correlations in the nonlinear case (large A) without using the assumption of Gaussian displacements and the Corrsin approximation (McComb 1990). The idea was to identify the exact equation from which the Corrsin approximation originates, to replace it by a similar exact equation and finally to find a simple approximation for this last equation. The Lagrangian correlation (LC) can be related to the conditional correlation defined by

$$L^{(x(t)=\zeta)}(0,t) \equiv \langle v(x(0),0)\, v(x(t),t)\rangle|_{x(t)=\zeta}$$

and representing the Lagrangian correlation on the subensemble characterized by the condition that the trajectories starting from $x(0) = 0$ reach the point ζ at time t:

$$L(0,t) = \int_0^\infty d\zeta P(\zeta,t)\, L^{(x(t)=\zeta)}(0,t). \tag{3.5}$$

Equation (3.5) is an exact equation and represents the decomposition of the LC

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on the class of non-intersecting subensembles characterized by the condition $x(t) = \zeta$. The weighting factor in this decomposition, $P(\zeta, t)$, is the probability density for the displacements ζ in the time intervals t. The Corrsin approximation would consist in replacing the conditional correlation $L^{(x(t)=\zeta)}(0,t)$ by the Eulerian correlation. A similar decomposition of the LC can be done on another class of subensembles, which is characterized by another parameter of the trajectories, namely the initial velocity $\dot{x}\,(0) = v_0$:

$$L(0,t) = \int_0^\infty dv_0\, P_1(v_0)\, L^{(\dot{x}(0)=v_0)}\,(0,t) \tag{3.6}$$

Like (3.5), this is an exact equation. The weighting factor $P_1(v_0)$ is the probability density for the Eulerian velocities, and the conditional correlation is defined by $L^{(\dot{x}(0)=v_0)}(0,t) \equiv \langle v(x(0),0)\, v(x(t),t)\rangle|_{x(0)=v_0}$. There are two important advantages of (3.6) compared with (3.5). First, concerning the weighting factors appearing in the two equations, one can see that $P_1(v_0)$ is known, while an approximation of $P(\zeta,t)$ beyond the Gaussian probability (which is not appropriate at large amplitude) is very difficult to find. Secondly, the condition defining the subensembles in (3.6) is of Eulerian type: it is just the velocity at a fixed point $x = 0, t = 0$. In (3.5) the condition is Lagrangian: the subensembles are classified according to the point reached by the trajectory at a given time t.

To find an approximation for the conditional correlation $L^{(\dot{x}(0)=v_0)}(0,t)$, we calculate the conditional average of the velocity at a point (x_2,t_2), given the velocity v_1 at the point (x_1,t_1), $\langle v(x_2,t_2)\rangle_{v(x_1,t_1)=v_1}$ (using the Gaussian conditional probability density). Then, solving the deterministic nonlinear equation

$$\frac{dx_2}{dt} = \langle v(x_2,t_2)\rangle_{v(x_1,t_1)=v_1} = v_1 \frac{E(x_2-x_1, t_2-t_1)}{E(0,0)}, \tag{3.7}$$

we obtain a trajectory that represents the decorrelation path. The average Lagrangian velocity in the subensemble characterized by $v(x_1,t_1) = v_1$ is approximated by the conditional average velocity calculated on the decorrelation path. For instance, taking the Eulerian correlation $E(x,t) = V^2 \exp(-|x|/\lambda)\exp(-|t|/\tau_c)$, the decorrelation path is

$$\zeta(t; v_1) = \operatorname{sign}(v_1)\,\lambda \ln\left\{1 + \frac{|v_1|\,\tau_c}{\lambda}\left[1 - \exp\left(-\frac{t}{\tau_c}\right)\right]\right\}. \tag{3.8}$$

The average Lagrangian velocity on the subensemble is approximated by

$$v_L(t_2) = v_1\left\{1 + \frac{|v_1|\tau_c}{\lambda}\left[1 - \exp\left(-\frac{t}{\tau_c}\right)\right]\right\}^{-1},$$

and the LC is estimated using (3.6) as:

$$L(t_1,t_2) = \int_0^\infty dv_0\, P_1(v_0,t)\, v_L(t_1)\, v_L(t_2) \exp\left(-\frac{|t_1-t_2|}{\tau_c}\right). \tag{3.9}$$

The diffusion coefficient is obtained by integrating the LC:

$$D = \frac{\tau_c}{(2\pi)^{1/2}\, V}\int_{-\infty}^{\infty} dv_0 \exp\left(-\frac{v_0^2}{2V^2}\right)\frac{v_0^2}{(1+|v_0|\tau_c/\lambda)^2}. \tag{3.10}$$

Although a rather crude approximation has been used, the result correctly

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reproduces the two extreme limits. For $\tau_c \to 0$ (small Kubo numbers A), (3.10) reduces to the quasilinear diffusion coefficient. For frozen turbulence (τ_c, $A \to \infty$), the total trapping diffusion coefficient $D_t = \lambda^2/\tau_c$ is obtained, which vanishes in this limit. This is expected since in a frozen one-dimensional velocity field all trajectories are trapped in the first zero (with negative slope) of the velocity. Equation (3.10) determines the diffusion coefficient for all values of the Kubo number describing the process of trajectory trapping and release due to time variation of the velocity field.

3.4. CTRW, non-local and strange transport

A very important class of random walks has recently been revived, namely continuous-time random walks (CTRW), in which displacements occur continuously in time (Montroll and Weiss 1965). The ingredients are the probability density function (PDF) of the displacements, $f(\mathbf{r})$, and the waiting-time distribution $\Psi(t)$, defined as the PDF of a pause of duration t. From these quantities, one can describe the density profile $n(t)$, i.e. the PDF that a particle starting from $\mathbf{x} = 0$ at time $t = 0$ will be at $\mathbf{x}$ at time t. This profile function has been shown (Montroll and Schlesinger 1984) to satisfy an interesting integro-differential equation that is non-local in space and non-Markovian in time. As a consequence, it may lead to a non-local generalized diffusion equation, associated with non-Gaussian and strange diffusion. The usual Gaussian case is recovered only when the transition probability $f(\mathbf{x})$ has at least finite first and second moments, and when the waiting-time distribution $\Psi(t)$ has at least a finite first moment.

Balescu (1995) has studied the possibility of describing this subdiffusive behaviour of particle evolution in a stochastic magnetic field (Section 3.2) by a CTRW. He has defined a standard long-tail (SLT) CTRW in which the Laplace transform of the waiting-time distribution has an asymptotic expansion $\hat{\Psi}(s) \sim 1 - s^\alpha$, with an exponent $0 < \alpha < 1$, and a second moment of $f(\mathbf{x})$ characterized by a finite value σ^2. Such an SLT-CTRW is shown to predict subdiffusive behaviour with an MSD displacement $\langle x^2(t) \rangle \propto t^{\alpha<1}$. When this process is applied to the magnetic diffusion problem, it correctly reproduces the results of Balescu *et al.* (1994) when $\alpha = \frac{1}{2}$ and $\sigma \propto \beta\lambda_\parallel$. But the important point is that this theory is able to yield more than the second moment: the SLT-CTRW is also able to yield the full profile function $n(t)$. The latter satisfies a non-Markovian equation in which the kernel has a long-time tail, preventing a possible Markovianization. This appears to be in contradiction with the description by Rechester and Rosenbluth (1978), which corresponds to a Markovian equation with a time-dependent diffusion coefficient to represent subdiffusion. In other words, beyond any Langevin description, a CTRW model for magnetic diffusion is able to take into account not only strange transport ($\alpha = \frac{1}{2}$), but also non-Gaussian, non-local and non-Markovian features of the complete statistics, far beyond only the second moment corresponding to the calculation of the MSD displacement.

Actually, the transport in the real magnetic configuration of a tokamak can be incredibly more complex than in the usual neoclassical theory (Balescu 1988), involving beside regular regions of undestroyed tori, not only magnetic islands, but also braided magnetic lines, stochastic seas, cantori or stochastic layers. In order to study the nature of particle transport in a finite-size

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stochastic layer, an interesting application of CTRW has been performed in Balescu (1997*a*), concerning the subdiffusive behaviour of magnetic lines in a finite-size stochastic layer described by the standard map. By applying this iteration over long time sequences for a given small stochasticity parameter, three different kind of motion are seen to alternate randomly in time: trapped motion around the island remnant ('O' basin), and passing motion above and below the island ('H' and 'L' basins). The waiting-time distributions in each basin $i = O, H, L$ and the transition frequencies have been measured by a systematic method based on symbolic dynamics (Misguich *et al.* 1998). It has been found that an average residence time exists (typically a few hundred iterations), since the exponents of the various $\hat{\Psi}_i(s)$ are found to be larger than unity ($\alpha_O = 1.1$, $\alpha_H = \alpha_L = 1.34$). By solving the corresponding equation (Montroll and Weiss 1965), the asymptotic time dependence of the density profile has been obtained analytically for each kind of motion (O, H and L). The asymptotic probability to be in a given kind of basin is simply distributed according to the time spent in that basin, multiplied by the number of sojourns in that kind of basin. Finally, the time dependence of the (subdiffusive) running diffusion coefficient has been shown to tend asymptotically to zero (as it should in a finite layer) with a long-time tail $D(t) \sim t^{-\alpha_0 = -1.1}$.

This clearly shows that stochastic layers may act as local trapping regions where the dispersive motion is strongly reduced, and that global diffusion of particles across the various magnetic structures in a tokamak cannot be described everywhere by a classical diffusive motion. Non-local, non-Gaussian, non-Markovian and trapping effects can be responsible for strange transport.

4. Conclusions

Studying various aspects of the anomalous transport induced by electrostatic or magnetic turbulence in magnetically confined plasmas, we have found that the trapping of the trajectories generally determines a decrease of the effective diffusion. In the case of electrostatic turbulence, our extended numerical simulations have led us to the conclusion that the observed temporary trapping on small, almost-closed trajectories around the extrema of the potential produces a deviation from Bohm to percolation scaling, confirming the prediction of Isichenko (1992). The statistics of the displacements and the Lagrangian correlation of the velocities are strongly influenced. The effect of trapping is even more dramatic for the evolution of collisional particles in stochastic magnetic fields. When the small perpendicular collisional velocity can be neglected, the effective diffusion is hindered and the process is subdiffusive in the absence of a release mechanism.

Concerning the theoretical models, our conclusion is that the extrinsic trapping can be described rather well. In contrast, the intrinsic trapping and the Lagrangian nonlinearity are far from being understood, although important progress has been achieved in the last decade.

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