Ergod. Th. & Dynam. Sys. (1997), **17**, 625–642
Printed in the United Kingdom © *1997 Cambridge University Press*

Ratio ergodic theorems for maps with indifferent fixed points

TOMOKI INOUE

Section of Applied Mathematics, Department of Computer Science, Faculty of Engineering, Ehime University, Matsuyama 790-77, Japan

(*Received* 20 *January* 1995 *and revised* 15 *April* 1996)

Abstract. We study one-dimensional transformations T with indifferent fixed points p and q (p and q are fixed points with $T'(p) = T'(q) = 1$). In the case when T has a Lebesgue equivalent infinite invariant measure, we give the ratio ergodic theorems which describe the limit value of the ratio of the sojourn time of the trajectory $\{T^k(x)\}_{k=0}^n$ in a small neighborhood of p to that in a small neighborhood of q.

1. *Introduction*

For an interval map T, a fixed point p is called indifferent if

$$\lim_{x\to p} T(x) = p \quad \text{and} \quad \lim_{x\to p} |T'(x)| = 1.$$

Maps with indifferent fixed points are related to intermittency. A typical example of an interval map with indifferent fixed points is the following transformation $T : [0, 1] \to [0, 1]$, which is related to continued fractions (see [**IY**]):

$$T(x) = \begin{cases} \dfrac{x}{1-x} & \text{for } x \in [0, \frac{1}{2}) \\ \dfrac{2x-1}{x} & \text{for } x \in [\frac{1}{2}, 1]. \end{cases} \tag{1.1}$$

This map T has the indifferent fixed points 0 and 1. The behavior of the typical trajectory is as follows. Once the trajectory $\{T^k(x)\}_{k=0}^n$ enters a small neighborhood of 0 or 1, the trajectory stays in the neighborhood for an extremely long time and eventually goes out of the neighborhood. After this, the trajectory oscillates chaotically and next enters a small neighborhood of 0 or 1. The trajectory repeats this kind of behavior. This behavior results from indifferent fixed points 0 and 1.

We have the following questions. Does the limit value

$$\lim_{n\to\infty} \frac{\sharp\{k \le n;\, T^k(x) \in [0, \varepsilon)\}}{\sharp\{k \le n;\, T^k(x) \in (1-\delta, 1]\}} \tag{1.2}$$

exist? What is the limit value if it exists? ($\sharp S$ stands for the number of elements of the set S, and ε and δ are positive small constants.)

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

The transformation (1.1) has a Lebesgue equivalent ergodic σ-finite invariant measure μ. As a classical result of the ergodic theory, we have the Hopf ratio ergodic theorem [**H**, **K**], by which we have

$$\lim_{n\to\infty} \frac{\sharp\{k \le n; T^k(x) \in U\}}{\sharp\{k \le n; T^k(x) \in V\}} = \frac{\mu(U)}{\mu(V)} \quad \text{if } \mu(U) < \infty \text{ and } \mu(V) < \infty.$$

But, in this case, since $\mu([0, \varepsilon)) = \infty$ and $\mu((1-\delta, 1]) = \infty$, we cannot obtain the limit value (1.2) by the Hopf ratio ergodic theorem.

In this paper, we study more general transformations T with indifferent fixed points p and q, and give the ratio ergodic theorems which describe limit values like (1.2).

2. *Setting and results*

In this section, we consider the following class of transformations including the previous transformation (1.1). Let $T : [0, 1] \to [0, 1]$ be a transformation satisfying the following conditions (i)–(v):

(i) there exists a partition $0 = c_0 = p_1 < c_1 < p_2 < c_2 < \cdots < p_{r-1} < c_{r-1} < c_r = p_r = 1$ of $[0, 1]$ such that the restrictions of T to the interval (c_{i-1}, p_i) and (p_i, c_i) are C^2 functions for each i;

(ii) $T(p_i) = p_i$ and T is continuous at p_i for each i;

(iii) $T(c_{i-1}, c_i) = (0, 1)$ for each i;

(iv) there exists a positive constant c such that $|T''(x)|/(T'(x))^2 < c < \infty$ for $x \notin \{p_1, \ldots, p_r, c_1, \ldots, c_{r-1}\}$;

(v) $T'(x) > 1$ for $x \notin \{p_1, \ldots, p_r\}$.

Remark 1. We do not assume $T'(p_i) > 1$.

Remark 2. T satisfying the above conditions (i)–(v) has a Lebesgue equivalent ergodic σ-finite invariant measure μ whose density function is bounded away from 0 and continuous except on the set of the indifferent fixed points (cf. [**T1**]).

Put $p = p_i$ and $q = p_j$ (it is not necessary that $i \neq j$). Let U and V be a small one-sided neighborhood of p and a small one-sided neighborhood of q respectively. (The closure of U does not contain any fixed points except p.) We consider the ratio of the sojourn time of the trajectory $\{T^k(x)\}_{k=0}^n$ in U to that in V. (If $p = q$ $(i = j)$, we consider a left neighborhood of p and a right neighborhood of p.) If we consider a small left neighborhood of p, assume that

$$\lim_{x\to p-0} T'(x) = 1.$$

If we consider a small right neighborhood of p, assume that

$$\lim_{x\to p+0} T'(x) = 1.$$

We make the same assumption for q. Further, assume that

$$|T(x) - x| = \theta_U |x - p|^{d_U} + o(|x - p|^{d_U}) \quad \text{in } U \tag{2.1}$$

and

$$|T(x) - x| = \theta_V |x - q|^{d_V} + o(|x - q|^{d_V}) \quad \text{in } V, \tag{2.2}$$

where θ_U, θ_V, d_U and d_V are constants.

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

Remark 3. Let μ be the invariant measure as in Remark 2. Then, $\mu(U) = \infty$ if and only if $d_U \geq 2$ (see Lemma 4.2 and Remark 4.1 in [**I1**] and Theorem 1.1 in [**I2**]).

By this remark and the Hopf ratio ergodic theorem, if d_U and d_V are less than 2, we have

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_U(T^k(x))}{\sum_{k=0}^{n} 1_V(T^k(x))} = \frac{\mu(U)}{\mu(V)} \quad \text{for Leb-a.e. } x,$$

where 1_U is the indicator function on U and Leb is the Lebesgue measure. Further, a basic corollary of the Hopf ratio ergodic theorem (Lemma 3.1) implies that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_U(T^k(x))}{\sum_{k=0}^{n} 1_V(T^k(x))} = 0 \quad \text{for Leb-a.e. } x$$

if $d_U < 2$ and $d_V \geq 2$. So, we only consider the case $2 \leq d_U \leq d_V$.

Now we give the two main theorems in this paper.

THEOREM 1. *If* $d_V > d_U \geq 2$,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_U(T^k(x))}{\sum_{k=0}^{n} 1_V(T^k(x))} = 0 \quad \textit{for Leb-a.e. } x.$$

This gives the limit in the case $d_U \neq d_V$. On the other hand, in the case $d_U = d_V$, we have the following theorem.

THEOREM 2. *If* $d_U = d_V = 2$,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_U(T^k(x))}{\sum_{k=0}^{n} 1_V(T^k(x))} = \rho \quad \textit{for Leb-a.e. } x,$$

where ρ is a positive finite constant. In particular:

(1) *if $p = q$ (U is a small left neighborhood of p, and V is a small right neighborhood of p), then*

$$\rho = \frac{\theta_V}{\theta_U};$$

(2) *if $r = 2$, $p = 0$ and $q = 1$, then*

$$\rho = \frac{\theta_V M_1}{\theta_U M_2},$$

where

$$M_1 = \lim_{x\uparrow c_1} T'(x) \quad \textit{and} \quad M_2 = \lim_{x\downarrow c_1} T'(x).$$

From this it follows that the limit value (1.2) for T defined by (1.1) is equal to 1, since $d_U = d_V = 2$, $\theta_U = \theta_V$ and $M_1 = M_2$.

In the case $d_U = d_V > 2$, the limit value of the ratio does not exist, which will be proved in another paper. Further, we have the following conjecture.

Conjecture. If $d_U = d_V > 2$, the 'limit distribution'

$$G(s) = \lim_{n\to\infty} \text{Leb}\left\{x; \frac{\sum_{k=0}^{n} 1_U(T^k(x))}{\sum_{k=0}^{n} 1_V(T^k(x))} < s\right\}$$

exists.

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

3. *Preliminaries*

In this section we give the definition of ergodicity and a basic corollary of the Hopf ratio ergodic theorem, and we state a basic property of the first return maps which are important tools in this paper. Further, we prepare a notation which we will often use. First we define the ergodicity for a transformation T on a σ-finite measure space $(X, \mathcal{F}, \mu)$.

Definition 3.1. (T, μ) is called *ergodic* if $\mu(A) = 0$ or $\mu(X \setminus A) = 0$ for every $A \in \mathcal{F}$ with $T^{-1}A = A$ μ-a.e.

The following lemma is a basic corollary of the Hopf ratio ergodic theorem.

LEMMA 3.1. *Let T be a measure-preserving ergodic transformation on a σ-finite measure space $(X, \mathcal{F}, \mu)$, let $f \in L^1(\mu)$ and let $\int g\, d\mu = \infty$, $g \geq 0$. Then*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} f(T^k(x))}{\sum_{k=0}^{n} g(T^k(x))} = 0 \quad \textit{for } \mu\textit{-a.e. } x.$$

The proof of this is easy (see Lemma 2.1 in [**I2**]).

The first return map of T on A is defined as $T^{n(x)}(x)$, where $n(x)$ is $\inf\{n \geq 1;\ T^n(x) \in A\}$. In the following lemma, let T be a transformation on a σ-finite measure space $(X, \mathcal{F}, \mu)$ and let $A \subset X$ be a measurable set with $A \subset \bigcup_{n=1}^{\infty} T^{-n}(A)$. Then the first return map is well defined.

LEMMA 3.2. *Assume that μ is a T-ergodic σ-finite invariant measure. Let $\mu|_A$ be the restriction of μ to A. Then the first return map R of T on A is $\mu|_A$-ergodic invariant. As a consequence, if $\tilde{\mu}$ is a μ-absolutely continuous σ-finite R-invariant measure, then $\tilde{\mu} = constant \cdot \mu|_A$.*

The proof of this is a minor modification of the proof of Lemma 1 in [**P**].

The following notation will be used in the rest of this paper.

Notation. $v_n \sim w_n$ stands for $\lim_{n\to\infty}(v_n/w_n) = 1$.

4. *Probabilistic lemmas*

In this section we give probabilistic lemmas, which will be used in the next section. Let L be a subset of N, where N is the set of natural numbers.

LEMMA 4.1. *Let $\xi_{z,k}$ be a random variable on a probability space $(X, \mathcal{F}, \mu)$ for $k = 0, 1, \ldots, z$. Assume that*

$$\sum_{z\in L} \frac{1}{z^2(\mathrm{E}(\xi_{z,0}))^2} < \infty \quad \textit{and} \quad \mathrm{E}(\xi_{z,i} \cdot \xi_{z,j}) \leq \textit{constant} \cdot (\mathrm{E}(\xi_{z,0}))^2.$$

Then,

$$\lim_{L\ni z\to\infty} \frac{1}{(z+1)^2(\mathrm{E}(\xi_{z,0}))^2} \sum_{k=0}^{z} \xi_{z,k} = 0 \quad \mu\textit{-a.e.}$$

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

Proof. Let $\varepsilon > 0$ be arbitrary. Put $p_z = \mathrm{E}(\xi_{z,0})$. Then it follows from the Chebyshev inequality that

$$\begin{aligned}\mu\left(\left\{\left|\frac{1}{((z+1)\cdot p_z)^2}\sum_{k=0}^{z}\xi_{z,k}\right| > \varepsilon\right\}\right) &\leq \frac{1}{\varepsilon^2(z+1)^4\cdot {p_z}^4}\,\mathrm{E}\left(\left|\sum_{k=0}^{z}\xi_{z,k}\right|^2\right)\\ &\leq \text{constant}\cdot\varepsilon^{-2}(z+1)^{-2}\cdot {p_z}^{-2}.\end{aligned}$$

By the Borel–Cantelli lemma we obtain Lemma 4.1. □

LEMMA 4.2. *Let $u : N \to N$ be a function. Let $\zeta_{z,k}$ be a random variable on a probability space $(X, \mathcal{F}, \mu)$ for $k = 0, \ldots, z$. Assume that*

$$\mathrm{E}(\zeta_{z,0}) = \mathrm{E}(\zeta_{z,k}) \geq 0, \quad \sum_{z\in L}\frac{u(z)}{z\mathrm{E}(\zeta_{z,0})} < \infty \quad \textit{and}$$
$$\mathrm{E}(\zeta_{z,k}\cdot\zeta_{z,i}) - \mathrm{E}(\zeta_{z,0})^2 \leq K\gamma^{-|k-i|}u(z)\mathrm{E}(\zeta_{z,0}),$$

where $K > 0$ and $\gamma > 1$ are constants. Then,

$$\lim_{L\ni z\to\infty}\frac{1}{(z+1)\mathrm{E}(\zeta_{z,0})}\sum_{k=0}^{z}\zeta_{z,k} = 1 \quad \mu\textit{-a.e.}$$

Proof. Let $\varepsilon > 0$ be arbitrary. By the Chebyshev inequality we have

$$\begin{aligned}&\mu\left(\left\{\left|\frac{1}{(z+1)\mathrm{E}(\zeta_{z,0})}\sum_{k=0}^{z}\zeta_{z,k} - 1\right| > \varepsilon\right\}\right)\\ &\leq \left(\frac{1}{\varepsilon(z+1)\mathrm{E}(\zeta_{z,0})}\right)^2\int\left|\sum_{k=0}^{z}\zeta_{z,k} - (z+1)\mathrm{E}(\zeta_{z,0})\right|^2 d\mu.\end{aligned}$$

Since it follows from the first and third assumptions of the lemma that

$$\begin{aligned}\sum_{0\leq k,i\leq z}\mathrm{E}(\zeta_{z,k}\cdot\zeta_{z,i}) &\leq \sum_{0\leq k,i\leq z}\mathrm{E}(\zeta_{z,0})^2 + \sum_{0\leq k,i\leq z}K\gamma^{-|k-i|}u(z)\mathrm{E}(\zeta_{z,0})\\ &= (z+1)^2\mathrm{E}(\zeta_{z,0})^2 + K\sum_{i=0}^{z}(i+1)\gamma^{-z+i}u(z)\mathrm{E}(\zeta_{z,0})\\ &\leq (z+1)^2\mathrm{E}(\zeta_{z,0})^2 + \text{constant}\cdot(z+1)u(z)\mathrm{E}(\zeta_{z,0}),\end{aligned}$$

we have

$$\begin{aligned}&\mu\left(\left\{\left|\frac{1}{(z+1)\mathrm{E}(\zeta_{z,0})}\sum_{k=0}^{z}\zeta_{z,k} - 1\right| > \varepsilon\right\}\right)\\ &\leq \left(\frac{1}{\varepsilon(z+1)\mathrm{E}(\zeta_{z,0})}\right)^2\left(\text{constant}\cdot(z+1)u(z)\mathrm{E}(\zeta_{z,0})\right)\\ &\leq \frac{\text{constant}\cdot u(z)}{\varepsilon^2(z+1)\mathrm{E}(\zeta_{z,0})}.\end{aligned}$$

The lemma follows from the second assumption of the lemma and the Borel–Cantelli lemma. □

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

5. *Ergodic properties for piecewise monotonic maps with countable partitions*

We devote this section to the investigation of some ergodic properties for piecewise monotonic maps with countable partitions which naturally arise from the first return maps.

Let J be the union of disjoint intervals $J_1, \dots, J_n$ with $\mathrm{Leb}(J) < \infty$. We assume that $S : J \to J$ satisfies the following conditions (1)–(4):

(1) there exists a partition $\{I_\lambda\}_{\lambda\in\Lambda}$ of J such that the restriction of S to the interval I_λ is a C^2 function for each λ;

(2) for each λ there exists an i such that $S(I_\lambda) = J_i$;

(3) there exists a positive constant K_0 such that

$$|S''(x)|/(S'(x))^2 < K_0 < \infty,$$

whenever the first and the second derivatives of S are defined;

(4) there exists a constant $K_1 > 1$ with $S'(x) > K_1$, whenever the first derivative of S is defined.

Then we have the following lemma.

LEMMA 5.1. (See [**R**].) *S has an absolutely continuous invariant probability measure μ and there exists a non-negative function g of bounded variation such that $\mu(U) = \int_U g(x)\,\mathrm{Leb}(dx)$ for all measurable set U.*

Notice that $S^k : J \to J$ satisfies the conditions (1)–(4) replacing S with S^k. Let $\{I_\lambda^{(k)}\}_{\lambda\in\Lambda^k}$ be the partition associated with S^k and let $\varphi_\lambda^{(k)}$ be the inverse function of the restriction to $I_\lambda^{(k)}$ of S^k. Let v_λ and w_λ be the left and right endpoints of $S^k(I_\lambda)$ respectively.

LEMMA 5.2. *$\varphi_\lambda^{(k)\prime}(x) \le K_2\,\mathrm{Leb}(I_\lambda^{(k)})$ for $x \in S^k(I_\lambda^{(k)})$, where K_2 is a constant independent of k and λ.*

Proof. For simplicity, we omit the index (k) in the proof. Let $x, y \in (v_\lambda, w_\lambda)$. By the mean value theorem there exists an ω such that

$$\log\frac{\varphi_\lambda'(x)}{\varphi_\lambda'(y)} = \frac{\varphi_\lambda''(\omega)}{\varphi_\lambda'(\omega)}(x-y) \le K_0(w_\lambda - v_\lambda).$$

Since there exists a u_λ such that $\varphi_\lambda(w_\lambda) - \varphi_\lambda(v_\lambda) = \varphi_\lambda'(u_\lambda)(w_\lambda - v_\lambda)$, it follows from the previous inequality that

$$\begin{aligned}\varphi_\lambda'(x) &\le \exp(K_0(w_\lambda - v_\lambda))\varphi_\lambda'(u_\lambda)\\ &= \exp(K_0(w_\lambda - v_\lambda))\frac{\varphi_\lambda(w_\lambda) - \varphi_\lambda(v_\lambda)}{w_\lambda - v_\lambda}\\ &= \frac{\exp(K_0(w_\lambda - v_\lambda))}{w_\lambda - v_\lambda}\cdot\mathrm{Leb}(I_\lambda).\end{aligned}$$

Put $K_2 = \exp(K_0\,\mathrm{Leb}(J))(\min_i\{\mathrm{Leb}(J_i)\})^{-1}$. Then we obtain the lemma. □

LEMMA 5.3. (I) *Assume that the density g satisfies*

$$0 < \frac{K_4}{\mathrm{Leb}(J)} \le g \le \frac{K_5}{\mathrm{Leb}(J)}.$$

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

Let A be a union of disjoint intervals, that is, $A = \cup_j (x_j, y_j)$, and let $B = \cup_{\lambda\in\Lambda_B} I_\lambda^{(k)}$, where Λ_B is a subset of Λ^k. Then

$$\mu(S^{-k}A \cap B) \le K_3 \cdot \mu(A) \cdot \mu(B),$$

where K_3 is a constant independent of A, B and k.

(II) *Let $\hat{J}_{i,n}$ be a decreasing sequence of the intervals with respect to n such that* $\mathrm{Leb}(\hat{J}_{i,n} \cap \hat{J}_{j,n}) = 0$ $(i \ne j)$. *Put $\hat{J}_n = \cup_i \hat{J}_{i,n}$. Consider $\hat{S}_n : \hat{J}_n \to \hat{J}_n$ instead of $S : J \to J$. Assume that $\hat{S}_n : \hat{J}_n \to \hat{J}_n$ satisfies conditions* (1)–(4), *and assume that* $\mathrm{Leb}(\hat{J}_n)/\mathrm{Leb}(\hat{J}_{i,n})$ *is bounded for all i, n. Then, the assertion corresponding to* (I) *holds and the constant corresponding to K_3 is finite even if* $\mathrm{Leb}(\hat{J}_{i,n}) \to 0$ *for all i as $n \to \infty$.*

Proof. For simplicity, we omit the index (k) in the proof. By the assumption on g we have

$$\frac{\mu(S^{-k}A \cap B)}{\mu(B)} \le \frac{K_5 \sum_{\lambda\in\Lambda_B} \sum_j (\varphi_\lambda(y_j) - \varphi_\lambda(x_j))}{K_4 \,\mathrm{Leb}(B)}.$$

By the mean value theorem, there exists $z_j \in (x_j, y_j)$ such that

$$\varphi_\lambda(y_j) - \varphi_\lambda(x_j) = \varphi_\lambda'(z_j)(y_j - x_j).$$

This equality and the previous lemma imply

$$\begin{aligned} \frac{\mu(S^{-k}A \cap B)}{\mu(B)} &\le \frac{K_5 K_2 \sum_{\lambda\in\Lambda_B} \sum_j \mathrm{Leb}(I_\lambda)(y_j - x_j)}{K_4 \,\mathrm{Leb}(B)} \\ &= \frac{K_5 K_2 \,\mathrm{Leb}(A)}{K_4} \le \frac{K_5 K_2 \,\mathrm{Leb}(J)\mu(A)}{K_4^2}, \end{aligned}$$

which shows the first half of the lemma. The latter half of the lemma follows from the end of the proof of Lemma 5.2. □

Let $\{\Lambda_l\}$ be a decreasing sequence of subsets of Λ. Put

$$E_l = \cup_{\lambda\in\Lambda_l} I_\lambda.$$

Let $\{\Lambda_l^k\}$ be the decreasing sequence of the subsets of Λ^k such that

$$E_l = \cup_{\lambda\in\Lambda_l^k} I_\lambda^{(k)}.$$

LEMMA 5.4. *Let $u : N \to N$ be a function satisfying*

$$z \cdot \mu(E_{u(z)}) \sim \textit{constant} \cdot \log_2(\log_2 z), \tag{5.1}$$

and let

$$L = \{z; \log_2(\log_2 z) \textit{ is a positive integer}\}. \tag{5.2}$$

Assume that the density g is bounded away from 0. Then

$$\lim_{L\ni z\to\infty} \frac{1}{(z+1)^2(\mu(E_{u(z)}))^2} \sum_{k=0}^{z} 1_{E_{u(z)}}(S^k(x)) = 0 \quad \textit{for } \mu\textit{-a.e. } x.$$

Proof. We easily obtain this from Lemmas 4.1 and 5.3. □

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

Now put

$$\hat{\zeta}_{u,k}(x) = \sum_{m=1}^{u} 1_{E_m}(S^k(x)),$$

where u is an integer.

Let P be the Frobenius–Perron operator corresponding to (S, Leb) (the definition of the Frobenius–Perron operator is found in [**LM**]).

LEMMA 5.5. *Assume that S is ergodic and that P is asymptotically periodic, that is, for any probability density function f,*

$$\lim_{n\to\infty} \left\| P^n f - \sum_{i=1}^{M} \lambda_i(f) g_{\alpha^n(i)} \right\|_{L^1(\mathrm{Leb})} = 0,$$

where $g_1, \ldots, g_M$ are probability density functions with $g_i(x) \cdot g_j(x) = 0$ for all $i \neq j$ and α is a cyclic permutation of $\{1, \ldots, M\}$. Then

$$\mathrm{E}(\hat{\zeta}_{u,k} \cdot \hat{\zeta}_{u,i}) - \mathrm{E}(\hat{\zeta}_{u,0})^2 \leq K_6 \gamma^{-|k-i|} u \mathrm{E}(\hat{\zeta}_{u,0}),$$

where $\mathrm{E}(\zeta)$ is the expectation of ζ on the probability space (J, μ), $K_6 > 0$ and $\gamma > 1$ are constants independent of i, k and u.

Proof. It is sufficient to prove that

$$\mathrm{E}(\hat{\zeta}_{u,k} \cdot \hat{\zeta}_{u,0}) - \mathrm{E}(\hat{\zeta}_{u,0})^2 \leq K_6 \gamma^{-k} u \mathrm{E}(\hat{\zeta}_{u,0}).$$

Put $g = M^{-1} \sum_{i=1}^{M} g_i$. Then g is the Radon–Nikodym derivative of μ. Put $C_i = \{x; g_i(x) > 0\}$ and $\mu_i(E) = \mu(E \cap C_i)$. We first estimate $\mathrm{E}(\hat{\zeta}_{u,k} \cdot \hat{\zeta}_{u,0})$:

$$\begin{aligned}
\mathrm{E}(\hat{\zeta}_{u,k} \cdot \hat{\zeta}_{u,0}) &= \sum_{l=1}^{u} \sum_{i=1}^{M} \sum_{m=1}^{u} \int_J 1_{E_l \cap C_i}(x) 1_{E_m}(S^k(x)) \mu(dx) \\
&= \sum_{l=1}^{u} \sum_{i=1}^{M} \sum_{m=1}^{u} \int_{S^{-k}(E_m)} 1_{E_l \cap C_i}(x) g(x) \,\mathrm{Leb}(dx) \\
&= \frac{1}{M} \sum_{l=1}^{u} \sum_{i=1}^{M} \sum_{m=1}^{u} \int_{S^{-k}(E_m)} 1_{E_l}(x) g_i(x) \,\mathrm{Leb}(dx) \\
&= \frac{1}{M} \sum_{l=1}^{u} \sum_{i=1}^{M} \sum_{m=1}^{u} \int_{E_m} P^k(1_{E_l} \cdot g_i)(x) \,\mathrm{Leb}(dx) \\
&= \frac{1}{M} \sum_{l=1}^{u} \sum_{i}^{\star} \sum_{m=1}^{u} \mu_i(E_l) \int_{E_m} P^k \left(\frac{1}{\mu_i(E_l)} 1_{E_l} \cdot g_i \right)(x) \,\mathrm{Leb}(dx),
\end{aligned} \tag{5.3}$$

where $\sum_i^{\star}$ is the sum over i with $\mu_i(E_l) > 0$. Next we estimate $\mathrm{E}(\hat{\zeta}_{u,0})^2$:

$$\mathrm{E}(\hat{\zeta}_{u,0})^2 = \sum_{l=1}^{u} \sum_{m=1}^{u} \mu(E_l) \int_{E_m} g(x) \,\mathrm{Leb}(dx)$$

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

$$= \frac{1}{M}\sum_{l=1}^{u}\sum_{m=1}^{u}\mu(E_l)\int_{E_m}\sum_{i=1}^{M} g_{\alpha^j(i)}(x)\,\mathrm{Leb}(dx)$$
$$\geq \frac{1}{M}\sum_{l=1}^{u}\sum_{m=1}^{u}\sum_{i=1}^{M}\mu_i(E_l)\int_{E_m} g_{\alpha^j(i)}(x)\,\mathrm{Leb}(dx), \tag{5.4}$$

where j is the integer satisfying $k = j \pmod{M}$, $0 \leq j < M$. Since

$$\int_{E_m}\left\{P^k\left(\frac{1}{\mu_i(E_l)}1_{E_l}\cdot g_i\right)(x) - g_{\alpha^j(i)}(x)\right\}\mathrm{Leb}(dx)$$
$$\leq \left\|P^k\left(\frac{1}{\mu_i(E_l)}1_{E_l}\cdot g_i\right) - g_{\alpha^j(i)}\right\|_{L^1(\mathrm{Leb})} \quad \text{for } i \text{ with } \mu_i(E_l) > 0,$$

it follows from (5.3) and (5.4) that

$$\mathrm{E}(\hat{\zeta}_{u,k}\cdot\hat{\zeta}_{u,0}) - \mathrm{E}(\hat{\zeta}_{u,0})^2 \leq \frac{u}{M}\sum_{l=1}^{u}\sum_{i}^{\star}\mu_i(E_l)\left\|P^k\left(\frac{1}{\mu_i(E_l)}1_{E_l}\cdot g_i\right) - g_{\alpha^j(i)}\right\|_{L^1(\mathrm{Leb})}.$$

Now we denote the norm $\|\cdot\|_V$ by

$$\|f\|_V = \max(\|f\|_{L^1(\mathrm{Leb})}, \inf\{\mathrm{Var}(\widetilde{f});\ \widetilde{f} \text{ is a version of } f\}),$$

where $\mathrm{Var}(f)$ is the variation of f.

From Theorem 1 in [**R**] it follows that there exists a $\gamma > 1$ and a $K_7 > 0$ such that

$$\left\|P^k\left(\frac{1}{\mu_i(E_l)}1_{E_l}\cdot g_i\right) - g_{\alpha^j(i)}\right\|_{L^1(\mathrm{Leb})} \leq K_7\gamma^{-k}\left(\left\|\frac{1}{\mu_i(E_l)}P^j\left(1_{E_l}\cdot g_i\right)\right\|_V + \left\|g_{\alpha^j(i)}\right\|_V\right),$$

where we consider $P^k g_i$, $P^j g_i$ and $g_{\alpha^j(i)}$ as functions on $C(\alpha^j(i)) = C_{\alpha^j(i)}$. We are going to estimate the variation of $P^j(1_{E_l}\cdot g_i)$. For simplicity we omit (j) of $\varphi_\lambda^{(j)}$ and $I_\lambda^{(j)}$, and j of Λ_l^j.

$$\begin{aligned}
\mathrm{Var}_{C(\alpha^j(i))}(P^j(1_{E_l}\cdot g_i)) &\leq \mathrm{Var}(P^j(1_{E_l}\cdot g)) = \mathrm{Var}\left(\sum_{\lambda\in\Lambda_l}\varphi_\lambda'\cdot g(\varphi_\lambda)\right)\\
&\leq \sum_{\lambda\in\Lambda_l}\left(\sup_{x\in J}\varphi_\lambda'(x)\cdot\mathrm{Var}(g(\varphi_\lambda)) + \int g(\varphi_\lambda(x))|\varphi_\lambda''(x)|\,dx\right)\\
&\leq \mathrm{Var}(g)\sum_{\lambda\in\Lambda_l}\sup_{x\in J}\varphi_\lambda'(x) + \sup_{x\in J}g(x)\sum_{\lambda\in\Lambda_l}\int|\varphi_\lambda''(x)|\,dx\\
&\leq \mathrm{Var}(g)K_3\sum_{\lambda\in\Lambda_l}\mathrm{Leb}(I_\lambda) + \sup_{x\in J}g(x)\sum_{\lambda\in\Lambda_l}K_0\,\mathrm{Leb}(I_\lambda)\\
&= (K_3\,\mathrm{Var}(g) + K_0\sup_{x\in J}g(x))\,\mathrm{Leb}(E_l).
\end{aligned}$$

We remark that $\mathrm{Leb}(E_l)/\mu(E_l) \leq K_4^{-1}$, since $g \geq K_4$.

Put

$$K_6 = K_7((K_3\,\mathrm{Var}(g) + K_0\sup_{x\in J}g(x))K_4^{-1} + \mathrm{Var}(g) + 2).$$

Then we have

$$\mathrm{E}(\hat{\zeta}_{u,k}\cdot\hat{\zeta}_{u,0}) - \mathrm{E}(\hat{\zeta}_{u,0})^2 \leq uK_6\gamma^{-k}\mathrm{E}(\hat{\zeta}_{u,0}),$$

and hence we obtain the lemma. □

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

LEMMA 5.6. *Let $u : N \to N$ be a function satisfying* (5.1) *and* (5.2) *and let S satisfy the assumption as in Lemma 5.5. Assume that $\mu(E_l) \sim constant \cdot l^{-s}$ $(0 < s \leq 1)$. Then*

$$\lim_{L \ni z \to \infty} \frac{1}{(z+1)\sum_{m=1}^{u(z)} \mu(E_m)} \sum_{k=0}^{z} \sum_{m=1}^{u(z)} 1_{E_m}(S^k(x)) = 1 \quad \textit{for } \mu\textit{-a.e. } x.$$

Proof. In the case $s = 1$, putting $\zeta_{z,k} = \hat{\zeta}_{u(z),k}$ we easily obtain this by Lemmas 4.2 and 5.5. In the case $s < 1$ we obtain this by a similar technique. □

6. *Proof of the theorems*

In this section we are going to prove Theorems 1 and 2. Let T be a map satisfying the conditions (i)–(v) of §2. Let T_i be the restriction of T to (c_{i-1}, c_i). Set

$$T_i(c_{i-1}) = \lim_{x \to c_{i-1}} T_i(x) \quad \text{and} \quad T_i(c_i) = \lim_{x \to c_i} T_i(x).$$

Put

$$\begin{aligned} a_{j,n} &= T_j^{-n}(c_{j-1}) \quad \text{for } j = 2, \ldots, r, \\ b_{i,n} &= T_i^{-n}(c_i) \quad \text{for } i = 1, \ldots, r-1, \\ \alpha_{j,\kappa,n} &= T_\kappa^{-1}(a_{j,n}) \quad \text{for } \kappa \neq j \text{ and } n \geq 0, \\ \beta_{i,\kappa,n} &= T_\kappa^{-1}(b_{i,n}) \quad \text{for } \kappa \neq i \text{ and } n \geq 0. \end{aligned}$$

Then the first return map R of T on $\Omega = [0,1] \setminus (\cup_i [p_i, b_{i,1}] \cup \cup_j [a_{j,1}, p_j])$ can be represented in the following form. For $n \geq 1$,

$$R(x) = T^n(x) \quad \text{if } x \in \bigcup_i \left(\bigcup_{\kappa \neq j} (\alpha_{j,\kappa,n-1}, \alpha_{j,\kappa,n}) \right) \cup \bigcup_j \left(\bigcup_{\kappa \neq i} (\beta_{i,\kappa,n}, \beta_{i,\kappa,n-1}) \right).$$

It is clear that $R(x)$ is defined except on the set of the endpoints of the countable partition of Ω. It is easy to see that $S = R$ satisfies conditions (1)–(4) in §5 and the assumption of Lemma 5.5.

Let μ be a Lebesgue equivalent T-ergodic invariant measure (such a measure exists as we have already remarked in §2) and let $\tilde{\mu}$ be an absolutely continuous R-ergodic invariant measure. We remark that $\tilde{\mu} = \text{constant} \cdot \mu|_\Omega$ (Lemma 3.2).

Now fix i and j. We first consider the case when U is a small right neighborhood of p and V is a small left neighborhood of q, where $p = p_i$ and $q = p_j$. Put $c = c_i$ and $d = c_{j-1}$.

We prepare some notation.

Notation.

$$\begin{gathered} F(t,x) := \frac{\sum_{k=0}^{t} 1_{(p,c)}(T^k(x))}{\sum_{k=0}^{t} 1_{(d,q)}(T^k(x))}, \\ \alpha_{\kappa,n} := \alpha_{j,\kappa,n}, \quad \beta_{\kappa,n} := \beta_{i,\kappa,n}, \\ E(\alpha_n) := \bigcup_{\kappa \neq j} [\alpha_{\kappa,n}, T_\kappa^{-1}(q)], \quad E(\beta_n) := \bigcup_{\kappa \neq i} [T_\kappa^{-1}(p), \beta_{\kappa,n}], \\ \tau_t := \max \left\{ \tau; \sum_{k=1}^{\tau} 1_\Omega(T^k(x)) = t \right\}, \\ t^* : \text{the integer satisfying } \log_2(\log_2 t^*) = [\log_2(\log_2 t)] + 1, \end{gathered}$$

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

where [ ] is the Gaussian symbol,

$$n(t) : \text{the maximal integer satisfying } \log_2(\log_2 t^*) \le t^*(\mu(E(\alpha_{n(t)})) + \mu(E(\beta_{n(t)}))),$$

$$H_p(t,x) := \sum_{k=0}^{\tau_t} 1_{[p,b_{i,n(t)})}(T^k(x)), \quad H_c(t,x) := \sum_{k=0}^{\tau_t} 1_{[b_{i,n(t)},c)}(T^k(x)),$$

$$H_d(t,x) := \sum_{k=0}^{\tau_t} 1_{(d,a_{j,n(t)}]}(T^k(x)), \quad H_q(t,x) := \sum_{k=0}^{\tau_t} 1_{(a_{j,n(t)},q]}(T^k(x)),$$

$$\eta(t,x) := \frac{H_c(t,x)}{H_d(t,x)}.$$

As we see in the following lemma, it is essential for our purpose to calculate the limit value of $F(t,x)$.

LEMMA 6.1. *Assume that*

$$\mu(U) = \infty, \quad \mu(V) = \infty \quad \textit{and} \quad \mu(((p,c) \setminus U) \cup ((d,q) \setminus V)) < \infty$$

and that $\lim_{t\to\infty} F(t,x)$ *exists. Then*

$$\lim_{t\to\infty} F(t,x) = \lim_{t\to\infty} \frac{\sum_{k=0}^{t} 1_U(T^k(x))}{\sum_{k=0}^{t} 1_V(T^k(x))} \quad \textit{for } \mu\textit{-a.e. } x.$$

Proof. We easily obtain this by Lemma 3.1. □

In the rest of this section we assume (2.1) and (2.2). The following lemma is important in determining the limit value of $F(t,x)$.

LEMMA 6.2. *Assume that* $2 \le d_U \le d_V$. *Then*

$$\mu(E(\alpha_n)) \sim \alpha' n^{-1/(d_V-1)}, \quad \mu(E(\beta_n)) \sim \beta' n^{-1/(d_U-1)},$$

where α' *and* β' *are positive constants. In addition*

$$\rho = \lim_{n\to\infty} \frac{\mu(E(\beta_n))}{\mu(E(\alpha_n))} \tag{6.1}$$

exists and $0 \le \rho < \infty$. *Further, let* g *be the Radon–Nikodym derivative of the invariant measure* μ. *Then* ρ *is given by*

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{\sum_{\kappa\neq i} g(T_\kappa^{-1}(p)) \operatorname{Leb}([T_\kappa^{-1}(p), \beta_{\kappa,n}])}{\sum_{\kappa\neq j} g(T_\kappa^{-1}(q)) \operatorname{Leb}([\alpha_{\kappa,n}, T_\kappa^{-1}(q)])} \\ &= \lim_{n\to\infty} \frac{\sum_{\kappa\neq i} g(T_\kappa^{-1}(p))(T_\kappa'(T_\kappa^{-1}(p)))^{-1}(\theta_V(d_V-1)n)^{1/(d_V-1)}}{\sum_{\kappa\neq j} g(T_\kappa^{-1}(q))(T_\kappa'(T_\kappa^{-1}(q)))^{-1}(\theta_U(d_U-1)n)^{1/(d_U-1)}}. \end{aligned} \tag{6.2}$$

As a consequence:

(1) $\rho = 0$ *if* $d_U < d_V$;
(2) $0 < \rho < \infty$ *if* $d_U = d_V$;
(3) *if* $p = q$ *and* $d_U = d_V = 2$, *then* $\rho = \theta_V/\theta_U$;
(4) *if* $r = 2$, $p = 0$, $q = 1$ *and* $d_U = d_V = 2$, *then* $\rho = (T_1'(c_1)\theta_V)/(T_2'(c_1)\theta_U)$.

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

To prove this we need the following lemma.

LEMMA 6.3. (See [**T2**, Corollary, §3].)

$$\mathrm{Leb}([p, b_n]) \sim (\theta_U(d_U - 1)n)^{-1/(d_U-1)}.$$

Proof of Lemma 6.2. Since T_κ is a C^1 function in the neighborhood of $T_\kappa^{-1}(p)$ for each κ, $\mathrm{Leb}([T_\kappa^{-1}(p), \beta_{\kappa,n}]) \sim (T_\kappa'(T_\kappa^{-1}(p)))^{-1}\,\mathrm{Leb}([p, b_n])$ for each κ. Thus it follows from Lemma 6.3 that

$$\mathrm{Leb}([T_\kappa^{-1}(p), \beta_{\kappa,n}]) \sim (T_\kappa'(T_\kappa^{-1}(p)))^{-1}(\theta_U(d_U - 1)n)^{-1/(d_U-1)}$$

for each $\kappa \neq i$. Similarly, we have

$$\mathrm{Leb}([\alpha_{\kappa,n}, T_\kappa^{-1}(q)]) \sim (T_\kappa'(T_\kappa^{-1}(q)))^{-1}(\theta_V(d_V - 1)n)^{-1/(d_V-1)}$$

for each $\kappa \neq j$. So, the limit

$$\lim_{n\to\infty} \frac{\sum_{\kappa\neq i} g(T_\kappa^{-1}(p))\mathrm{Leb}([T_\kappa^{-1}(p), \beta_{\kappa,n}])}{\sum_{\kappa\neq j} g(T_\kappa^{-1}(q))\,\mathrm{Leb}([\alpha_{\kappa,n}, T_\kappa^{-1}(q)])}$$

exists. Since g is continuous and bounded away from 0 at $T_\kappa^{-1}(p)$ for $\kappa \neq i$ and at $T_\kappa^{-1}(q)$ for $\kappa \neq j$, we have

$$\lim_{n\to\infty} \frac{\sum_{\kappa\neq i} \mu([T_\kappa^{-1}(p), \beta_{\kappa,n}])}{\sum_{\kappa\neq j} \mu([\alpha_{\kappa,n}, T_\kappa^{-1}(q)])} = \lim_{n\to\infty} \frac{\sum_{\kappa\neq i} g(T_\kappa^{-1}(p))\,\mathrm{Leb}([T_\kappa^{-1}(p), \beta_{\kappa,n}])}{\sum_{\kappa\neq j} g(T_\kappa^{-1}(q))\,\mathrm{Leb}([\alpha_{\kappa,n}, T_\kappa^{-1}(q)])}.$$

Therefore $\lim_{n\to\infty} \mu(E(\beta_n))/\mu(E(\alpha_n))$ exists and ρ is given by equality (6.2). From the assumption $2 \leq d_U \leq d_V$ it follows that $0 \leq \rho < \infty$. If $2 \leq d_U < d_V$, we have $\rho = 0$ by equality (6.2). So we have (1).

If $2 \leq d_U = d_V$, we have $0 < \rho < \infty$ by the equality (6.2). So we have (2).

Now we prove (3). Since $p = q$, we have

$$\rho = \lim_{n\to\infty} \frac{\sum_{\kappa\neq i} g(T_\kappa^{-1}(p)) \cdot (T_\kappa'(T_\kappa^{-1}(p)))^{-1}(\theta_V(d_V - 1)n)^{1/(d_V-1)}}{\sum_{\kappa\neq i} g(T_\kappa^{-1}(p)) \cdot (T_\kappa'(T_\kappa^{-1}(p)))^{-1}(\theta_U(d_U - 1)n)^{1/(d_U-1)}}.$$

Thus, since $d_U = d_V = 2$, we have $\rho = \theta_V/\theta_U$.

Finally, we show (4). Since $r = 2$, $p = 0$, $q = 1$ and $d_U = d_V = 2$, we have

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{g(T_2^{-1}(0)) \cdot (T_2'(T_2^{-1}(0)))^{-1}(\theta_V n)}{g(T_1^{-1}(1)) \cdot (T_1'(T_1^{-1}(1)))^{-1}(\theta_U n)} \\ &= \frac{\theta_V \cdot g(T_2^{-1}(0)) \cdot T_1'(T_1^{-1}(1))}{\theta_U \cdot g(T_1^{-1}(1)) \cdot T_2'(T_2^{-1}(0))}. \end{aligned}$$

Since g is essentially continuous at $T_1^{-1}(1) = T_2^{-1}(0) = c_1$, we obtain $\rho = T_1'(c_1)\theta_V / T_2'(c_1)\theta_U$. □

LEMMA 6.4. *ρ in* (6.1) *satisfies the following property:*

$$\lim_{t\to\infty} \frac{\sum_{l=1}^{n(t)} \sum_{k=0}^{t} 1_{E(\beta_l)}(R^k(x))}{\sum_{l=1}^{n(t)} \sum_{k=0}^{t} 1_{E(\alpha_l)}(R^k(x))} = \rho \quad \textit{for } \mu\textit{-a.e. } x. \tag{6.3}$$

As a consequence, $\eta(t, x)$ converges to ρ for μ-a.e. x.

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

Proof. Notice that

$$\lim_{n\to\infty}\frac{\sum_{l=1}^{n}\mu(E(\beta_l))}{\sum_{l=1}^{n}\mu(E(\alpha_l))}=\lim_{n\to\infty}\frac{\mu(E(\beta_n))}{\mu(E(\alpha_n))}=\rho,$$

since $\sum_{l=1}^{\infty}\mu(E(\alpha_l))=\sum_{l=1}^{\infty}\mu(E(\beta_l))=\infty$. Then, by Lemma 5.6 we can easily obtain equality (6.3).

We remark that

$$\begin{aligned}H_d(t,x) &= \sum_{k=0}^{\tau_t}\sum_{l=1}^{n(t)}1_{(a_{j,l-1},a_{j,l}]}(T^k(x))=\sum_{k=0}^{t}\sum_{l=1}^{n(t)}1_{E(\alpha_{l-1})}(R^k(x)),\\ H_c(t,x) &= \sum_{k=0}^{t}\sum_{l=1}^{n(t)}1_{E(\beta_{l-1})}(R^k(x)).\end{aligned}$$

From this fact and equality (6.3) it follows that $\eta(t,x)$ converges to ρ for μ-a.e. x. □

We are going to show the following proposition.

PROPOSITION 6.5. *$F(t,x)$ converges to ρ for μ-a.e. x if $2\le d_U<d_V$ or $d_U=d_V=2$.*

In order to show this, we prove the following lemma.

LEMMA 6.6. *If $2\le d_U<d_V$ or $d_U=d_V=2$, then*

$$\mu|_\Omega\left(\bigcap_{k=0}^{\infty}\bigcup_{t=k}^{\infty}\{x;\,|F(\tau_t,x)-\rho|>\varepsilon\}\right)=0\quad\text{for any }\varepsilon>0.$$

To prove Lemma 6.6 we prepare the following lemma.

LEMMA 6.7. *If $d_U=2$, $H_c(t,x)\sim$ constant $\cdot\, t\log n(t)$.*

Proof. Since $H_c(t,x)\sim\sum_{k=0}^{t}\sum_{l=1}^{n(t)}1_{E(\beta_l)}(R^k(x))$, it follows from Lemma 5.6 that

$$H_c(t,x)\sim t\sum_{l=1}^{n(t)}\mu(E(\beta_l))\sim\text{constant}\cdot t\sum_{l=1}^{n(t)}l^{-1}\sim\text{constant}\cdot t\log n(t).$$

This completes the proof. □

Proof of Lemma 6.6. Put

$$E_{n(t)}:=E(n(t)):=E(\alpha_{n(t)})\cup E(\beta_{n(t)}).$$

Let $R_{n(t)}$ be the first return map on $E(n(t))$ of R and let $\mu_{E(n(t))}$ be the Lebesgue equivalent $R_{n(t)}$-invariant ergodic probability measure. For simplicity of notation, put

$$\mathrm{ll}(t)=\log_2(\log_2 t^*).$$

First we consider the case $d_U=d_V=2$. We remark that

$$F(\tau_t,x)=\frac{H_p(t,x)+H_c(t,x)}{H_d(t,x)+H_q(t,x)}.$$

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

Roughly speaking, $\eta(t,x)$ converges to ρ as $t \to \infty$ (Lemma 6.4), and hence it is sufficient to show that $H_c(t,x)$ and $H_d(t,x)$ are much greater than $H_p(t,x)$ and $H_q(t,x)$ for sufficiently large t. Put

$$C_t := \left\{ x \in \Omega; \sum_{l=1}^{t^*} 1_{E(n(t))}(R^l(x)) > (\mathrm{ll}(t))^2 \right\},$$

$$D_{n(t)} := \{ x \in E_{n(t)}; R^l_{n(t)}(x) \in (E_{(n(t)+M(t))})^C \text{ for all } 1 \le l \le (\mathrm{ll}(t))^2 \},$$

where $M_t = [(t^*(\log n(t))^{1/2})/(\mathrm{ll}(t))^2]$,

$$E^1_{n(t)} := E_{n(t)}, \quad E^h_{n(t)} := R^{-h+1}E_{n(t)} \setminus \left(\bigcup_{k=1}^{h-1} E^k_{n(t)} \right) \quad \text{for } h \ge 2,$$

$$D^1_{n(t)} := D_{n(t)}, \quad D^h_{n(t)} := R^{-h+1}D_{n(t)} \setminus \left(\bigcup_{k=1}^{h-1} E^k_{n(t)} \right) \quad \text{for } h \ge 2.$$

Then, since

$$\{ x \in E^h_{n(t)}; H_p(t^*,x) + H_q(t^*,x) \ge t^*(\log n(t))^{1/2} \} \cap C_t^C \subset E^h_{n(t)} \setminus D^h_{n(t)},$$

we have

$$\begin{aligned}
&\mu|_\Omega\left(\bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} \{ x \in \Omega; H_p(t^*,x) + H_q(t^*,x) \ge t^*(\log n(t))^{1/2} \} \right) \\
&= \mu|_\Omega\left(\bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} \bigcup_{h=1}^{\infty} \{ x \in E^h_{n(t)}; H_p(t^*,x) + H_q(t^*,x) \ge t^*(\log n(t))^{1/2} \} \right) \\
&\le \mu|_\Omega\left(\bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} \bigcup_{h=1}^{\infty} (E^h_{n(t)} \setminus D^h_{n(t)}) \right) + \mu|_\Omega\left(\bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} C_t \right). \qquad (6.4)
\end{aligned}$$

Applying Lemma 5.3 we have

$$\begin{aligned}
&\mu|_\Omega\left(\bigcup_{h=1}^{\infty} (E^h_{n(t)} \setminus D^h_{n(t)}) \right) \\
&= \sum_{h=1}^{\infty} \mu|_\Omega(E^h_{n(t)} \setminus D^h_{n(t)}) \\
&= \sum_{h=1}^{\infty} \mu|_\Omega(E^h_{n(t)}) \cdot (\mu|_\Omega(E^h_{n(t)} \setminus D^h_{n(t)})/\mu|_\Omega(E^h_{n(t)})) \\
&\le \text{constant} \cdot \sum_{h=1}^{\infty} \mu|_\Omega(E^h_{n(t)}) \cdot (\mu|_\Omega(E_{n(t)} \setminus D_{n(t)})/\mu|_\Omega(E_{n(t)})) \\
&= \text{constant} \cdot \mu|_\Omega(E_{n(t)} \setminus D_{n(t)})/\mu|_\Omega(E_{n(t)}) = \text{constant} \cdot \mu_{E(n(t))}({D_{n(t)}}^C).
\end{aligned}$$

Hence, since $n(t)$ is a constant for $2^{2^n} < t \le 2^{2^{n+1}}$, we have

$$\mu|_\Omega\left(\bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} \bigcup_{h=1}^{\infty} (E^h_{n(t)} \setminus D^h_{n(t)}) \right) \le \text{constant} \cdot \lim_{k\to\infty} \sum_{t \in L_k} \mu_{E(n(t))}({D_{n(t)}}^C), \qquad (6.5)$$

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

where $L_k = \{t \geq k; \log_2(\log_2 t) \text{ is an integer}\}$. Since Lemma 5.4 implies

$$\mu|_\Omega\left(\bigcap_{k=0}^{\infty}\bigcup_{t=k}^{\infty} C_t\right) = 0,$$

it follows from (6.4) and (6.5) that

$$\begin{aligned}\mu|_\Omega&\left(\bigcap_{k=0}^{\infty}\bigcup_{t=k}^{\infty}\{x \in \Omega;\, H_p(t^*, x) + H_q(t^*, x) \geq t^*(\log n(t))^{1/2}\}\right)\\ &\leq \text{constant} \cdot \lim_{k\to\infty}\sum_{t\in L_k} \mu_{E(n(t))}(D_{n(t)}{}^C).\end{aligned}$$

Since $t \leq t^*$, the above inequality implies

$$\begin{aligned}\mu|_\Omega&\left(\bigcap_{k=0}^{\infty}\bigcup_{t=k}^{\infty}\{x \in \Omega;\, H_p(t, x) + H_q(t, x) \geq t^*(\log n(t))^{1/2}\}\right)\\ &\leq \text{constant} \cdot \lim_{k\to\infty}\sum_{t\in L_k} \mu_{E(n(t))}(D_{n(t)}{}^C).\end{aligned}$$

To estimate

$$\mu_{E(n(t))}(D_{n(t)}{}^C)$$

we first estimate

$$\mu_{E(n(t))}(E_{(n(t)+M(t))}).$$

Put $\nu = \alpha' + \beta'$. Since $n(t) \sim \nu t^*/\mathrm{ll}(t)$ holds by Lemma 6.2 if $d_U = d_V = 2$ and since

$$n(t) + M(t) \sim \frac{\nu t^* + (\mathrm{ll}(t))^{-1} t^* (\log(\nu t^*/\mathrm{ll}(t)))^{1/2}}{\mathrm{ll}(t)}$$

holds, we have

$$\mu_{E(n(t))}(E_{(n(t)+M(t))}) = \frac{\mu(E_{(n(t)+M(t))})}{\mu(E_{n(t)})} \sim \frac{\mathrm{ll}(t)}{\nu}\left(\log\left(\frac{\nu t^*}{\mathrm{ll}(t)}\right)\right)^{-1/2}.$$

Thus, applying Lemma 5.3 we have

$$\begin{aligned}\mu_{E(n(t))}(D_{n(t)}{}^C) &= 1 - \mu_{E(n(t))}(D_{n(t)})\\ &\leq 1 - (1 - \text{constant} \cdot \mu_{E(n(t))}(E_{(n(t)+M(t))}))^{(\mathrm{ll}(t))^2}\\ &\sim \frac{\text{constant} \cdot (\mathrm{ll}(t))^3}{\nu(\log(\nu t^*/\mathrm{ll}(t)))^{1/2}}.\end{aligned}$$

Therefore we obtain

$$\lim_{k\to\infty}\sum_{t\in L_k} \mu_{E(n(t))}(D_{n(t)}{}^C) = 0.$$

Hence

$$\lim_{t\to\infty}\frac{H_p(t, x) + H_q(t, x)}{t(\log n(t))} = 0 \quad \text{for } \mu|_\Omega\text{-a.e. } x,$$

which implies the lemma in this case.

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

Next we consider the case $2 \le d_U < d_V$. Similar to the first case, it is sufficient to prove that

$$H_p(t,x)/(H_p(t,x)+H_q(t,x)) \to 0 \quad (t \to \infty).$$

Put

$$G(\beta_{n(t)}) := \{x \in E_{n(t)};\ R^l_{n(t)}(x) \in E(\beta_{n(t)}) \text{ for some } 1 \le l \le (\mathrm{ll}(t))^2\}.$$

If we show

$$\lim_{k\to\infty} \sum_{t\in L_k} \mu_{E(n(t))}(G(\beta_{n(t)})) = 0, \tag{6.6}$$

we obtain the lemma in this case in a way similar to the first case. So, we are going to prove (6.6). By Lemma 6.2 and by the definition of $n(t)$ we have

$$\begin{aligned} \mu(E(\beta_{n(t)})) &\sim \beta' n(t)^{-1/(d_U-1)} \sim \beta'((\alpha')^{-1}\mu(E(\alpha_{n(t)})))^{(d_V-1)/(d_U-1)} \\ &\sim \text{constant} \cdot (\mathrm{ll}(t)/t^*)^{(d_V-1)/(d_U-1)}. \end{aligned}$$

Thus we have

$$\begin{aligned} \mu_{E(n(t))}(G(\beta_{n(t)})) &= \frac{\mu(G(\beta_{n(t)}))}{\mu(E_{n(t)})} \le \frac{(\mathrm{ll}(t))^2\mu(E(\beta_{n(t)}))}{\mu(E_{n(t)})} \\ &\sim \mathrm{ll}(t)\mu(E(\beta_{n(t)}))t^* \sim \text{constant} \cdot (\mathrm{ll}(t)/t^*)^{(d_V-1)/(d_U-1)}\mathrm{ll}(t)t^*. \end{aligned}$$

Since $2 \le d_U < d_V$, we obtain (6.6). Thus the proof of the lemma is complete. □

Proof of Proposition 6.5. Let Q be an arbitrary set with $\mu|_\Omega(Q) = 0$. Since $\mathrm{Leb}(\bigcup_{n=0}^\infty T^{-n}\Omega) = 1$ and $\mathrm{Leb}(\bigcup_{n=0}^\infty T^{-n}Q) = 0$, it is sufficient to show that

$$\mu|_\Omega\left(\bigcap_{k=0}^\infty \bigcup_{t=k}^\infty \{x;\ |F(t,x)-\rho| > \varepsilon\}\right) = 0 \quad \text{for any } \varepsilon > 0.$$

Obviously, this is equivalent to

$$\mu|_\Omega\left(\bigcap_{k=0}^\infty \bigcup_{t=k}^\infty \{x;\ |F(\tau_t,x)-\rho| > \varepsilon\}\right) = 0 \quad \text{for any } \varepsilon > 0.$$

Thus the proposition follows from Lemma 6.6. □

Proof of Theorems 1 and 2. In the case when U is a small right neighborhood of p and when V is a small left neighborhood of q, Theorems 1 and 2 follow from Lemmas 6.1 and 6.2 and Proposition 6.5. The other cases are similarly proved. □

7. *Corollaries*

In this section we give easy corollaries of our main theorems, which are analogues to the Birkhoff individual ergodic theorem and results concerning asymptotic measures.

Let $T : [0,1] \to [0,1]$ be a map satisfying (i)–(v) in §2.

COROLLARY OF THEOREM 1. *Let U be a one-sided small neighborhood of a fixed point p and assume that condition* (2.1) *with $d_U \ge 2$ is satisfied.*

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

(1) *If we have constants* $d_0 < d_U$ *and* $\theta_0 > 0$ *such that*

$$|T(x) - x| > \theta_0 |x - q|^{d_0} + o(|x - q|^{d_0}) \tag{7.1}$$

in a one-sided small neighborhood of $q = p$ *which is opposite to* U *and in a small neighborhood of each fixed point* $q \neq p$*, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_U(T^k(x)) = 1 \quad \textit{for Leb-a.e. } x.$$

(2) *If we have constants* $d_0 < d_U$ *and* $\theta_0 > 0$ *satisfying (7.1) in a small neighborhood of each fixed point* $q \neq p$*, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k(x)} = \delta_p \quad \textit{weakly for Leb-a.e. } x,$$

where δ_x *is the Dirac delta measure on* x.

COROLLARY OF THEOREM 2. *Let* $U_1, U_2, \ldots, U_m, \ldots, U_{2r-2}$ *be one-sided small neighborhoods of fixed points* $p_{i_1}, p_{i_2}, \ldots, p_{i_m}, \ldots, p_{i_{2r-2}}$ *respectively* $(\mathrm{Leb}(U_j \cap U_k) = 0$ *if* $j \neq k)$*. Assume that* (2.1) *with* $d_U = 2$ *is satisfied in* $U = U_i$ *for each* $i \leq m$*, and assume that* (7.1) *with* $d_0 < 2$ *and* $\theta_0 > 0$ *is satisfied in* U_i *for each* $i > m$*. Then:*
(1)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{U_j}(T^k(x)) = s_j \quad \textit{for Leb-a.e. } x,$$

where s_j *is a positive constant for each* $j \leq m$ *and* $s_j = 0$ *for each* $j > m$.
(2)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k(x)} = \sum_{j=1}^{m} s_j \delta_{p(i_j)} \quad \textit{weakly for Leb-a.e. } x,$$

where $\sum_{j=1}^{m} s_j = 1$ *and* $p(i_j) = p_{i_j}$.

8. *Remark*

Until now we have assumed that T is a piecewise increasing function. In this section we consider the map T which is not piecewise increasing.

Assume that conditions (i) and (v) in §2 are replaced by conditions (i)* and (v)* respectively.

(i)* There exists a partition $0 = c_0 \leq p_1 < c_1 < p_2 < c_2 < \cdots < p_{r-1} < c_{r-1} < p_r \leq c_r = 1$ of $[0, 1]$ such that the restrictions of T to the interval (c_{i-1}, p_i) and (p_i, c_i) are C^2 functions for each i.

(v)* $|T'(x)| > 1$ for $x \notin \{p_1, \ldots, p_r\}$.

In (i)* we allow $0 < p_1$ and $p_r < 1$. Let U, V, d_U and d_V be as in §2. Let $\hat{U}$ and $\hat{V}$ be small one-sided neighborhoods of p and q which are opposite to U and V respectively. (If U is a right neighborhood of 0 or a left neighborhood of 1, we do not consider $\hat{U}$.) Define

$$\bar{d}_U = \begin{cases} \min(d_U, d_{\hat{U}}), & \text{if } T'(x) < 0 \text{ in } U \\ d_U, & \text{if } T'(x) > 0 \text{ in } U. \end{cases}$$

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press

In the same way we define $\bar{d}_V$. Then:

- if $2 \leq \bar{d}_U < \bar{d}_V$, the conclusion of Theorem 1 holds;
- if $2 = \bar{d}_U = \bar{d}_V$, the conclusion of Theorem 2 holds without the 'in particular' part.

These kind of facts can be proved in essentially the same way as Theorems 1 and 2.

Acknowledgement. The author thanks Professor Izumi Kubo for his kind remarks to the preliminary version of the manuscript.

REFERENCES

[H] E. Hopf. *Ergodentheorie*. Chelsea, New York, 1948.

[I1] T. Inoue. Weakly attracting repellors for piecewise convex maps. *Japan J. Indust. Appl. Math.* **9** (1992), 413–430.

[I2] T. Inoue. Ergodic theorems for piecewise affine Markov maps with indifferent fixed points. *Hiroshima Math. J.* **24** (1994), 447–471.

[IY] Sh. Ito and S.Yasutomi. On continued fractions, substitutions and characteristic sequences $[nx + y] - [(n-1)x + y]$. *Japanese J. Math.* **16** (1990), 287–306.

[K] U. Krengel. *Ergodic Theorems (de Gruyter Studies in Math. 6)*. de Gruyter, Berlin, New York, 1985.

[LM] A. Lasota and M. C. Mackey. Chaos, fractals, and noise. *Stochastic Aspects of Dynamics (Appl. Math. Sci. 97)*. Springer, New York, Berlin, Heidelberg, 1994.

[P] G. Pianigiani. First return map and invariant measures. *Israel J. Math.* **35** (1980), 32–48.

[R] M. Rychlik. Bounded variation and invariant measures. *Studia Math.* **LXXVI** (1983), 69–80.

[T1] M. Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.* **37** (1980), 304–314.

[T2] M. Thaler. Transformations on [0, 1] with infinite invariant measures. *Israel J. Math.* **46** (1983), 67–96.

https://doi.org/10.1017/S0143385797084952 Published online by Cambridge University Press