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On Dimensions of a Random Solid Diagram

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For Béla Bollobás on his 60th birthday

A solid diagram of volume n is a packing of n unit cubes into a corner so that the heights of vertical stacks of cubes do not increase in either of two horizontal directions away from the corner. An asymptotic distribution of the dimensions – heights, depths, and widths – of the diagram chosen uniformly at random among all such diagrams is studied. For each k, the planar base of k tallest stacks is shown to be Plancherel distributed in the limit $n \to \infty$.

1. Introduction

A solid diagram of volume n is defined as a set $\mathscr{P}$ of n positive integer lattice points $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{N}^3$, such that if $\mathbf{x} \in \mathscr{P}$, $\mathbf{x}' \leqslant \mathbf{x}$ then $\mathbf{x}' \in \mathscr{P}$ too. Such a diagram can also be interpreted as a descending plane partition of n, *i.e.*, a set of nonnegative integers p_{ij}, $(i, j \geqslant 1)$, such that $\sum_{i,j} p_{ij} = n$, and p_{ij} decreases with i (j resp.) for each fixed j (i resp.) Indeed, p_{ij} is the height of the column of unit cubes stacked along the vertical line $x_1 = i, x_2 = j$, and the solid diagram is the union of all such columns. Given positive integers r, s, t, let $B(r, s, t)$ stand for the parallelepiped $\{\mathbf{x} \in \mathbb{N}^3 : x_1 \leqslant r, x_2 \leqslant s, x_3 \leqslant t\}$. Let $|\mathscr{P}|$ denote the volume of a diagram $\mathscr{P}$. According to MacMahon [19], for $|q| < 1$,

$$p_{r,s,t}(q) \stackrel{\text{def}}{=} \sum_{\mathscr{P} \subseteq B(r,s,t)} q^{|\mathscr{P}|} = \prod_{\mathbf{x} \in B(r,s,t)} \frac{1 - q^{|\mathbf{x}|-1}}{1 - q^{|\mathbf{x}|-2}}, \qquad |\mathbf{x}| = x_1 + x_2 + x_3. \tag{1.1}$$

(We refer the reader to Andrews [1], Gessel and Vienott [11], Stanley [28] and Bressoud [3] for numerous references and eminently readable accounts of subsequent research on

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plane partitions spawned by MacMahon's studies.) Here is a more explicit version of (1.1):

$$\begin{aligned} p_{r,s,t}(q) = \prod_{\ell=1}^{\infty}(1-q^\ell)^{-\ell} \\ \times \prod_{\ell>r}(1-q^\ell)^{\ell-r}\cdot\prod_{\ell>s}(1-q^\ell)^{\ell-s}\cdot\prod_{\ell>t}(1-q^\ell)^{\ell-t} \\ \times \prod_{\ell>r+s}(1-q^\ell)^{r+s-\ell}\cdot\prod_{\ell>r+t}(1-q^\ell)^{r+t-\ell}\prod_{\ell>s+t}(1-q^\ell)^{s+t-\ell} \\ \times \prod_{\ell>r+s+t}(1-q^\ell)^{\ell-r-s-t}. \end{aligned} \tag{1.2}$$

Sending r, s, t to infinity, we recover MacMahon's

$$p(q) = \prod_{\ell\geqslant 1}(1-q^\ell)^{-\ell} \tag{1.3}$$

as the generating function of unrestricted diagrams. Likewise, fixing r and letting $s, t \to \infty$, we obtain that

$$p_r(q) = \prod_{\ell\geqslant 1}(1-q^\ell)^{-\min(\ell,r)} \tag{1.4}$$

is the generating function of diagrams with the x_1-axis dimension at most r (see [19]). The formula (1.2) is ideally suited to our purposes. To prove it, we notice that, introducing

$$C_\ell = C_\ell(r,s,t) = |\{\mathbf{x}\in B(r,s,t) : |\mathbf{x}| = \ell\}|,$$

we transform (1.1) into

$$\sum_{\mathscr{P}\subseteq B(r,s,t)} q^{|\mathscr{P}|} = \prod_{\ell\geqslant 1}(1-q^\ell)^{C_{\ell+1}-C_{\ell+2}}. \tag{1.5}$$

Further, as

$$|\{\mathbf{y}\in\mathbb{N}^3 : |\mathbf{y}| = \mu\}| = \binom{\mu-1}{2},$$

by the inclusion–exclusion principle we have

$$\begin{aligned} C_v = \binom{v-1}{2} - \binom{v-r-1}{2} - \binom{v-s-1}{2} - \binom{v-t-1}{2} \\ + \binom{v-r-s-1}{2} + \binom{v-r-t-1}{2} + \binom{v-s-t-1}{2} - \binom{v-r-s-t}{2}. \end{aligned} \tag{1.6}$$

Combining (1.5), (1.6) and

$$\binom{a}{2} - \binom{a-1}{2} = \binom{a-1}{1} = (a-1)^+,$$

we obtain (1.2).

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Wright [30] used (1.3) to derive an asymptotic formula for $p_n = [q^n]p(q)$, the total number of plane partitions of n or the total number of solid diagrams of volume n:

$$\begin{aligned} p_n &= an^{-25/36}e^{bn^{2/3}}\left(1+\sum_{h=1}^{r}\delta_h n^{-2h/3}+O(n^{-2(r+1)/3})\right);\\ a &= \frac{(2^{25}\zeta(3)^7)^{1/36}}{2\pi^{1/2}}e^c,\\ b &= 32^{-2/3}\zeta(3)^{1/3},\\ c &= 2\int_0^\infty \frac{y\log y}{e^{2\pi y}-1}\,dy. \end{aligned} \tag{1.7}$$

Here $\zeta(\cdot)$ is the Riemann ζ-function, and δ_h are certain constants. Wright's powerful argument is quite technical, and adapting it for the more general case of restricted diagrams may well be quite problematic. However, a cruder formula – with the last factor of the form $1+O(n^{-1/3})$ – is quite sufficient for our purposes. We will give a much simpler, and self-contained, proof of such a 'poor man's' asymptotic formula for both unrestricted and restricted diagrams. (We could also have used a rather general approach due to Meinardus – see [1] – but it is markedly more complex and yields a worse error bound.) In particular, we will determine an asymptotic formula for the total number of diagrams $\mathscr{P}$ that fit inside $B(r,s,t)$, with r,s,t all of order $n^{1/3}\log n$. Endowing the set of solid diagrams of volume n with uniform probability measure, we will then deduce the following probabilistic statement.

Theorem 1.1. *Let H_n, W_n, D_n denote the dimensions – height, width and depth – of the uniformly random diagram $\mathscr{P}$ of volume n. Define h_n, w_n, d_n by*

$$H_n = \frac{n^{1/3}}{\beta}\log\frac{\left(\frac{n^{1/3}}{\beta}\right)^2}{h_n}, \qquad W_n = \frac{n^{1/3}}{\beta}\log\frac{\left(\frac{n^{1/3}}{\beta}\right)^2}{w_n}, \qquad D_n = \frac{n^{1/3}}{\beta}\log\frac{\left(\frac{n^{1/3}}{\beta}\right)^2}{d_n}, \tag{1.8}$$

where $\beta = (2\zeta(3))^{1/3}$. In the limit $n\to\infty$, h_n, w_n, d_n are asymptotically independent, and exponentially distributed, with parameter 1 each.

Note. For the random planar diagram of area n (linear partition of n) the corresponding result for the width W_n and the depth D_n, *i.e.*, the lengths of the longest row and the longest column, was obtained by Erdős and Lehner [7], and Auluck, Chowla and Gupta [2]. It states that $\mathscr{W}_n, \mathscr{D}_n$ defined by

$$W_n = \frac{n^{1/2}}{\alpha}\left(\log\frac{n^{1/2}}{\alpha}+\mathscr{W}_n\right), \qquad D_n = \frac{n^{1/2}}{\alpha}\left(\log\frac{n^{1/2}}{\alpha}+\mathscr{D}_n\right) \tag{1.9}$$

($\alpha = \sqrt{\zeta(2)} = \frac{\pi}{6^{1/2}}$), are asymptotically independent, and *double*-exponentially distributed, *i.e.*,

$$\mathrm{P}(\mathscr{W}_n \leqslant x),\ \mathrm{P}(\mathscr{D}_n \leqslant x) \to e^{-e^{-x}}, \quad \forall\, x\in R.$$

If we define w_n, d_n by $\log 1/w_n = \mathscr{W}_n$, $\log 1/d_n = \mathscr{D}_n$, thus expressing W_n, D_n in the form of (1.8), then in the limit w_n, d_n are independent, exponentially distributed, with parameter 1.

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The analogy is undeniably striking, and one is tempted to conjecture similar results for higher-dimensional diagrams. Any attempts to prove such an extension appear to be doomed at present, as even the basic problem of enumerating diagrams for four or more dimensions is wide open.

Using a conditioning device, Fristedt [10] considerably extended the results in [2, 7] and determined the limiting joint distribution of the first $k = o(n^{1/4})$ largest widths $W_n^{(i)}$ (row lengths) of the random planar diagram. The sequence of random variables $\{\mathscr{W}_n^{(i)}\}_{i\leqslant k}$, that are to $W_n^{(i)}$ what $\mathscr{W}_n = W_n^{(1)}$ is to W_n, was shown to be asymptotic to the Markov chain $\{Y_i\}_{i\leqslant k}$; here the density of Y_i conditioned on $Y_{i-1} = x$ is $e^{-y-e^{-y}+e^{-x}}$, $y \leqslant x$, and the density of Y_1 is $e^{-x-e^{-x}}$. By duality, the sequence $\{D_n^{(i)}\}$ of largest depths (column lengths) and $\{W_n^{(i)}\}$ are equidistributed, so the corresponding $\{\mathscr{D}_n^{(i)}\}_{i\leqslant k}$ is asymptotic to a copy of the above Markov chain. For the case $k = O(1)$, this result *plus* mutual independence of the two Markov chains are implicit in Erdős and Richmond [8]. In [24] it was found that $\{Y_i\}$ coincides, in distribution, with $\{\log(\sum_{j=1}^{i} E_i)^{-1}\}$, where $E_1, E_2, \ldots$ are independent copies of the exponential E. This means that, for k fixed, say, $\{w_n^{(i)}\}_{i\leqslant k}$, $\{d_n^{(i)}\}_{i\leqslant k}$ defined by

$$W_n^{(i)} = \frac{n^{1/2}}{\alpha} \log \frac{\frac{n^{1/2}}{\alpha}}{w_n^{(i)}}, \qquad D_n^{(i)} = \frac{n^{1/2}}{\alpha} \log \frac{\frac{n^{1/2}}{\alpha}}{d_n^{(i)}}$$

are asymptotic to $\{\sum_{j=1}^{i} E_i\}_{i\leqslant k}$ and $\{\sum_{j=1}^{i} E_i'\}_{i\leqslant k}$, where $E_1, E_1', E_2, E_2', \ldots$ are independent copies of E.

Extending the proof of Theorem 1.1, we will obtain the following 3-dimensional counterpart of the cited results for the random planar diagram.

Theorem 1.2. *Let $H_n^{(i)}$ denote the length of the ith-tallest vertical 'stack' of unit cubes in the random diagram of volume n. Let $W_n^{(i)}$ ($D_n^{(i)}$ resp.) denote the length of the ith-longest horizontal 'beam' of unit cubes which is parallel to the x_2-axis (to the x_1-axis resp.), $i = 1, 2 \ldots$. Define $h_n^{(i)}, w_n^{(i)}, d_n^{(i)}$ by formulas analogous to* (1.8). *Then, for each k, the sequences $\{h_n^{(i)} : i \leqslant k\}$, $\{w_n^{(i)} : i \leqslant k\}$, $\{d_n^{(i)} : i \leqslant k\}$ are asymptotic, in distribution, to $\{\sum_{j=1}^{i} E_j : i \leqslant k\}$, $\{\sum_{j=1}^{i} E_j' : i \leqslant k\}$, $\{\sum_{j=1}^{i} E_j'' : i \leqslant k\}$ respectively, where $\{E_i : i \leqslant k\}$, $\{E_i' : i \leqslant k\}$, $\{E_i'' : i \leqslant k\}$ are independent sequences, each formed by the independent copies of the random variable E.*

We could have stated and proved a stronger version of this theorem, covering $k = o(n^{\sigma})$, for some $\sigma \in (0, 1/3)$, but decided to stick with k fixed to avoid clouding the basic ideas.

Our results imply that, with probability approaching 1, the k largest heights, widths, depths are all distinct. It's a dog-eat-dog world at each of the three 'extremities' of the diagram. Unlike linear partitions though, knowing the (limiting) distribution of, say, k tallest stacks of unit cubes gives us no information regarding the shape of the summit. The next theorem fills the gap.

First some definitions and facts. Given a planar (Young–Ferrers) diagram λ of area k, let $d(\lambda)$ denote the total number of ways to fill its k squares by numbers $1, \ldots, k$ so that the entries increase in every row and column in the direction away from the diagram's corner. The value of $d(\lambda)$ is given by a celebrated hook formula, due to Frame, Robinson

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and Thrall [9], namely

$$d(\lambda) = \frac{k!}{\prod_{\square \in \lambda} h(\square)}, \tag{1.10}$$

where $h(\square)$ is the area of the hook that consists of the $\square$ itself, and of the squares right of and down from the $\square$. Remarkably, given k, $\{d^2(\lambda)/k!\}$ is a probability (Plancherel) distribution on the set Λ_k of all diagrams of area k, that is,

$$\sum_{\lambda \in \Lambda_k} \frac{d^2(\lambda)}{k!} = 1. \tag{1.11}$$

This is a special case of a general theorem due to Burnside: the sum of squared dimensions of all irreducible representations of a group equals the group's order. (There is a bijection between the set of all irreducible representations of the symmetric group S_k (of order $k!$) and the set Λ_k, such that the degree of a representation corresponding to $\lambda \in \Lambda_k$ is $d(\lambda)$, Ledermann [17], Diaconis [6], Sagan [26].) Combinatorially, (1.11) is a consequence of a bijection, via the Robinson–Schensted algorithm, between elements of S_k and the set of pairs of Young tableaux of the same shape λ, $\lambda \in \Lambda_k$ (Knuth [15, Section 5.1.4], [28]).

Theorem 1.3. *Let $\mu_n^{(k)}$ denote the 'footprint' of the k tallest stacks, i.e., the planar diagram formed by the stacks bases. Then $\mu_n^{(k)}$ is, in the limit, Plancherel distributed on Λ_k, i.e.,*

$$\lim_{n\to\infty} P(\mu_n^{(k)} = \lambda) = \frac{d^2(\lambda)}{k!}, \quad \lambda \in \Lambda_k. \tag{1.12}$$

Furthermore, conditioned on $\mu_n^{(k)} = \lambda$, in the limit $n \to \infty$ each of $d(\lambda)$ ways to allocate the bases of the k stacks among the k squares of λ is equally likely. Consequently, $\{\mu_n^{(i)}\}_{i \leqslant k}$ is asymptotic in distribution to the Markov chain $\{\mu^{(i)}\}_{i \leqslant k}$ such that, for every $\lambda \in \Lambda_i$ and $\lambda' \in \Lambda_{i+1}$ obtained from λ by adding an external square,

$$P(\mu^{(i+1)} = \lambda' \,|\, \mu^{(i)} = \lambda) = \frac{d(\lambda')}{(i+1)d(\lambda)}. \tag{1.13}$$

Needless to say, analogous results hold for two other k-size projections, on the planes $x_1 = 0$ and $x_2 = 0$. In the spirit of Theorem 1.2, we conjecture that all three k-size projections are asymptotically independent.

Notes. **(1)** A new key ingredient of the proof is a formula, established independently by Krattenhaler [16, Section 6, Theorem 6.8] and Kadell [14, Section 6, Theorem 8] for the generating function of skew planar partitions $\mathscr{P}$ of shape $[1, r] \times [1, s] \setminus \mu$ where μ is a given (finite) diagram, such that the parts of $\mathscr{P}$ in the jth column are $t + \mu'_j$ at most. Here μ'_j is the number of squares in the jth column of μ. (Krattenhaler's result is slightly more general.) In the limit $r = s = \infty$, the formula becomes

$$\sum_{\mathscr{P}} q^{|\mathscr{P}|} = p_t(q) \prod_{\square \in \mu} (1 - q^{h(\square)})^{-1} \tag{1.14}$$

(*cf.* [28, Chapter 7, Exercise 7.107(b)]); see (1.4) for $p_t(q)$. (Remarkably, the hook product factor in (1.14) had long been known as the generating function of reverse plane partitions

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of shape μ; see Stanley [27].) Ideally one would have preferred to have such a formula for the parts uniformly bounded by t, but the beautiful identity (1.14) will do the job perfectly anyway, as in our case $t \gg \max_j \mu'_j$.

(2) Greene, Nijenhuis and Wilf [12] discovered a random (hook) walk executed on a given (planar) diagram that provided a probabilistic proof for the hook formula. Subsequently [13] they showed how the hook walk can be used to generate the Markov chain $\{\mu^{(i)}\}$ that we have encountered in Theorem 1.3. Briefly, given the current diagram $\mu^{(i)}$, $\mu^{(i+1)}$ is obtained by executing the hook walk on the 'complementary' diagram, that starts at the corner, and adding the terminal square to $\mu^{(i)}$. Inspired by [12], and independently from [13], we found [22] a different hook-driven walk that builds $\mu^{(i+1)}$ from $\mu^{(i)}$. Unlike [13], this walk is executed on $\mu = \mu^{(i)}$ itself, starting with the corner square $(1,1)$. This is how it works. Call $\square \notin \mu$ 'external' if $\mu \cup \square$ is a diagram again, and $\square = (\alpha, \beta) \in \mu$ 'admissible' if both $(\alpha, \mu_\alpha + 1)$ and $(\mu'_\beta + 1, \beta)$ are external. The corner square is one of the admissible squares. Recursively, let an admissible (α, β) be the current state of the walk. Choose uniformly at random a square (γ, δ) belonging to the 'extended' hook, that is, the union of the hook of (α, β) in μ and two external squares, $(\alpha, \mu_\alpha + 1)$ and $(\mu'_\beta + 1, \beta)$. If (γ, δ) is external, then stop the walk and form a diagram $\mu \cup (\gamma, \delta)$. If (γ, δ) is admissible, then continue the walk from (γ, δ). Suppose (γ, δ) is not admissible, so that, say, $(\mu'_\delta + 1, \delta)$ is not external, in which case $\gamma = \alpha$. Then the next (admissible) state is set to be $(\mu'_\delta + 1, \beta)$, and the walk continues. Once an external $\square$ is hit, the walk stops, and we set $\mu^{(i+1)} = \mu \cup \square$. In light of Theorem 1.3, it is hard to resist interpreting the walk as a peculiar random descent from a mountain top.

(3) In a recent work McKay, Morse and Wilf [18] showed that the *uniformly* random Young tableau contains a subtableau of a shape λ with the limiting probability $d^2(\lambda)/|\lambda|!$. Then Stanley [29] derived an exact formula, which can be used, in principle, to obtain an asymptotic expansion of this probability.

We should mention that Cerf and Kenyon [4] were able to determine the limiting shape of the bulk of the random solid diagram (scaled by $n^{1/3}$), the deep analysis being based on connections with statistical physics, namely a 3-dimensional Ising model.

In [23] it was shown that the boundary of the random planar diagram undergoes, asymptotically, Gaussian fluctuations of order $n^{1/4}$ around its deterministic limiting shape. The corresponding problem for the solid diagram is open. It would seem plausible, by analogy with the planar partitions, that the fluctuations around the limiting shape should be of order $n^{1/6}$. However, Rick Kenyon (private communication) insisted that, on the basis of numerical experiments, the variance order is logarithmic, at most.

Cohn, Larsen and Propp [5] studied the limiting shape of a solid diagram chosen uniformly at random among all diagrams 'boxed' in $B(r,s,t)$, for large r,s,t, all of the same order of magnitude. At the core of their approach is the remarkable product formula for the number of Gelfand–Tsetlin patterns. Very recently Okounkov and Reshetikhin [21] studied the limiting shape and the asymptotic correlations in the bulk of the random solid diagram $\mathscr{P}$ with distribution proportional to $q^{|\mathscr{P}|}$, and $q \uparrow 1$. The analysis is based on a determinantal formula for the correlation functions of the Schur process; see Okounkov [20].

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2. Auxiliary estimates

We will use the generating functions $p(q)$, $p_r(q)$, $p_{r,s,t}(q)$ (see (1.1)–(1.4)) to estimate the total numbers of unrestricted and restricted diagrams via the Cauchy integral formula. To this end we will need a sharp asymptotic formula for $p(q)$ for a complex-valued q close to 1.

Lemma 2.1. *Let $q = e^{-u}$. Suppose $\operatorname{Re} u > 0$ and $|\operatorname{Im} u| \leqslant \sigma \operatorname{Re} u$, where $\sigma > 0$ is fixed. Then*

$$\ln p(q) = \frac{\zeta(3)}{u^2} - \frac{1}{12} \ln \frac{1}{u} + c + O(|u|), \quad u \to 0, \tag{2.1}$$

where ln *stands for the principal value of the logarithm, the implied constant in $O(|u|)$ depends on σ only, and*

$$c = \int_0^\infty \left[\frac{e^z}{z(e^z - 1)^2} - \frac{1}{z^3} + \frac{e^{-z}}{12z} \right] dz. \tag{2.2}$$

Note. The integral in (2.2) is finite as the integrand is $O(z^{-3})$ for $z \to \infty$, and $-1/12 + O(z)$ for $z \to 0$.

Proof of Lemma 2.1. The argument is a considerably shortened variation of the proof of an asymptotic formula for $\prod_{\ell \geqslant 1}(1 - q^\ell)^{-1}$, the generating function of the linear partitions, due to Freiman; see Postnikov [25, Section 2.7].

Using (1.3),

$$\begin{aligned} \ln p(e^{-u}) &= \sum_{m \geqslant 1} m \ln \frac{1}{1 - e^{-mu}} = \sum_{m \geqslant 1} m \sum_{j \geqslant 1} \frac{e^{-muj}}{j} \\ &= \sum_{j \geqslant 1} \frac{1}{j} \sum_{m \geqslant 1} m e^{-muj} = \sum_{j \geqslant 1} \frac{1}{j} \frac{e^{-uj}}{(1 - e^{-uj})^2} \\ &= u \sum_{j \geqslant 1} \frac{1}{uj} \frac{e^{uj}}{(e^{uj} - 1)^2}. \end{aligned}$$

Now, for $z \to 0$,

$$\frac{e^z}{z(e^z - 1)^2} = \frac{1}{z^3} - \frac{1}{12z} + O(|z|) = \frac{1}{z^3} - \frac{e^{-z}}{12z} + O(1).$$

So, introducing

$$\phi(z) = \frac{e^z}{z(e^z - 1)^2} - \frac{1}{z^3} + \frac{e^{-z}}{12z},$$

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we obtain

$$\ln p(e^{-u}) = u^{-2}\sum_{j\geqslant 1}\frac{1}{j^3} - \frac{1}{12}\sum_{j\geqslant 1}\frac{e^{-uj}}{j} + u\sum_{j\geqslant 1}\phi(uj)$$
$$= u^{-2}\zeta(3) - \frac{1}{12}\ln\frac{1}{1-e^{-u}} + u\sum_{j\geqslant 1}\phi(uj)$$
$$= u^{-2}\zeta(3) - \frac{1}{12}\ln\frac{1}{u} + u\sum_{j\geqslant 1}\phi(uj) + O(|u|). \tag{2.3}$$

The key point of this identity is that the first two terms sharply capture the singular behaviour of $\ln p(e^{-u})$ for $u \to 0$, and the series term will turn out to be bounded for u in the wedge $\{u : \operatorname{Re} u > 0, |\operatorname{Im} u| \leqslant \sigma \operatorname{Re} u\}$.

Introduce $L = \{z = tu : t \in (0,\infty)\}$, the ray starting at $z = 0$ and passing through the point $z_0 = u$. Let L_n denote the part of L for t from $(n-1)$ to n. A standard argument shows that

$$\left|\int_L \phi(z)\,dz - u\sum_{j=1}^{\infty}\phi(uj)\right| \leqslant |u|^2\sum_{j=1}^{\infty}\max_{z\in L_n}|\phi'(z)|. \tag{2.4}$$

Since $\phi(z)$ is regular for $|z| < 2\pi$, $|\phi'(z)| \leqslant K$ for some K and all z with $|z| \leqslant 1$. Let $z = \xi + i\eta$ be in the wedge; then $\xi \geqslant |z|(1+\sigma^2)^{-1/2}$. Using

$$|e^z| = e^{\xi}, \quad |e^z - 1| \geqslant ||e^z| - 1| = e^{\xi} - 1,$$

we then have: for $|z| \geqslant 1$,

$$|\phi'(z)| \leqslant 2\frac{e^{\xi}}{(e^{\xi}-1)^2} + 2\frac{e^{2\xi}}{(e^{\xi}-1)^3} + \frac{3}{|z|^4} + |e^{-\xi}| = O(|z|^{-4}).$$

Thus, for all z in the wedge,

$$|\phi'(z)| \leqslant \frac{K_1}{1+|z|^4},$$

for some absolute constant $K_1 = K_1(\sigma)$. Hence the right-hand expression in (2.4) is of order

$$|u|^2\sum_{n=0}^{\infty}\frac{1}{1+n^4|u|^4} \leqslant |u|^2\left(1 + \int_0^{\infty}\frac{dx}{1+x^4|u|^4}\right)$$
$$= |u|^2 + |u|\int_0^{\infty}\frac{dy}{1+y^4}$$
$$= O(|u|), \quad u \to 0.$$

In addition,

$$\lim_{R\to\infty}\int_{\substack{z=Re^{i\theta}\\ \theta\in[0,\operatorname{Arg} u]}}\phi(z)\,dz = 0,$$

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as $|\phi(z)| \leqslant K_2 R^{-3}$ on the integration arc, $K_2 = K_2(\sigma)$. Therefore

$$\int_L \phi(z)\,dz = \int_0^\infty \phi(z)\,dz.$$

This completes the proof. □

Besides (2.1), we will need a bound for $|p(q)|$ when q is not necessarily close to 1.

Lemma 2.2. *Let $q = re^{i\theta}$, $0 < r < 1$, $\theta \in (-\pi, \pi]$. Then*

$$|p(q)| \leqslant p(r) \exp\left(-\frac{\alpha r \theta^2}{(1-r^2)^2\left(\alpha r\theta^2 + (1-r)^2\right)}\right), \quad \alpha = 4/\pi^2. \tag{2.5}$$

Note. This bound indicates that, on the circle of radius r, $|p(q)|$ attains its only maximum at $q = r$, and the maximum is sharply pronounced if r is close to 1.

Proof of Lemma 2.2. Using an inequality

$$\left|\frac{1}{1-z}\right| \leqslant \frac{1}{1-|z|} \exp(\operatorname{Re} z - |z|), \quad (|z| < 1),$$

(see [23]), we obtain

$$|p(re^{i\theta})| \leqslant p(r) \exp\left(\sum_{\ell \geqslant 1} \ell r^\ell (\cos(\theta\ell) - 1)\right).$$

Here

$$\begin{aligned}\sum_{\ell \geqslant 1} \ell r^\ell (\cos(\theta\ell) - 1) &= -(1-r)^{-2} + \operatorname{Re}(1 - re^{i\theta})^{-2} \\ &\leqslant -(1-r)^{-2} + (1 - 2r\cos\theta + r^2)^{-1} \\ &\leqslant -\frac{2r(1-\cos\theta)}{(1-r^2)^2\left(2r(1-\cos\theta) + (1-r)^2\right)},\end{aligned}$$

and it remains to notice that $1 - \cos\theta \geqslant 2\theta^2/\pi^2$. □

3. Proof of Theorem 1.1

We begin with an asymptotic estimate of p_n, the total number of unrestricted diagrams of volume n.

Lemma 3.1. *Let a, b be defined by the first two relations in* (1.7)*, with c defined in* (2.2)*. Then*

$$p_n = a n^{-25/36} e^{bn^{2/3}} (1 + O(n^{-1/3})). \tag{3.1}$$

Note. Comparing two formulas for c, (1.7) and (2.2), we thus obtain an identity

$$2\int_0^\infty \frac{y \ln y}{e^{2\pi y} - 1}\,dy = \int_0^\infty \left[\frac{e^z}{z(e^z-1)^2} - \frac{1}{z^3} + \frac{e^{-z}}{12z}\right] dz.$$

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We have not tried to prove this identity directly. According to Maple, the approximate values of the integrals are −0.1654211437 and −0.1654211434, respectively.

Proof of Lemma 3.1. Given $0 < \rho < 1$, by Cauchy's integral formula,

$$p_n = (2\pi i)^{-1} \oint\limits_{\substack{z=\rho e^{i\theta} \\ \theta\in(-\pi,\pi]}} z^{-(n+1)} p(z)\, dz. \tag{3.2}$$

Predictably, we want to choose ρ close to the root of $(\rho^{-n}p(\rho))' = 0$, or setting $\rho = e^{-\xi}$,

$$\sum_{\ell\geqslant 1} \frac{\ell^2}{e^{\ell\xi}-1} = n \longrightarrow \xi^{-3}\left[\int_0^\infty \frac{y^2\, dy}{e^y-1} + O(\xi)\right] = n.$$

Since the integral equals $2\zeta(3)$, we select $\xi = (2\zeta(3)/n)^{1/3}$. Break the interval $(-\pi,\pi]$ in two parts, $[-n^{-\delta}, n^{-\delta}]$ and $[-n^{-\delta}, n^{-\delta}]^c$, where $\delta \in (5/9, 2/3)$. By (2.1), and Lemma 2.2,

$$\begin{aligned}\left|\int\limits_{|\theta|\geqslant n^{-\delta}} z^{-(n+1)} p(z)\, dz\right| &\leqslant \rho^{-n}p(\rho) \int\limits_{|\theta|\geqslant n^{-\delta}} e^{-\alpha_1 n^{4/3-2\delta}}\, d\theta \quad (\alpha_1 > 0) \\ &= \exp(bn^{2/3} - \alpha_2 n^{4/3-2\delta}) \quad (\alpha_2 > 0). \end{aligned} \tag{3.3}$$

Consider $|\theta| \leqslant n^{-\delta}$. Since $|\theta| = o(\xi)$, we apply (2.1) for $u = \xi - i\theta$ and after simple algebra obtain

$$\begin{aligned}\ln \frac{p(\rho e^{i\theta})}{(\rho e^{i\theta})^n} &= bn^{2/3} - \frac{1}{36}\ln\frac{n}{2\zeta(3)} + c - \frac{1}{12\xi} i\theta - \theta^2 n^{4/3}\gamma_n \\ &\quad + i\theta^3 n^{5/3}\gamma_n' + O(\theta^4 n^2) + O(n^{-1/3}), \end{aligned} \tag{3.4}$$

where

$$\gamma_n = \zeta(3)^{-1/3} 2^{-4/3} n^{4/3}(1 + O(n^{-2(\delta-1/3)})),$$

$\gamma_n' = O(1)$, and the last remainder term comes from $O(|u|)$ in (2.1). We notice that, by the definition of δ, $2(\delta - 1/3) > 1/3$ and $|\theta|^3 n^{5/3} = o(1)$, $\theta^4 n^2 = o(1)$, uniformly for θs in question. Using (3.4) and

$$\exp\left(-\frac{1}{12\xi} i\theta + i\theta^3 n^{5/3}\gamma_n' + O(\theta^4 n^2)\right) = 1 - \frac{i\theta}{12\xi} + i\theta^3 n^{5/3}\gamma_n' + O(\theta^4 n^2 + n^{-1/3}),$$

we obtain

$$\begin{aligned}&\oint\limits_{|\theta|\leqslant n^{-\delta}} z^{-(n+1)} p(z)\, dz \\ &\quad = \frac{e^{bn^{2/3}+c}}{2\pi\left(\frac{n}{2\zeta(3)}\right)^{1/36}} \times \int\limits_{|\theta|\leqslant n^{-\delta}} e^{-\gamma_n\theta^2}\left(1 + i\theta^3 n^{5/3}\gamma_n' + O(\theta^4 n^2 + n^{-1/3})\right) d\theta. \end{aligned} \tag{3.5}$$

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Here

$$\int_{|\theta|\leqslant n^{-\delta}} e^{-\gamma_n\theta^2}\,d\theta = (\pi/\gamma_n)^{1/2}(1+o(n^{-2/3})),$$

$$\int_{|\theta|\leqslant n^{-\delta}} e^{-\gamma_n\theta^2}\left(-\frac{i\theta}{12\xi}+i\theta^3 n^{5/3}\gamma_n'\right)d\theta = 0,$$

$$\int_{|\theta|\leqslant n^{-\delta}} e^{-\gamma_n\theta^2}(\theta^4 n^2 + n^{-1/3})\,d\theta = O(\gamma_n^{-1/2}n^{-1/3}).$$

Hence the integral in (3.5) equals, within a factor $1+O(n^{-1/3})$,

$$\frac{e^{bn^{2/3}+c}}{2\pi\left(\frac{n}{2\zeta(3)}\right)^{1/36}}\cdot\frac{\pi^{1/2}}{\sqrt{\zeta(3)^{-1/3}2^{-4/3}n^{4/3}}} = an^{-25/36}e^{bn^{2/3}}.$$

The proof of Lemma 3.1 is complete. □

Turn now to $p_n(r,s,t)$, the total number of diagrams of volume n that fit inside the parallelepiped $B_{r,s,t}$.

Lemma 3.2. *Given $h,w,d>0$, let r,s,t be the integer parts of*

$$\frac{n^{1/3}}{\beta}\ln\left[\frac{\left(\frac{n^{1/3}}{\beta}\right)^2}{h}\right],\qquad \frac{n^{1/3}}{\beta}\ln\left[\frac{\left(\frac{n^{1/3}}{\beta}\right)^2}{w}\right],\qquad \textit{and}\qquad \frac{n^{1/3}}{\beta}\ln\left[\frac{\left(\frac{n^{1/3}}{\beta}\right)^2}{d}\right],$$

respectively. Then

$$p_n(r,s,t) = an^{-25/36}e^{bn^{2/3}}e^{-(h+w+d)}(1+O(n^{-1/3})). \tag{3.6}$$

Using Lemmas 3.1 and 3.2 we obtain immediately that the probability that the (uniformly) random diagram fits into the parallelepiped $B(r,s,t)$, with r,s,t defined in Lemma 3.2, is asymptotic to $e^{-h}e^{-w}e^{-d}$. This proves Theorem 1.1.

Proof of Lemma 3.2. The number $p_n(r,s,t)$ is given by the Cauchy integral formula, like (3.2), with $p_{r,s,t}(q)$ instead of $p(q)$. We choose again the circle of radius $\rho = e^{-\xi}$. On this circle, the total of the last four products in the formula (2.2) for $p_{r,s,t}(q), (q=\rho e^{i\theta})$, is (uniformly) asymptotic to

$$\exp\left(1+O\left(\frac{\rho^{r+s}+\rho^{r+t}+\rho^{s+t}}{(1-\rho)^2}\right)\right)\cdot\exp\left(1+O\left(\frac{\rho^{r+s+t}}{(1-\rho)^2}\right)\right) = 1+O(n^{-2/3}). \tag{3.7}$$

Furthermore, the total of the three products in the second line of (1.2) is asymptotic to

$$\exp\left(-\frac{q^r+q^s+q^t}{(1-q)^2}+O\left(\frac{\rho^{2r}+\rho^{2s}+\rho^{2t}}{(1-\rho^2)^2}\right)\right) = \exp\left(-\frac{q^r+q^s+q^t}{(1-q)^2}+O(n^{-2/3})\right)$$
$$= \exp(O(1)). \tag{3.8}$$

Using (3.7), (3.8), we see that – analogously to (3.3) – the contribution of the set $\{\theta : |\theta| \geqslant n^{-\delta}\}$ to the value of the contour integral is of order $\exp(bn^{2/3}-\alpha_3 n^{4/3-2\delta})$, at

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most. Now, for $|\theta| \leqslant n^{-\delta}$,

$$\frac{q^r+q^s+q^t}{(1-q)^2} = \frac{e^{-\xi r}+e^{-\xi s}+e^{-\xi t}}{(1-e^{-\xi})^2} + i\theta(r+s+t+6/\xi) + O(n^{-1/3})$$
$$= (h+w+d) + i\theta(r+s+t+6/\xi) + O(n^{-1/3}). \tag{3.9}$$

From (3.7)–(3.9), it follows that the contribution of the interval $[-n^{-\delta}, n^{-\delta}]$ to the contour integral is obtained via inserting an extra factor $e^{-(h+w+d)}(1+i\theta(r+s+t+6/\xi)+O(n^{-1/3}))$ into the integrand on the right-hand side of (3.5). So, arguing analogously, we obtain that

$$p_{r,s,t}(n) = an^{-25/36}e^{bn^{2/3}}e^{-(h+w+d)}(1+O(n^{-1/3})).$$

Lemma 3.2 is proved. □

4. Proof of Theorem 1.2

Consider first the joint distribution of the largest heights $H_n^{(i)}$ alone. (By symmetry, the distributions of the largest widths $W_n^{(i)}$ and the largest depths $D_n^{(i)}$ are exactly the same.)

Given $k \geqslant 1$, $\mathbf{d}^{(k)} = (d_1, \dots, d_k)$ where $d_1 > \cdots > d_k \geqslant 1$, let $f(q; \mathbf{d}^{(k)})$ denote the generating function of the diagrams with the k largest heights equal to $d_1, \dots, d_k$. For $k \geqslant 2$, introduce also $F(q; \mathbf{d}^{(k)})$, the generating function of the diagrams with the $(k-1)$ largest heights equal $d_1, \dots, d_{k-1}$, and all other heights being d_k, at most. Define $F(q; d) = p_d(q)$, so $F(q; d)$ is the generating function of diagrams with all heights being d, at most. Clearly

$$f(q; \mathbf{d}^{(k)}) = F(q; \mathbf{d}^{(k)}) - F(q; \tilde{\mathbf{d}}^{(k)}), \qquad \tilde{\mathbf{d}}^{(k)} = (d_1, \dots, d_{k-1}, d_k - 1). \tag{4.1}$$

Furthermore, given $d_1 > d_2 > \cdots > d_k > d_{k+1}$, the ratio of $f(q; d_1, \dots, d_{k-1}, d_{k+1})$ to $q^{d_{k+1}}$ is the generating function of partitions with the k largest heights equal to $d_1, \dots, d_{k-1}, d_{k+1}$, enumerated according to the sum of all heights, excluding the height d_{k+1}. (In all such partitions, the $(k+1)$th-largest height is d_{k+1} at most.) Multiplying this fraction by q^{d_k}, we get the generating function of all partitions with the k largest heights equal to $d_1, \dots, d_{k-1}, d_k$, and all other heights being d_{k+1}, at most, which is $F(q; \mathbf{d}^{(k+1)})$. Thus

$$F(q; \mathbf{d}^{(k+1)}) = q^{d_k - d_{k+1}} f(q; \hat{\mathbf{d}}^{(k)}), \qquad \hat{\mathbf{d}}^{(k)} = (d_1, \dots, d_{k-1}, d_{k+1}). \tag{4.2}$$

The equations (4.1), (4.2) allow us to express $f(q; d^{(k)})$ through $p_d(q)$ directly.

Lemma 4.1. *Let $d_1 > \cdots > d_k \geqslant k$. Then*

$$f(q; d^{(k)}) = q^{\sum_{t=1}^{k-1}(d_t - d_k)} \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q p_{d_k - j}(q), \tag{4.3}$$

where $\begin{bmatrix} k \\ j \end{bmatrix}_q$ is the q-binomial coefficient,

$$\begin{bmatrix} k \\ j \end{bmatrix}_q = \frac{[k!]_q}{[j!]_q [(k-j)!]_q}, \qquad [m!]_q \stackrel{\text{def}}{=} (1-q)(1-q^2)\cdots(1-q^m).$$

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Notes. **(1)** The coefficients of p_{d_k-j} in the (4.3) sum are exactly those in the q-binomial theorem, [1], namely

$$\prod_{j=0}^{k-1}(1-xq^j)=\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_q x^j, \tag{4.4}$$

a fact of critical importance to us later.

(2) The derivation of (4.3) works without any changes for the ordinary, linear, partitions, in which case $p_d(q)$ is the generating function of partitions with all parts d, at most. Here

$$p_d(q)=\prod_{j=1}^{d}(1-q^j)^{-1},$$

and, without using (4.3),

$$f(q;d^{(k)})=q^{d_1+\cdots+d_k}p_{d_k}(q)=q^{d_1+\cdots+d_k}\prod_{j=1}^{d_k}(1-q^j)^{-1}.$$

Plugging both relations into (4.3), after cancellations and setting $d_k=d$, we obtain an identity

$$\frac{q^{kd}}{[d!]_q}=\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_q\frac{1}{[(d-j)!]_q},\quad d\geqslant k.$$

Has the reader seen this identity anywhere? (Christian Krattenhaler has informed me that this identity is a special case of the q-Chu–Vandermonde summation formula.)

Proof of Lemma 4.1. Since $\begin{bmatrix}1\\ 0\end{bmatrix}_q=\begin{bmatrix}1\\ 1\end{bmatrix}_q=1$, (4.3) and (4.2) coincide for $k=1$. Suppose (4.3) holds for some $k\geqslant 1$. Using (4.2) and (4.3), we then obtain

$$F(q;\mathbf{d}^{(k+1)})=q^{\sum_{t=1}^{k}(d_t-d_{k+1})}\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_q p_{d_{k+1}-j}(q).$$

So, by (4.1) with $k+1$ instead of k,

$$\begin{aligned}f(q;\mathbf{d}^{(k+1)})&=q^{\sum_{t=1}^{k}(d_t-d_{k+1})}\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_q p_{d_{k+1}-j}\\ &\quad-q^{\sum_{t=1}^{k}(d_t-(d_{k+1}-1))}\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_q p_{d_{k+1}-(1+j)}\\ &=q^{\sum_{t=1}^{k}(d_t-d_{k+1})}\sum_{j=0}^{k+1}(-1)^j q^{\binom{j}{2}}\left(\begin{bmatrix}k\\ j\end{bmatrix}_q+q^{k-(j-1)}\begin{bmatrix}k\\ j-1\end{bmatrix}_q\right)p_{d_{k+1}-j},\end{aligned}$$

($\begin{bmatrix}k\\ -1\end{bmatrix}_q=0$ by definition). It remains to use the basic recurrence for q-binomial coefficients, namely

$$\begin{bmatrix}k+1\\ j\end{bmatrix}_q=\begin{bmatrix}k\\ j\end{bmatrix}_q+q^{k+1-j}\begin{bmatrix}k\\ j-1\end{bmatrix}_q. \tag{4.5}$$

So (4.3) holds for $k+1$, too. Lemma 4.1 is proved. □

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While Lemma 4.1 holds for both linear and planar partitions, the next statement is specific for planar partitions.

Corollary 4.2.

$$\begin{aligned} f(q;\mathbf{d}^{(k)}) &= q^{\sum_{t=1}^{k-1}(d_t-d_k)}p_{d_k}(q)F_k(q;d_k),\\ F_k(q;d) &:= \sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_q\prod_{v=1}^{j}x(d-v),\\ x(\tau) &:= \prod_{\ell>\tau}(1-q^\ell). \end{aligned} \tag{4.6}$$

Proof of Corollary 4.2. In view of Lemma 4.1, and

$$\frac{p_{d-j}}{p_d(q)} = \prod_{v=1}^{j}\frac{p_{d-v}(q)}{p_{d-v+1}(q)},$$

it suffices to show that

$$\frac{p_{d-v}(q)}{p_{d-v+1}(q)} = x(d-v), \tag{4.7}$$

which in turn follows directly from the formula (1.4) for $p_r(q)$. The corollary is proved. □

However explicit, the formula (4.6) is unsuitable for use in the Cauchy integral formula. The reason is that the terms' signs alternate, and the absolute value of $F_k(q;d)$ for $q\to 1$ falls far below the absolute values of individual terms, and of their differences. By analogy with the q-binomial theorem, one would expect a closed, product-type formula for $F_k(q;d)$, which would tell one how small $F_k(q;d)$ actually is. No such formula is forthcoming though. Still, the q-binomial nature of the coefficients in the sum for F_k leads to the following *asymptotic* formula.

Lemma 4.3. *Let $q\to 1$ and $d\to\infty$ in such a way that $1-q = O(|1-|q||)$, and $|q|^d = o(|1-q|)$. Then*

$$F_k(q;d) = \left(\frac{q^d}{1-q}\right)^k + O\left(\frac{|q|^{(k+1)d}}{|1-q|^{k+1}}\right). \tag{4.8}$$

Proof of Lemma 4.3. For $k=1$, we have

$$\begin{aligned} F_1(q;d) &= 1-x(d-1) = 1-\prod_{\ell\geqslant d}(1-q^\ell)\\ &= \sum_{t\geqslant 1}(-1)^{t-1}\sum_{\ell_t>\cdots>\ell_1\geqslant d} q^{\ell_t}\cdots q^{\ell_1} \end{aligned}$$

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$$= \sum_{t\geqslant 1}(-1)^{t-1}\frac{q^{td}}{1-q^t}\prod_{v=1}^{t-1}\frac{q^v}{1-q^v}$$
$$= \sum_{t\geqslant 1} g_{1,t}(q)\left(\frac{q^d}{1-q}\right)^t, \tag{4.9}$$

where

$$g_{1,t}(q) = (-1)^{t-1}q^{\binom{t}{2}}\prod_{v=1}^{t}(1+\cdots+q^{v-1})^{-1}.$$

In particular, $g_{1,1}(q) = 1$. The series in (4.8) converges absolutely. Indeed, using $1-q = O(|1-|q||)$, we have

$$|1+\cdots+q^{v-1}| = \frac{|1-q^v|}{|1-q|} \geqslant \omega^{-1}\left|\frac{1-|q|^v}{1-|q|}\right| \geqslant \omega^{-1},$$

for some constant $\omega > 0$, so that $|g_{1,t}(q)| \leqslant \omega^t$ and the t th term in (4.9) is of order $(o(1))^t$. Therefore (4.8) holds for $k = 1$. Suppose that, for some $k \geqslant 1$,

$$F_k(q;d) = \sum_{t\geqslant k} g_{k,t}(q)\left(\frac{q^d}{1-q}\right)^t, \tag{4.10}$$

where $g_{k,t}(q) = O(\omega_k^t)$ for q in question; $\omega_k > 0$ is a constant. (We have just established (4.10) for $k = 1$, with $\omega_1 = \omega$.) Let us prove (4.10) for $F_{k+1}(q;d)$. To this end, first we use (4.5) and (4.6) for $F_{k+1}(q;d)$ to get

$$F_{k+1}(q;d) = F_k(q;d) + q^k\sum_{j=1}^{k+1}(-1)^j q^{\frac{(j-1)(j-2)}{2}}\begin{bmatrix} k \\ j-1\end{bmatrix}\prod_{v=1}^{j}x(d-v)$$
$$= F_k(q;d) - q^k x(d-1)F_k(q;d-1).$$

So, using the inductive assumption (4.10) and the last equality in (4.9), we have

$$F_{k+1}(q;d) = \sum_{t\geqslant k} g_{k,t}\left(\frac{q^d}{1-q}\right)^t - q^k\left(1-\sum_{t_1\geqslant 1} g_{1,t}\left(\frac{q^d}{1-q}\right)^{t_1}\right)\sum_{t_2\geqslant k} g_{k,t_2}\left(\frac{q^{d-1}}{1-q}\right)^{t_2}$$
$$= \sum_{t\geqslant k} g_{k,t}(1-q^{k-t})\left(\frac{q^d}{1-q}\right)^t + \sum_{t\geqslant k+1}\left(\frac{q^d}{1-q}\right)^t\left(\sum_{\substack{t_1\geqslant 1, t_2\geqslant k\\ t_1+t_2=t}} q^{k-t}g_{1,t_1}g_{k,t_2}\right)$$
$$= \sum_{t\geqslant k+1} g_{k+1,t}\left(\frac{q^d}{1-q}\right)^t,$$

where

$$g_{k+1,t} = g_{k,t}(1-q^{k-t}) + \sum_{\substack{t_1\geqslant 1, t_2\geqslant k\\ t_1+t_2=t}} q^{k-t}g_{1,t_1}g_{k,t_2}, \quad t \geqslant k+1. \tag{4.11}$$

Consequently, $g_{k+1,t} = O(\omega_{k+1}^t)$, for $\omega_{k+1} = 2\max\{\omega_1, \omega_k\}$, say. So (4.10) holds for all $k \geqslant 1$.

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Setting $t = k+1$ in (4.11), and remembering $g_{1,1} = 1$, we get

$$g_{k+1,k+1} = q^{-1} g_{k,k} + g_{k,k+1}(1 - q^{-1}).$$

Therefore, after simple algebra,

$$g_{k+1,k+1} = q^{-k} + q^{-k}(q-1)\sum_{t=0}^{k-1} q^t g_{t+1,t+2} = 1 + O(|1-q|), \tag{4.12}$$

for every $k \geqslant 0$. The relations (4.10), (4.12) taken together complete the proof of (4.8). □

Corollary 4.2 and Lemma 4.3 imply the following.

Corollary 4.4. *Let $q \to 1$ and $d_k \to \infty$ in such a way that $1 - q = O(|1 - |q||)$ and $|q|^{d_k} = o(|1-q|)$. Then*

$$f(q; \mathbf{d}^{(k)}) = \frac{q^{\sum_{t=1}^k d_t}}{(1-q)^k}\, p_{d_k}(q)\left[1 + O\left(\frac{|q|^{d_k}}{|1-q|}\right)\right]. \tag{4.13}$$

Introduce $p_n(\mathbf{d}^{(k)})$, the total number of diagrams with the k largest heights equal $d_1, d_2, \ldots, d_k$. Of course, $p_n(\mathbf{d}^{(k)})$ is the coefficient by q^n in the generating function $f(q; \mathbf{d}^{(k)})$, expressed, as before, through the Cauchy integral

$$p_n(\mathbf{d}^{(k)}) = (2\pi i)^{-1} \oint\limits_{\substack{z = e^{-\xi + i\theta} \\ \theta \in (-\pi, \pi]}} z^{-(n+1)} f(z; \mathbf{d}^{(k)})\, dz. \tag{4.14}$$

Let the integers $d_1 > \cdots > d_k$ be such that

$$d_j = \frac{n^{1/3}}{\beta}\left[\log\left(\frac{n^{1/3}}{\beta}\right)^2 + u_j\right],$$

where $u_j = u_j(n) = O(1)$. Consider $\theta \in [-n^{-\delta}, n^{\delta}]$, $\delta \in (5/9, 2/3)$. For these θs, using $\xi = \beta / n^{1/3}$,

$$\begin{aligned} q^{\sum_{t=1}^k d_t} &= \exp\left((-\xi + i\theta)\sum_{t=1}^k \frac{n^{1/3}}{\beta}\left[\log\left(\frac{n^{1/3}}{\beta}\right)^2 + u_j\right]\right) \\ &= \left(\frac{\beta}{n^{1/3}}\right)^{2k} \exp\left(-\sum_{t=1}^k u_t\right)\left(1 + i\theta k \frac{n^{1/3}}{\beta}\log\left(\frac{n^{1/3}}{\beta}\right)^2 + O(n^{-1/3})\right), \end{aligned}$$

then

$$\begin{aligned} (1-q)^{-k} &= (1 - e^{-\xi + i\theta})^{-k} \\ &= (1 - e^{-\xi})^{-k}\left(1 + \frac{1 - e^{i\theta}}{e^{\xi} - 1}\right)^{-k} \\ &= \left(\frac{n^{1/3}}{\beta}\right)^k \left(1 - ik\frac{\theta}{\xi} + O(n^{-1/3})\right), \end{aligned}$$

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and (see the proof of Lemma 3.2)

$$p_{d_k}(e^{-\xi+i\theta}) = p(e^{-\xi+i\theta})\exp\left(-\frac{e^{-\xi d_k}}{(1-e^{-\xi})^2} + i\theta(d_k + 2/\xi) + O(n^{-1/3})\right)$$
$$= e^{-e^{-u_k}}\left(1 + i\theta(d_k + 2/\xi) + O(n^{-1/3})\right).$$

Therefore (4.13) becomes

$$f(e^{-\xi+i\theta}; \mathbf{d}^{(k)}) = \left(\frac{\beta}{n^{1/3}}\right)^k \exp\left(-\sum_{t=1}^{k} u_t - e^{-u_k}\right) p(e^{-\xi+i\theta})$$
$$\times\left(1 + i\theta\frac{n^{1/3}}{\beta}\left(2(k+1)\log\frac{n^{1/3}}{\beta} - (k-2)\right) + O(n^{-1/3})\right).$$

So, as in the proof of Lemma 3.2, the contribution of the small θs to the contour integral (4.14) is, within a factor $1 + O(n^{-1/3})$,

$$an^{-25/36}e^{bn^{2/3}} \cdot \left(\frac{\beta}{n^{1/3}}\right)^k \exp\left(-\sum_{t=1}^{k} u_t - e^{-u_k}\right). \tag{4.15}$$

As for the remaining θs, the exact formula (4.6) can be used to show – just as in the proofs of Lemmas 3.1 and 3.2 – that their overall contribution is by a factor $\exp(-\alpha^* n^{4/3-2\delta})$ smaller than the quantity in (4.15). Combining this with the asymptotic formula for p_n, the total number of solid diagrams, we have thus proved that the probability that the k largest heights are $(n^{1/3}/\beta)\left[\log(n^{1/3}/\beta)^2 + u_j\right]$, $u_j = O(1)$, $1 \leqslant j \leqslant k$, is asymptotic to

$$\left(\frac{\beta}{n^{1/3}}\right)^k \exp\left(-\sum_{t=1}^{k} u_t - e^{-u_k}\right). \tag{4.16}$$

Therefore we have proved that the sequence $\{\mathscr{H}_n^{(i)}\}_{i\leqslant k}$, defined by

$$H_n^{(i)} = (n^{1/3}/\beta)(\log(n^{1/3}/\beta) + \mathscr{H}_n^{(i)}),$$

converges – in distribution – to $\{Y_i\}_{i\leqslant k}$ with the density

$$\exp\left(-\sum_{t\leqslant k} u_t - e^{-u_k}\right), \quad u_1 \geqslant \cdots \geqslant u_k.$$

As we know, it is the Markov chain, with the density of Y_1 equal to $e^{-u-e^{-u}}$, and the conditional density of Y_i, given $Y_{i-1} = u$, equal to $e^{-v-e^{-v}+e^{-u}}$, $v \leqslant u$. Since $\{Y_i\}_{i\leqslant k}$ is distributed as $\{\log(\sum_{j\leqslant i} E_j)^{-1}\}_{i\leqslant k}$, $E_1, E_2, \ldots$ being independent copies of E, this establishes the part of Theorem 1.2 concerning the *marginal* distribution of the largest heights, whence – by symmetry – the largest widths, and the largest depths. What is still lacking is the proof that the *joint* distribution of $\{h_n^{(i)}, w_n^{(i)}, d_n^{(i)}\}_{i\leqslant k}$ converges to that of $\{\sum_{j\leqslant i} E_j, \sum_{j\leqslant i} E'_j, \sum_{j\leqslant i} E''_j\}_{i\leqslant k}\}$, where the sequences $\{E_j\}$, $\{E'_j\}$, $\{E''_j\}$ are mutually independent. Somewhat surprisingly, we will prove it by using – three times in succession – the core of the '1-dimensional' argument above.

First, notation. For $\mathbf{k} = (k_1, k_2, k_3) \geqslant (1,1,1)$, denote

$$\mathbf{d}_i^{(k_i)} = (d_{i,1}, \ldots, d_{i,k_i}), \quad (d_{i,1} > \cdots > d_{i,k_i} \geqslant 1), \quad i = 1,2,3.$$

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Set $\mathbf{d}^{(\mathbf{k})} = (\mathbf{d}_1^{(k_1)}, \mathbf{d}_2^{(k_2)}, \mathbf{d}_3^{(k_3)})$. Let $f(q;\mathbf{d}^{(\mathbf{k})})$ denote the generating function of the diagrams with k_1 largest heights, k_2 largest widths, and k_3 largest depths comprising the tuple $\mathbf{d}^{(\mathbf{k})}$. Let $F_1(q;\mathbf{d}^{(\mathbf{k})})$ denote the generating function of the diagrams with (k_1-1) largest heights, k_2 largest widths, and k_3 largest depths comprising the $((k_1-1)+k_2+k_3)$-tuple $(\mathbf{d}_1^{(k_1-1)}, \mathbf{d}_2^{(k_2)}, \mathbf{d}_3^{(k_3)})$, and the remaining heights being at most d_{1,k_1}. Define $F_2(q;\mathbf{d}^{(\mathbf{k})})$, $F_3(q,\mathbf{d}^{(\mathbf{k})})$ analogously.

As in (4.1),

$$\begin{aligned} f(q;\mathbf{d}^{(\mathbf{k})}) &= F_1(q;\mathbf{d}^{(\mathbf{k})}) - F_1(q;\mathbf{d}^{(\mathbf{k})} - \mathbf{e}_{k_1}) \\ &= F_2(q;\mathbf{d}^{(\mathbf{k})}) - F_2(q;\mathbf{d}^{(\mathbf{k})} - \mathbf{e}_{k_1+k_2}) \\ &= F_3(q;\mathbf{d}^{(\mathbf{k})}) - F_3(q;\mathbf{d}^{(\mathbf{k})} - \mathbf{e}_{k_1+k_2+k_3}), \end{aligned} \tag{4.17}$$

where $\mathbf{e}_j = (\overbrace{0,\dots,0}^{j-1}, 1, \overbrace{0,\dots,0}^{|\mathbf{k}|-j})$, $|\mathbf{k}| = k_1+k_2+k_3$. Furthermore, as in (4.2),

$$\begin{aligned} F_1(q;\mathbf{d}^{(\mathbf{k}+\mathbf{e}_1)}) &= q^{d_{1,k_1}-d_{1,k_1+1}} f(q;((\mathbf{d}_1^{(k_1-1)}, d_{1,k_1+1}), \mathbf{d}_2^{(k_2)}, \mathbf{d}_3^{(k_3)})), \\ F_2(q;\mathbf{d}^{(\mathbf{k}+\mathbf{e}_2)}) &= q^{d_{2,k_2}-d_{2,k_2+1}} f(q;(\mathbf{d}_1^{(k_1)}, (\mathbf{d}_2^{(k_2-1)}, d_{2,k_2+1}), \mathbf{d}_3^{(k_3)})), \\ F_3(q;\mathbf{d}^{(\mathbf{k}+\mathbf{e}_3)}) &= q^{d_{3,k_3}-d_{3,k_3+1}} f(q;(\mathbf{d}_1^{(k_1)}, \mathbf{d}_2^{(k_2)}, (\mathbf{d}_3^{(k_3-1)}, d_{3,k_3+1}))) \end{aligned} \tag{4.18}$$

($\mathbf{e}_i$ being ith unit vector in R^3), provided that $d_{k_i+1} \geqslant \max\{k_j : j \neq i\}$ for the ith identity, $i = 1,2,3$. Let us check, say, the first identity. Consider a diagram $\mathscr{P}$ with the parameter $((\mathbf{d}_1^{(k_1-1)}, d_{1,k_1+1}), \mathbf{d}_2^{(k_2)}, \mathbf{d}_3^{(k_3)})$. If we replace the k_1-tallest vertical stack of unit cubes (whose height is d_{1,k_1+1}) by the taller stack of height d_{1,k_1} we get another diagram $\mathscr{P}'$. Its heights with rank exceeding (strictly) k_1 are all d_{1,k_1+1} at most. Also, the depths and the widths in $\mathscr{P}$ which are affected by this operation are among those having rank exceeding d_{1,k_1+1}. Since $d_{1,k_1+1} \geqslant \max\{k_2,k_3\}$, the diagram $\mathscr{P}'$ inherits the parameters $\mathbf{d}_2^{(k_2)}, \mathbf{d}_3^{(k_3)}$ from $\mathscr{P}$ fully intact. Therefore the right-hand expression in (4.18) is indeed the generating function of all diagrams with k_1 largest heights, k_2 largest widths, and k_3 largest depths comprising $\mathbf{d}^{(\mathbf{k})}$, and all other heights being d_{1,k_1+1} at most, that is $F_1(q;(\mathbf{d}_1^{(k_1+1)}, \mathbf{d}_2^{(k_2)}, \mathbf{d}_3^{(k_3)})) = F_1(q;\mathbf{d}^{(\mathbf{k}+\mathbf{e}_1)})$.

Using (4.17), (4.18), we will prove the following counterpart of Corollary 4.4.

Lemma 4.5. *Let* $\mathbf{k}$ *be fixed. Let* $q \to 1$ *and* $d_{\min} = \min\{d_{i,k_i} : i = 1,2,3\} \to \infty$ *in such a way that* $1-q = O(|1-|q||)$ *and* $|q|^{d_{\min}} = O(|1-q|^2)$. *Then*

$$f(q;\mathbf{d}^{(\mathbf{k})}) = \prod_{i=1}^{3} \frac{q^{\sum_{t=1}^{k_i} d_{i,t}}}{(1-q)^{k_i}}\, p_{d_{1,k_1},d_{2,k_2},d_{3,k_3}}(q)\big(1+O(|1-q|)\big). \tag{4.19}$$

Proof of Lemma 4.5. As in the proofs of Lemma 4.3 and Corollary 4.4, first we need to show that

$$\begin{aligned} f(q;\mathbf{d}^{(\mathbf{k})}) = p_{d_{1,k_1},d_{2,k_2},d_{3,k_3}}(q) \prod_{i=1}^{3} q^{\sum_{t=1}^{k_i-1}(d_{i,t}-d_{i,k_i})} \\ \times \sum_{\mathbf{t}\geqslant(\mathbf{k},0)} h_{\mathbf{k},\mathbf{t}}(1-q)^{t_4} \prod_{i=1}^{3}\left(\frac{q^{d_{i,k_i}}}{1-q}\right)^{k_i}, \end{aligned} \tag{4.20}$$

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where $\mathbf{t} = (t_1, t_2, t_3, t_4)$, $|h_{\mathbf{k},\mathbf{t}}| = O(\omega_{\mathbf{k}}^{|\mathbf{t}|})$, ($|\mathbf{t}| = \sum_i t_i$), for some constant $\omega_{\mathbf{k}} > 0$ and all $\mathbf{t}$.

For $\mathbf{k} = (1,1,1)$, (4.20) follows from the obvious inclusion–exclusion formula

$$f(q;\mathbf{d}_1) = \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^3} (-1)^{|\boldsymbol{\varepsilon}|} p_{\mathbf{d}_1-\boldsymbol{\varepsilon}}(q), \quad \mathbf{d}_1 = (d_{1,1}, d_{1,2}, d_{1,3}),$$

and the relations

$$\begin{aligned}
\frac{p_{r-1,s,t}(q)}{p_{r,s,t}(q)} &= \frac{x(r-1)x(r+s+t-1)}{x(r+s-1)x(r+t-1)},\\
\frac{p_{r,s-1,t}(q)}{p_{r,s,t}(q)} &= \frac{x(s-1)x(r+s+t-1)}{x(r+s-1)x(s+t-1)},\\
\frac{p_{r,s,t-1}(q)}{p_{r,s,t}(q)} &= \frac{x(t-1)x(r+s+t-1)}{x(r+t-1)x(s+t-1)},
\end{aligned} \tag{4.21}$$

implied by (1.2). (See (4.6) for the definition of $x(\cdot)$.) Omitting somewhat tedious computations, we just mention that the leading coefficient $h_{\mathbf{k},(\mathbf{k},0)}$ is $1 + O(|1-q|)$ in this case because, for $(r,s,t) = (d_{1,1}, d_{2,1}, d_{3,1})$, the ratios in (4.21) are $1 - (q^{d_{i,1}}/(1-q))[1 + O(|1-q|)]$, $i = 1,2,3$.

Now consider $k_1 \geqslant 1$, but keep $k_2 = k_3 = 1$. Using (4.17), (4.18), we can use induction on k_1 to show that, analogously to (4.3),

$$\begin{aligned}
f(q;\mathbf{d}^{(\mathbf{k})}) &= q^{\sum_{t=1}^{k_1-1}(d_{1,t}-d_{1,k_1})}\\
&\quad \times \sum_{j=0}^{k_1-1}(-1)^j \begin{bmatrix} k_1-1 \\ j \end{bmatrix}_q q^{\binom{j+1}{2}} f(q;(d_{1,k_1}-j, d_{2,1}, d_{3,1})).
\end{aligned} \tag{4.22}$$

Combining (4.22) and (4.20) ($\mathbf{k} = (1,1,1)$), and arguing as in the proof of Corollary 4.2, we obtain

$$\begin{aligned}
f(q;\mathbf{d}^{(\mathbf{k})}) &= p_{d_{1,k_1},d_{2,1},d_{3,1}}(q)\cdot q^{\sum_{t=1}^{k_1-1}(d_{1,t}-d_{1,k_1})}\phi_{k_1}(q;(d_{1,k_1}, d_{2,1}, d_{3,1}));\\
\phi_{k_1}(q;(d_1,d_2,d_3)) &= \sum_{j=0}^{k_1}(-1)^j \begin{bmatrix} k_1 \\ j \end{bmatrix}_q \prod_{v=1}^{j}\xi(d_1-v, d_2, d_3)\\
&\quad \times \sum_{\mathbf{t}\geqslant(1,1,1,0)} h_{(1,1,1),\mathbf{t}}(1-q)^{t_4}\frac{q^{d-j}}{1-q}\prod_{i=2}^{3}\frac{q^{d_2}}{1-q},
\end{aligned} \tag{4.23}$$

where $\xi(r,s,t)$ is the first ratio in (4.21). In view of the first relation in (4.23), and (4.20), our task is to show that for $|q|^{\min_i d_i} = O(|1-q|^2)$,

$$\phi_{k_1}(q;(d_1,d_2,d_3)) = \sum_{\mathbf{t}\geqslant(\mathbf{k},0)} h_{\mathbf{k},\mathbf{t}}(1-q)^{t_4}\prod_{i=1}^{3}\left(\frac{q^{d_i}}{1-q}\right)^{k_i}. \tag{4.24}$$

For k_1 this definitely holds by the second relation in (4.23). Using (4.5) for $k = k_1 - 1$, by the definition of $\phi_\cdot(\cdot)$, we obtain

$$\phi_{k_1+1}(q;(d_1,d_2,d_3)) = \phi_{k_1}(q;(d_1,d_2,d_3)) - q^{k_1}\xi(d_1-1,d_2,d_3)\phi_{k_1}(q;(d_1-1,d_2,d_3)).$$

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Suppose that (4.24) holds for some $k_1 \geqslant 1$. Then, by the last recurrence,

$$\phi_{k_1+1}(q;(d_1,d_2,d_3)) = \sum_{t\geqslant(\mathbf{k},0)} h_{\mathbf{k},\mathbf{t}}(1-q)^{t_4}\prod_{i=1}^{3}\left(\frac{q^{d_i}}{1-q}\right)^{k_i} - q^k\xi(d_1-1,d_2,d_3)\sum_{\mathbf{t}\geqslant(\mathbf{k},0)} h_{\mathbf{k},\mathbf{t}}(1-q)^{t_4}\prod_{i=1}^{3}\left(\frac{q^{d_i-\delta(i,1)}}{1-q}\right)^{t_i}. \quad (4.25)$$

Here, by the definition of $\xi(\cdot)$ (see the comment following (4.21)),

$$\xi(d_1,d_2,d_3) = 1 - \frac{q^{d_1}}{1-q} + \sum_{t_1\geqslant 1,|\mathbf{t}|\geqslant 2} a_{\mathbf{t}}(1-q)^{t_4}\prod_{i=1}^{3}\left(\frac{q^{d_i}}{1-q}\right)^{t_i}. \quad (4.26)$$

So every coefficient with subindex $\mathbf{t} \geqslant (\mathbf{k},0)$ such that $t_1 = k_1$ in the resulting expansion of $\phi_{k_1+1}(q;(d_1,d_2,d_3))$ equals $1-1=0$. Therefore the expansion contains only the terms with $t_1 \geqslant k_1+1$. That there exists $\omega_{\mathbf{k}+\mathbf{e}_1}$ such that the $\mathbf{t}$th coefficient in ϕ_{k_1+1} is $O(\omega_{\mathbf{k}+\mathbf{e}_1}^{|\mathbf{t}|})$ is done as in the proof of Lemma 4.3.

In addition, let $c_{k_1,\mathbf{t}}$ denote the coefficient of the $\mathbf{t}$th term in the expansion of ϕ_{k_1}, $\mathbf{t} \geqslant (\mathbf{k},0)$, and let c_{k_1} be the first (potentially) nonzero coefficient, that is, $c_{k_1} = c_{k_1,(\mathbf{k},0)}$. From (4.25), (4.26) it follows that

$$c_{k_1+1} = q^{-1}c_{k_1} + (1-q^{-1})c_{k_1,(\mathbf{k}+\mathbf{e}_1,0)}.$$

Using this recurrence and $c_1 = 1+O(|1-q|)$, we can easily show that $c_{k_1} = 1+O(|1-q|)$. Thus we have proved (4.20) and $h_{\mathbf{k},(\mathbf{k},0)} = 1+O(|1-q|)$ for $\mathbf{k} = (k_1,1,1)$.

Now, only notational changes are needed to use this result as a basis for the inductive proof of (4.20) for $\mathbf{k} = (k_1,k_2,1)$, together with the estimate of the first nonzero coefficient $h_{\mathbf{k},(\mathbf{k},0)}$. Of course, this time we use the second relations in (4.17), (4.18), and (4.21). The final step is the third inductive proof, with the $\mathbf{k} = (k_1,k_2,1)$ as the basis and the third relations in (4.17), (4.18), and (4.21) coming into play, that establishes (4.20) for all $\mathbf{k} = (k_1,k_2,k_3) \geqslant (1,1,1)$, and the formula

$$h_{\mathbf{k},(\mathbf{k},0)} = 1+O(|1-q|).$$

Lemma 4.5 is proved completely. □

Now let $\mathbf{d}^{(\mathbf{k})}$ be such that

$$d_{i,t} = \frac{n^{1/3}}{\beta}\left[\log\left(\frac{n^{1/3}}{\beta}\right)^2 + u_{i,t}\right],$$

where $u_{i,t} = u_{i,t}(n) = O(1)$, $t \leqslant k_i$, $i = 1,2,3$. Closely following the lines of the proof of (4.16), with Lemma 4.5 replacing Corollary 4.4, we obtain that the probability that k_1 largest heights, k_2 largest widths, and k_3 largest depths comprise the tuple $\mathbf{d}^{(\mathbf{k})} = (\mathbf{d}_1^{(k_1)},\mathbf{d}_2^{(k_2)},\mathbf{d}_3^{(k_3)})$ is asymptotic to

$$\prod_{i=1}^{3}\left(\frac{\beta}{n^{1/3}}\right)^{k_i}\exp\left(-\sum_{t=1}^{k_i}u_{i,t} - e^{-u_{i,k_i}}\right).$$

This, of course, is equivalent to what Theorem 1.2 states. □

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5. Proof of Theorem 1.3

Let $d_1 > \cdots > d_k$ and λ be a planar diagram of area k. Let T be a Young tableau of shape λ with entries $1,\dots,k$. Introduce $f(q;\mathbf{d}^{(\mathbf{k})}, T)$, the generating function of the plane partitions with k largest heights $d_1,\dots,d_k$, such that the base of the ith-tallest stack is labelled i in $T^{(k)}$. Let $p_n(d^{(\mathbf{k})}, T)$ be the number of such solid diagrams of volume n, *i.e.*, $p_n(\mathbf{d}^{(\mathbf{k})}, T) = [q^n]f(q;\mathbf{d}^{(\mathbf{k})}, T)$. From Kadell's formula (1.14) it follows that

$$p_n(\mathbf{d}^{(\mathbf{k})}, T) \leqslant p_n^*(\mathbf{d}^{(\mathbf{k})}, T), \qquad p_n^*(\mathbf{d}^{(\mathbf{k})}, T) \stackrel{\text{def}}{=} [q^n]\frac{q^{\sum_{t=1}^k d_t}}{\prod_{\square\in\lambda^{(k)}}(1-q^{h(\square)})}\, p_{d_k}(q). \tag{5.1}$$

Observe that

$$\prod_{\square\in\lambda^{(k)}}(1-q^{h(\square)}) = (1-q)^k\left(\prod_{\square\in\lambda^{(k)}} h(\square)\right)(1+O(|1-q|)), \quad q\to 1.$$

So, by (4.13), for $|q|^{d_k} = O(|1-q|^2)$, $p_n^*(\mathbf{d}^{(\mathbf{k})}, T)$ differs from $f(q;\mathbf{d}^{(k)})$ by the factor $\prod_{\square\in\lambda^{(k)}} h^{-1}(\square) + O(|1-q|)$. Consequently, as in the derivation of (4.15), we obtain the following: let $u_1 > \cdots > u_k$, $u_j = O(1)$, be such that $d_j = (n^{1/3}/\beta)\big[\log(n^{1/3}/\beta)^2 + u_j\big]$ are integers; then, within a factor $1 + O(n^{-1/3})$,

$$p_n^*(\mathbf{d}^{(\mathbf{k})}, T) = an^{-25/36}e^{bn^{2/3}}\cdot\left(\frac{\beta}{n^{1/3}}\right)^k \times \exp\left(-\sum_{t=1}^k u_t - e^{-u_k}\right)\prod_{\square\in\lambda^{(k)}} h^{-1}(\square). \tag{5.2}$$

Therefore, the probability that the k tallest stacks have heights $d_1,\dots,d_k$ and their bases form the tableau T is asymptotically

$$\left(\frac{\beta}{n^{1/3}}\right)^k \exp\left(-\sum_{t=1}^k u_t - e^{-u_k}\right)\prod_{\square\in\lambda^{(k)}} h^{-1}(\square), \tag{5.3}$$

at most. To bound this probability from below, we use (1.14) again to obtain

$$p_n(\mathbf{d}^{(\mathbf{k})}, T) \geqslant p_n^{**}(\mathbf{d}^{(\mathbf{k})}, T), \quad p_n^{**}(\mathbf{d}^{(\mathbf{k})}, T) \stackrel{\text{def}}{=} [q^n]\frac{q^{\sum_{t=1}^{k-1} d_t + \tilde{d}_k}}{\prod_{\square\in\lambda}(1-q^{h(\square)})}\, p_{\tilde{d}_k}(q), \tag{5.4}$$

where $\tilde{d}_k = d_k - \max_j \lambda'_j$. Like $p_n^*(\mathbf{d}^{(\mathbf{k})}, T)$, $p_n^{**}(\mathbf{d}^{(k)}, T)$ is asymptotic to the right-hand side expression in (5.2). Therefore the probability in question is asymptotically *at least* as large as the expression in (5.3), whence it is equivalent to (5.3).

We know that $\{\mathscr{H}_n^{(i)}\}_{i\leqslant k}$ defined by $H_n^{(i)} = (n^{1/3}/\beta)\big[\log(n^{1/3}/\beta)^2 + \mathscr{H}_n^{(i)}\big]$ has the limiting density equal to the exponential factor in (5.3). So, summing over $u_1,\dots,u_k$, we obtain that

$$\lim_{n\to\infty} \mathrm{P}(\mathscr{T}_n^{(k)} = T) = \prod_{\square\in\lambda} h^{-1}(\square). \tag{5.5}$$

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Here $\mathscr{T}_n^{(k)}$ is the random Young tableau formed by the bases of the k tallest stacks. (To be sure, $\mathscr{T}_n^{(k)}$ is defined for all partitions $\mathscr{P}$ except a set of partitions whose probability approaches zero as $n \to \infty$.)

Denoting the shape of $\mathscr{T}_n^{(k)}$ by $\mu_n^{(k)}$ and using the hook formula (1.10), we then arrive at

$$\lim_{n\to\infty} \mathrm{P}(\mu_n^{(k)} = \lambda) = d(\lambda) \prod_{\square\in\lambda} h^{-1}(\square) = \frac{d^2(\lambda)}{k!}.$$

In particular, denoting the set of all Young diagrams of area k by Λ_k, we re-establish the identity

$$\sum_{\lambda\in\Lambda_k} \frac{d^2(\lambda)}{k!} = 1.$$

The proof of Theorem 1.3 is complete. □

Note. It seems certain that the condition 'k is fixed' can be relaxed considerably, especially if one is content with an integral version of Theorem 1.3. We conjecture that for $k = o(n^{1/3})$ and $\mathscr{M}_k \subseteq \Lambda_k$,

$$\mathrm{P}(\mu_n^{(k)} \in \mathscr{M}_k) \sim \sum_{\lambda\in\mathscr{M}_k} \frac{d^2(\lambda)}{k!},$$

as long as the sum on the right is bounded away from zero.

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