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Semigroup representation of the Vlasov evolution

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The well-known van Kampen–Case treatment of the Vlasov equation leads to a spectrum on the real axis. In this paper we show that, by going to a 'rigged' Hilbert space, we can derive a spectral representation that is complex and breaks time symmetry. This leads to a semigroup description in which the decay rates due to the Landau damping appear explicitly in the spectrum. Moreover, we can then define an entropy. In this way, the relation between Landau damping and irreversibility is made explicit. The analogy with the well-known Friedrichs model is stressed.

1. Introduction

It is a great pleasure and a great privilege to participate in this volume honouring Professor Radu Balescu, a friend and colleague. Moreovoer, he is one of the most gifted former students and long-time associate of the senior author (I. P.). The senior author still remembers vividly the deep impression produced by Balescu's early contribution to the kinetic theory of plasmas, in which he used in a masterly way the theory of Cauchy integrals and of Hilbert integral equations. His contribution is described in a lucid way in his early monograph (Balescu 1963), as well as in his subsequent textbooks (Balescu 1975, 1997). In these monographs, he also discusses in detail the Vlasov equation, which plays such an important role in plasmas. He calls it a 'mean field equation' in contrast to 'kinetic equations' (see Balescu 1963, p. 71). Still, the Vlasov equation may lead to irreversible processes associated with Landau damping. What is then the status of these processes? Can they be related to some entropy increase as is the case for collisional processes? This is the problem we shall consider here. Balescu (1963) describes the Van Kampen–Case treatment of the Vlasov equation. This leads to a spectrum of the Vlasov operator on the real axis if there is no Landau growth. In this case no irreversible processes appear in the spectrum. We shall show that there exists a second spectral representation outside the Hilbert space, which incorporates Landau damping and leads to a semigroup description. As for 'kinetic equations', we may then introduce an H function. In this sense, the Vlasov mean field equation comes closer to kinetic equations.

Our method follows our presentation of the well-known Friedrichs model for spontaneous emission in quantum mechanics (Petrosky *et al.* 1991; Antoniou and Prigogine 1993; see also Sudarshan *et al.* 1978). There also exist two spectral representations: the traditional solution due to Friedrichs, leading to a spectrum of the

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Hamiltonian on the real axis, and a second solution corresponding to a semigroup in which the lifetime of the unstable particle appears explicitly in the spectrum. This alternative solution acquires meaning in a 'rigged' Hilbert space based on the Gelfand triplets (Gelfand and Vilenkin 1964; Bohm and Gadella 1989; Antoniou and Prigogine 1993). In this short paper mathematical questions are reduced to the minimum (for the mathematical aspects, see especially Antoniou and Prigogine 1993; Bohm and Gadella 1989). Because of space limitations, we shall only present the main results; more details, as well as the proofs, will be published by one of us (T. P.), who was the first to consider the Vlasov equation from the point of view of 'Gamow vectors' in the rigged Hilbert space.

2. The Vlasov operator

We start with the eigenvalue problem for the linearized Vlasov operator for a classical plasma as presented by Balescu (1963):

$$\mathbf{k}\cdot\mathbf{v}\Phi_{\mathbf{k},\mu}(\mathbf{v}) - \frac{\omega_p^2}{k^2}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}f_0(\mathbf{v})\int d\mathbf{v}'\,\Phi_{\mathbf{k},\mu}(\mathbf{v}') = k\mu\Phi_{\mathbf{k},\mu}(\mathbf{v}), \tag{1}$$

where $k = |\mathbf{k}|$, and $\omega_p = (4\pi e^2 c/m)^{1/2}$ is the plasma frequency, involving the concentration c of the electrons. $f_0(\mathbf{v})$ is the initial velocity distribution, which remains invariant. It should be emphasized that this equation is valid only in the case when the eigenstates of the Liouville operator for the N-particle system have a delta-function singularity in the Fourier representation in the thermodynamic limit (Petrosky and Prigogine 1996; see also Balescu 1963).

Following Van Kampen (1955, 1957) and Case (1959), we decompose the velocity $\mathbf{v}$ into two components: v, which is parallel to the wave vector $\mathbf{k}$, and $\tilde{v}$, which is perpendicular to $\mathbf{k}$. Integrating over the perpendicular component,

$$g_{k\mu}(v) \equiv \int_{-\infty}^{+\infty} d\tilde{v}\,\Phi_{\mathbf{k},\mu}(\mathbf{v}). \tag{2}$$

Equation (1) then reduces to

$$K_k g_{k\mu}(v) = \mu g_{k\mu}(v). \tag{3}$$

Here K_k is the Van Kampen operator, defined by

$$K_k g_{k\mu}(v) \equiv v g_{k\mu}(v) - \eta_k(v)\int_{-\infty}^{+\infty} dv'\,g_{k\mu}(v'), \tag{4}$$

where

$$\eta_k(v) \equiv \frac{\omega_p^2}{k^2}\frac{\partial}{\partial v}\int_{-\infty}^{+\infty} d\tilde{v}\,f_0(\mathbf{v}). \tag{5}$$

To avoid complicated notation, we shall drop the parameter k in the following expressions. Let us define the following indefinite two-form:

$$h(f_1, f_2) = \langle f_1|f_2\rangle \equiv \int_{-\infty}^{+\infty}\frac{dv}{\eta(v)}f_1^*(v)f_2(v), \tag{6}$$

where f^* is the complex conjugate of f. For simplicity, we keep Dirac notation for the indefinite two-form. The two-form makes K a symmetric operator:

$$K^\dagger = K, \tag{7}$$

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where $K^\dagger$ is the Hermitian conjugate of K, defined as usual by $\langle f_1|K^\dagger|f_2\rangle \equiv (\langle f_2|K|f_1\rangle)^*$. Therefore if $\langle g_\mu|g_\mu\rangle \neq 0$ then the eigenvalue μ in (3) is real, while if $\langle g_\mu|g_\mu\rangle = 0$ then this eigenvalue may be a complex number.

In the following discussion it is also convenient to introduce the notation $g_\mu(v) \equiv \langle v|g_\mu\rangle$ and $g_\mu^*(v) \equiv \langle g_\mu|v\rangle$. Then we have

$$\int_{-\infty}^{+\infty} \frac{dv}{\eta(v)} |v\rangle\langle v| = 1, \qquad \langle v|v'\rangle = \eta(v)\delta(v - v'), \tag{8}$$

as well as

$$K = \int \frac{dv}{\eta(v)} |v\rangle v\langle v| - \int dv \int dv' \, |v\rangle\langle v'|. \tag{9}$$

The appearance of the function $\eta(v)$ in the denominator in (6) seems to impose too strong restrictions; for example, $1/\eta(v) = v^{-1}\exp(v^2)$ obtained from a Maxwellian distribution function diverges for large values of $|v|$. However, if we use our formalism together with the eigenfunctions of the Van Kampen operator given in (10) (see also (15)), this problem does not arise. Indeed, because of the factor $\eta(v)$ in (10), as well as the factor $1/C(\mu)$ in (15), the singular factor $1/\eta(v)$ disappears from the spectral decomposition of the evolution operator in (15). Therefore the spectral decomposition (15) based on the indefinite two-form (6) is applicable to a wide class of initial conditions.

Moreover, we note a distinct advantage from the introduction of the indefinite two-form. Indeed, the introduction of this two-form immediately distinguishes functions with a pure exponentially damped oscillation (or more generally a superposition of exponentially damped or growing oscillations) from those with power-law decays or with non-damped oscillations. Indeed, the operator is symmetric, and a necessary condition to have complex eigenvalues is that the states f be outside the Hilbert space. That requires that the indefinite two-form $\langle f|f\rangle$ corresponding to the Hilbert norm vanish. For example, let us consider an initial distribution $f(v, t = 0)$ that vanishes sufficiently rapidly for $|v| \to \infty$. Moreover, if $f_0(\mathbf{v})$ is a Maxwellian distribution then, for all functions $f(v, 0)$ symmetric about the origin, we have $\langle f|f\rangle = 0$, and the evolution of $f(v, t)$ is associated with exponentially damped oscillations. The introduction of the indefinite two-form makes explicit the reason why the Vlasov operator may have decaying (or growing) modes, as well as real eigenvalues. In this way, the question of the Landau damping depends essentially on the extension of the Hilbert space.

As already mentioned, the choice of the initial velocity distribution function $f_0(\mathbf{v})$ may lead to a situation where $\langle g_\mu|g_\mu\rangle = 0$ for certain values of μ. For these function the eigenvalue may be a complex number with a *positive* imaginary part. Indeed, this corresponds to the case where instability of the plasma (i.e. Landau growth) occurs. In the following discussion we shall not consider the unstable case, and restrict ourselves to the stable case, so that there is only Landau damping. Then the eigenstates have been constructed by Van Kampen; they are

$$\langle v|g_\mu\rangle = P\frac{\eta(v)}{v - \mu} + \varepsilon_P(v)\delta(v - \mu), \tag{10}$$

corresponding to continuous real eigenvalue μ. The eigenfunctions satisfy the

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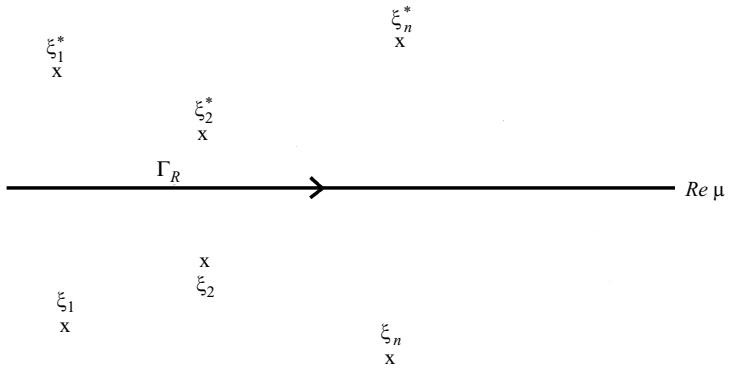


Figure 1. The contour Γ_R: ξ_j indicate the locations of the poles of $1/\varepsilon^+(z)$ in the lower half-plane.

relations† (c.f. (24.3) and (25.11) in Balescu 1963.)

$$\int_{-\infty}^{+\infty} \frac{d\mu}{C(\mu)} |g_\mu\rangle\langle g_\mu| = 1, \qquad \langle g_\mu | g_\nu\rangle = C(\mu)\delta(\mu - \nu), \tag{11}$$

as well as

$$\int_{-\infty}^{+\infty} dv\, g_\mu(v) = 1, \tag{12}$$

where

$$C(\mu) \equiv \frac{\varepsilon^+(\mu)\varepsilon^-(\mu)}{\eta(\mu)}. \tag{13}$$

Here P stands for the principal part, $\varepsilon_P(\mu) \equiv \frac{1}{2}[\varepsilon^-(\mu) + \varepsilon^+(\mu)]$, $\varepsilon^\pm(\mu) \equiv \varepsilon(\mu \pm i0)$, and the 'dielectric constant' is defined by

$$\varepsilon(z) \equiv 1 - \int_{-\infty}^{+\infty} dv\, \frac{\eta(v)}{v - z}. \tag{14}$$

The time evolution of the Vlasov evolution according to Van Kampen and Case is then given by

$$|f(t)\rangle = e^{-ikKt}|f(0)\rangle = \int_{\Gamma_R} \frac{d\mu}{C(\mu)} e^{-ik\mu t} |g_\mu\rangle\langle g_\mu | f(0)\rangle, \tag{15}$$

where the contour Γ_R is defined on the real axis of μ from $-\infty$ to $+\infty$ (see Fig. 1).

For the stable situation, the inverse of the dielectric constant $1/\varepsilon(z)$ has no poles in the complex plane, but has a cut on the real axis. Singularities appear in the lower half-plane for $1/\varepsilon^+(z)$ at $z = \xi_i$ (and in the upper half-plane for $1/\varepsilon^-(z)$ at $z = \xi_i^*$), where $1/\varepsilon^+(z)$ is defined as an analytically continued function of $1/\varepsilon(z)$ starting from the upper half-plane (and $1/\varepsilon^-(z)$ from the lower half-plane). In this paper we consider the case where all singularities of the analytically continued functions are simple poles (see Fig. 1). For $t > 0$, the contribution of these singularities to the evolution in (15) comes only from the poles lying in the lower half-plane. They correspond to Landau damping.

We note that the eigenstates in (10) are time-symmetric, since both analytic

† In this paper we do not consider the case where there are discrete real eigenvalues, since we are not discussing the problem of the transition from the stable to the unstable plasma.

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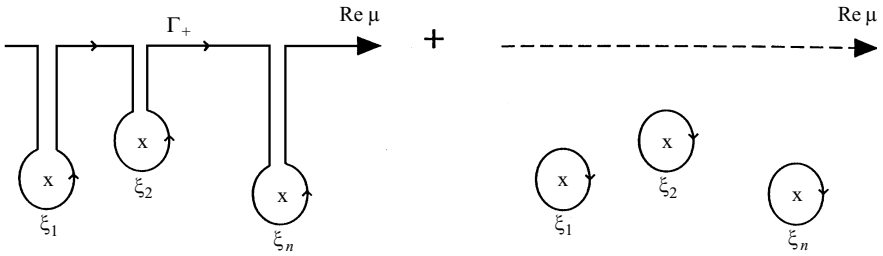


Figure 2. The contours Γ_+ and Γ_i: the contours Γ_i are indicated in the second term in this figure.

continuations from upper and lower half-planes appear symmetrically. As a result of the symmetric property (7), the evolution $\exp(-ikKt)$ is described as a group evolution. In this formulation, the Landau damping is a result of a superposition of time-symmetric oscillations $\exp(-ik\mu t)$.

For the stable case, the indefinite two-forms of the Van Kampen solution (10) do not vanish. Now we show that there is a new class of eigenfunctions of the Vlasov operator. These are functions that lead to two semigroup representations; one is oriented to the future ($t > 0$) and the other to the past ($t < 0$). The group splits into two time-oriented semigroups. Hence we have a breaking of time symmetry. This situation is quite analogous to the case of the spontaneous emission of a photon from an exited atom (Petrosky *et al.* 1991; Antoniou and Prigogine 1993), as well as to the exponential decay of autocorrelation functions in deterministic chaos (Hasegawa and Saphir 1991), where both a group and a semigroup description are possible. The semigroup representation has some advantages over the group representation, since, as we shall see later, the former admits a dynamical definition of Lyapunov functions (or H functions) that evolve monotonically in time. Moreover, the decay rates appear explicitly in the spectrum of K. Therefore the relation to the second law of thermodynamics and the irreversibility of the Landau damping is made explicit in this new representation.

For $t > 0$ the semigroup representation of K is obtained by a contour deformation of Γ_R in Fig. 1 to $\Gamma_+ + \sum_{i=1}^{n} \Gamma_i$ in the lower half-plane of μ in Fig. 2, where n is the number of poles in $1/\varepsilon^+(z)$ in the lower half-plane. Then we obtain from (15) that

$$|f(t)\rangle = \sum_{j=1}^{n} e^{-ik\xi_j t}|G_j\rangle\langle\tilde{G}_j|f(0)\rangle + \int_{-\infty}^{+\infty} d\mu\, e^{-ik\mu t}|G_\mu\rangle\langle\tilde{G}_\mu|f(0)\rangle, \tag{16}$$

where the first summation corresponds to the residues of the poles, and the last term is the background integral corresponding to the contour Γ_+ in Fig. 2. For $t < 0$, the contour should deform into the upper half-plane.

The definitions of the states G and $\tilde{G}$ in (16) are given by

$$\langle v|G_j\rangle = (N_j)^{1/2}\langle v|g_{\xi_j}\rangle = (N_j)^{1/2}\frac{\eta(v)}{(v-\xi_j)^+}, \tag{17a}$$

$$\langle\tilde{G}_j|v\rangle = (N_j)^{1/2}\langle g_{\xi_j}|v\rangle = \langle v|G_j\rangle, \tag{17b}$$

$$\langle v|G_\mu\rangle = \frac{1}{C_d^+(\mu)}\langle v|g_\mu\rangle, \tag{17c}$$

$$\langle\tilde{G}_\mu|v\rangle = \langle g_\mu|v\rangle, \tag{17d}$$

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where the normalization constant is N_j, defined by

$$N_j^{-1} = \frac{1}{\varepsilon^{+\prime}(\xi_j)} \frac{\eta(\xi_j)}{\varepsilon^-(\xi_j)}, \tag{18}$$

with $\varepsilon'(z) \equiv d\varepsilon(z)/dz$, and

$$\frac{1}{C_d^+(\mu)} \equiv \frac{\eta(\mu)}{\varepsilon_d^+(\mu)\varepsilon^-(\mu)}, \tag{19}$$

$$\frac{1}{\varepsilon_d^+(\mu)} \equiv \frac{1}{\varepsilon^+(\mu)} \frac{\prod_{j=1}^n (\mu - \xi_j)}{\prod_{j=1}^n (\mu - \xi_j)^+} \tag{20}$$

Here $1/(v-z)^+$ means that this is the analytically continued function of $1/(v-z)$ from the upper half-plane of z, when this appears in the integral over the real variable v. The subscript d in C_d^+ and ε_d^+ indicates that they are evaluated at the 'delayed' analytic continuation, i.e. one has to evaluate the integration over μ on the real axis in (16) by first shifting all complex numbers ξ_j to the upper half-plane, then analytically continue to the true values of ξ_j with $\operatorname{Im}\xi_j < 0$ in the lower half-plane after the integration.

Note the similarity of the formula (17) with the corresponding formula obtained for the Friedrich model (see (3.8a, b) and (3.11a, b) in Petrosky *et al.* 1991). As we are dealing with complex distributions, we need an appropriate rigged space that is defined by test functions belonging to the Hardy class (see Bohm and Gadella 1989; Antoniou and Prigogine 1993), but we shall not go into a discussion of the mathematical aspects here.

The crucial property of this new class of eigenstates is that the indefinite two-form of the eigenstate with the complex eigenvalue vanishes:

$$\langle G_j | G_j \rangle = \langle \tilde{G}_j | \tilde{G}_j \rangle = 0. \tag{21}$$

This property can be proven with the aid of the relations $\varepsilon^+(\xi_j) = \varepsilon^-(\xi_j^*) = 0$ and

$$\frac{\varepsilon^+(\xi_j) - \varepsilon^-(\xi_j^*)}{\xi_j - \xi_j^*} = \int_{-\infty}^{+\infty} dv \, \frac{\eta(v)}{(v-\xi_j)^+(v-\xi_j^*)^-} = 0. \tag{22}$$

Again (21) shows that our semigroup representation lies inside the rigged Hilbert space. The eigenfunctions G_j and $\tilde{G}_j$ are called the Gamow vectors (Bohm and Gadella 1989).

Moreover, applying the same contour deformation as in Fig. 2 to the spectral representation of the resolvent operator $R_K(z) = (z-K)^{-1}$, we obtain

$$R_K(z) = \int_{\Gamma_R} \frac{d\mu}{C(\mu)} \frac{1}{z-\mu} |g_\mu\rangle\langle g_\mu| = \sum_{j=1}^n \frac{1}{z-\xi_j} |G_j\rangle\langle \tilde{G}_j| + \int_{-\infty}^{+\infty} d\mu \, \frac{1}{z-\mu} |G_\mu\rangle\langle \tilde{G}_\mu|. \tag{23}$$

For the discrete spectrum of a given operator T, it is well known that the residue of $R_T(z)|f\rangle$ at the spectrum t_i of T gives the eigenstates of T with eigenvalues t_i, where f is an arbitrary state. It is easy to extend this theorem to the case where t_i is a complex or a continuous number (Petrosky and Prigogine 1991, Appendix B). Using this extension, one can show that (Prigogine and Petrosky 1996)

$$K|G_j\rangle = \xi_j |G_j\rangle, \qquad \langle \tilde{G}_j | K = \xi_j \langle \tilde{G}_j |, \tag{24a}$$

$$K|G_\mu\rangle = \mu |G_\mu\rangle, \qquad \langle \tilde{G}_\mu | K = \mu \langle \tilde{G}_\mu |. \tag{24b}$$

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Moreover, after a lengthy but straightforward calculation, one can show that

$$\langle \tilde{G}_i | G_j \rangle = \delta_{i,j}, \qquad \langle \tilde{G}_\nu | G_\mu \rangle = \delta(\nu - \mu), \tag{25a}$$

and (for $\alpha, \beta = 1, \ldots, n, \mu$)

$$\langle \tilde{G}_\alpha | G_\beta \rangle = 0 \qquad \text{for } \alpha \neq \beta. \tag{25b}$$

Therefore we obtain a bi-orthonormal spectral representation of the Vlasov operator with complex eigenvalues. In contrast to the group representation in (15), this representation breaks time symmetry, and is applicable only for $t > 0$.

An important consequence of our new representation of the Vlasov operator is that we can introduce a Lyapunov function defined by

$$\langle f(t) | \mathscr{H} | f(t) \rangle \equiv \sum_{j=1}^{n} \langle f(t) | \tilde{G}_j \rangle \langle \tilde{G}_j | f(t) \rangle + \int_{-\infty}^{+\infty} d\mu \, \langle f(t) | \tilde{G}_\mu \rangle \langle \tilde{G}_\mu | f(t) \rangle, \tag{26}$$

which is a nonlinear functional of the state $f(t)$. It evolves monotonically in time as

$$\begin{aligned} \langle f(0) | e^{+ikKt} \mathscr{H} e^{-ikKt} | f(0) \rangle = & \sum_{j=1}^{n} e^{-2k\gamma_j t} \langle f(0) | \tilde{G}_j \rangle \langle \tilde{G}_j | f(0) \rangle \\ & + \int_{-\infty}^{+\infty} d\mu \langle f(0) | \tilde{G}_\mu \rangle \langle \tilde{G}_\mu | f(0) \rangle, \end{aligned} \tag{27}$$

where $\gamma_j \equiv -\mathrm{Im}\, \xi_j > 0$. As already mentioned, the situation is quite parallel to the case of the Lyapunov function defined for the Friedrichs model (Petrosky *et al.* 1991; Antoniou and Prigogine 1993). This function plays the role of Boltzmann's $\mathscr{H}$ function.

3. Concluding remarks

Let us conclude with some general remarks. One of the main goals of non-equilibrium statistical mechanics is to derive 'irreversibility' starting with the time-symmetric description given by classical or quantum mechanics. It is therefore very interesting that even in simple situations (the Friedrichs model (Petrosky *et al.* 1991; Antoniou and Prigogine 1993), the Vlasov equation and the baker's transformation (Hasegawa and Saphir 1991)), we may obtain semigroup descriptions once we look for solutions outside the Hilbert space. In these simple cases, we have both a time-reversible Hilbert description and a rigged Hilbert description with broken time symmetry. The situation with 'Large Poincaré systems' is different (Petrosky and Prigogine 1996). In short, they are defined as large systems exhibiting resonances that prevent integrability. In the thermodynamic limit, the existence of a Hilbert-space spectral decomposition is quite doubtful, while we have established the existence in this limit of a spectral decomposition in both classical and quantum mechanics (Petrosky and Prigogine 1996, 1997; Prigogine and Petrosky 1996) outside the Hilbert space. Breaking of time symmetry then appears as a consequence of the thermodynamic limit – exactly like a phase transition in equilibrium statistical mechanics.

As anticipated by Boltzmann, the time-symmetry-broken description of nature is associated with a probabilistic description. Boltzmann failed, but the progress of functional analysis outside the Hilbert space has changed the situation. We can

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now realize Boltzmann's programme. The Vlasov equation that we have studied is a simple example. Of course, many mathematical problems remain open, but it is no longer necessary to introduce the probabilistic aspects through some independent postulate (as suggested by Balescu 1997, p. 21). The latter would mark a defeat for the whole program of statistical mechanics, which aims to deduce from microscopic dynamics the laws of nature, including irreversible processes. On the contrary, we believe that the recent advances in functional analysis outside the Hilbert space show that the realization of this goal is now possible. After decades of work, we remain more confident than ever. The senior author owes this feeling of confidence to the contributions of his many students, among whom Professor Radu Balescu occupies a central position.

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