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Isoperimetric Problems for r-sets

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How 'tightly' can we pack a given number of r-sets of an n-set? To be a little more precise, let $X=[n]=\{1,\dots,n\}$, and let $X^{(r)}=\{A\subset X:|A|=r\}$. For a set system $\mathscr{A}\subset X^{(r)}$, the *neighbourhood* of $\mathscr{A}$ is $N(\mathscr{A})=\{B\in X^{(r)}:|B\triangle A|\leqslant 2 \text{ for some } A\in\mathscr{A}\}$. In other words, $N(\mathscr{A})$ consists of those r-sets that are either in $\mathscr{A}$ or are 'adjacent' to it, in the sense that they are at minimal Hamming distance (i.e., distance 2) from some point of it. Given $|\mathscr{A}|$, how small can $|N(\mathscr{A})|$ be?

We start by giving a 'folklore' conjecture about this, and then go on to some related questions that may shed some light on it. We may assume, taking complements if necessary, that $r\leqslant n/2$. For $1\leqslant l\leqslant r$ and $1\leqslant k\leqslant n$, define $\mathscr{A}_{k,l}=\{A\in X^{(r)}:|A\cap[k]|\geqslant l\}$. (We have suppressed the dependence of $\mathscr{A}_{k,l}$ on n and r, as we keep them fixed throughout this note.) We call a set of the form $\mathscr{A}_{k,l}$ a *k-ball*. A conjecture that seems to have occurred independently to many people, but has not (to our knowledge) ever appeared in print, is that in order to minimize the neighbourhood of a given number of r-sets we cannot do much better than to take a k-ball for some $r\leqslant k\leqslant n-r$.

Conjecture 1. *Let $\mathscr{A}\subset X^{(r)}$. Then there is a set system $\mathscr{B}\subset X^{(r)}$, with $\mathscr{A}_{k,l}\subset\mathscr{B}\subset\mathscr{A}_{k,l-1}$ for some $1\leqslant l\leqslant r$ and $r\leqslant k\leqslant n-r$, such that $|\mathscr{B}|=|\mathscr{A}|$ and $|N(\mathscr{B})|\leqslant|N(\mathscr{A})|$.*

Of course, which k one should take will depend on $|\mathscr{A}|$. Actually, there is a 'fractional' version of this conjecture that is perhaps more appealing. For a fixed value of k, let us denote by f_k the function that satisfies $f_k(|\mathscr{A}_{k,l}|)=|\mathscr{A}_{k,l-1}|$ for all l and is piecewise linear in between these values. Then the fractional version, 'interpolating' between exact values at balls, would be as follows.

Conjecture 2. *Let $\mathscr{A}\subset X^{(r)}$. Then for some $r\leqslant k\leqslant n-r$ we have $|N(\mathscr{A})|\geqslant f_k(|\mathscr{A}|)$.*

These are very natural conjectures. For example, if we write ∂^+ for upper shadow and ∂^- for lower shadow, then we may express $N(\mathscr{A})$ as $\partial^+\partial^-\mathscr{A}$, or alternatively as $\partial^-\partial^+\mathscr{A}$. Now, the Kruskal-Katona theorem tells us that to minimize the lower shadow we should

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take initial segents of the colex order, and to minimize the upper shadow we should take initial segments of the lex order. (See Chapter 5 of [3] for general background about lex, colex and shadows.) Since balls of the form $\mathscr{A}_{k,1}$ are lex initial segments and balls of the form $\mathscr{A}_{k,r}$ are colex initial segments, it is not too surprising that when one mixes upper and lower shadows one needs balls $\mathscr{A}_{k,l}$ with intermediate values of l (and hence varying k as well).

Perhaps the most intriguing special case is the 'middle layer', when $r = n/2$, as then there is just one ball to consider. In addition, this is the one case when the neighbourhood of an extremal set would itself be extremal, so that the result would automatically strengthen itself to give bounds on iterated neighbourhoods.

Conjecture 3. *Let n be even, and let $\mathscr{A} \subset X^{(n/2)}$ with $|\mathscr{A}| \geqslant |\mathscr{A}_{n/2,l}|$. Then $|N(\mathscr{A})| \geqslant |\mathscr{A}_{n/2,l-1}|$.*

Often, isoperimetric inequalities have many consequences. Here, one would have a great strengthening of some intersection theorems. For example, Ahlswede and Khachatrian [2] proved the celebrated $4m - 2m$ conjecture of Erdős, Ko and Rado [5] giving the maximum size of a 2-intersecting family of $2m$-sets of a $4m$-set (namely all $2m$-sets meeting $[2m]$ in at least $m + 1$ points). But this would also be an instant consequence of Conjecture 3. Indeed, if $\mathscr{A} \subset [4m]^{(2m)}$, then $\mathscr{A}$ being 2-intersecting is equivalent to $N(\mathscr{A})$ being disjoint from the set of complements of sets in $\mathscr{A}$, from which the given bound follows immediately.

Ahlswede and Khachatrian [2] also proved the full Frankl conjecture on intersecting families, namely that the largest t-intersecting family of r-sets is always given by a ball (to be precise, the set $\mathscr{A}_{t+2a,t+a}$ for some a). Here is a new conjecture, generalising Conjecture 1, that would imply the Frankl conjecture in much the same way as Conjecture 3 implies the $4m - 2m$ conjecture. We state it in a 'non-fractional' form.

Conjecture 4. *Let $\mathscr{A} \subset X^{(r)}$. Then, for any s and t, there exist k and l with $|\mathscr{A}_{k,l}| \leqslant |\mathscr{A}| < |\mathscr{A}_{k,l-1}|$ and $|\partial^{+(s)}\partial^{-(t)}\mathscr{A}| \geqslant |\partial^{+(s)}\partial^{-(t)}\mathscr{A}_{k,l}|$.*

In other words, for a given number of r-sets, to minimize the number of q-sets at Hamming distance at most d from one of the family it is best to take some ball. Needless to say, we also conjecture the fractional version of this.

Let us close by giving a conjecture that at first sight appears simpler than Conjecture 1, in that the upper and lower shadows are merely added together rather than composed.

Conjecture 5. *Let $\mathscr{A} \subset X^{(r)}$. Then there exist $1 \leqslant l \leqslant r$ and $r \leqslant k \leqslant n - r$ with $|\mathscr{A}_{k,l}| \leqslant |\mathscr{A}| < |\mathscr{A}_{k,l-1}|$ and $|\partial^{+}\mathscr{A}| + |\partial^{-}\mathscr{A}| \geqslant |\partial^{+}\mathscr{A}_{k,l}| + |\partial^{-}\mathscr{A}_{k,l}|$.*

Although it might seem that this is unrelated to neighbourhood questions, in fact there is a close connection. For example, knowledge of this result for the middle layer would actually imply Conjecture 3. This is because, for any $\mathscr{A}$, the set systems $\mathscr{A}$ and the complement of $N(\mathscr{A})$ have disjoint upper shadows and disjoint lower shadows – a

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moment's thought shows that this yields the required bound. So perhaps Conjecture 5, in the special case of the middle layer, is the most tantalising problem of all.

Finally, although it is not our focus here, let us briefly mention the question of edge-isoperimetric (as opposed to vertex-isoperimetric) inequalities for r-sets. In this area, an elegant conjecture of Kleitman [8] was disproved by Ahlswede and Cai [1]. There is an approximate result by Harper [7], which is sharp for a continuous analogue of the problem, but for the discrete case the problem is still open. For general background on discrete isoperimetric problems, see Chapter 16 of [3], and also [9], and for more isoperimetric phenomena involving k-balls see [4].

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