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# Expansive attractors on surfaces

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*Abstract*. We obtain a local topological and dynamical description of expansive attractors on surfaces. The main result is that expansive attractors on surfaces are hyperbolic and have local product structure, except possibly at a finite number of periodic points, which can be either sinks, singularities or épines. Some open questions concerning this kind of dynamics are posed.

## 1. *Introduction*

This paper addresses the issue of the topological structure of attractors for homeomorphisms on surfaces. The situation is classical and very well understood in the case of hyperbolic attractors for diffeomorphisms (see, for instance, [**P, W2**]). Here we replace the hyperbolicity condition by a purely metric and more general condition, called *expansivity*, which is natural in the context of homeomorphisms.

More precisely, we obtain a local topological and dynamical description of expansive attractors of surfaces. By an *expansive attractor* of a homeomorphism $f$ of compact metric space $(X, d)$, we mean a compact invariant set $A$ for which there exists some constant $\alpha > 0$ verifying:

(1) $A = \bigcap_{n\geq 0} f^n(B_\alpha(A))$;

(2) $A$ is *expansive*: $\sup_{n\in\mathbb{Z}} d(f^n(x), f^n(y)) > \alpha$ for all $x \neq y \in A$.

All hyperbolic attractors, such as Plykin attractors [**P**] and DA attractors [**W1**], are examples of expansive attractors. However, there are also expansive attractors that are not hyperbolic: in §6 we build an example by perturbing a pseudo-Anosov homeomorphism on a surface. We call it *derived from pseudo-Anosov attractor*. Its construction is similar to that of a DA diffeomorphism. That is, we convert a periodic point of a pseudo-Anosov map into a repelling periodic point by a small perturbation, but the periodic point we consider for perturbation is *hyperbolic*. The attractor consists of the maximal invariant set of the complement of a small neighborhood of this point. It preserves the singularities of the original homeomorphisms, so it cannot be hyperbolic.

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It would be interesting to know if all *transitive* expansive attractors are conjugate to either hyperbolic or derived from pseudo-Anosov attractors. The main result we obtain here is that, at least locally, this seems to be the case. Indeed, we get that expansive attractors on surfaces are hyperbolic and have local product structure, except possibly at a finite number of periodic points.

THEOREM 1.1. *If $A$ is an expansive attractor of a homeomorphism $f$ of a compact surface $M$, then $A$ can be decomposed into the following disjoint invariant sets:*

$$A = pH \cup W^u(E) \cup S$$

*where*

(1) *$pH$ is a compact set consisting of hyperbolic points and a finite number of periodic singularities;*
(2) *$W^u(E)$ is a non-compact set that consists of the unstable manifolds of a finite and periodic set $E$, where $E$ is the set of épines of $A$; the rest of the points in $W^u(E)$ are hyperbolic;*
(3) *$S$ is the set of sinks, and is finite, periodic and isolated from the rest of $A$.*

*Moreover, in the case when $f$ is transitive on $A$ and $A$ is not a periodic sink, then $A = pH$.*

For the sake of accuracy, we remark that here a point $x \in A$ is called:

| | | |
|---|---|---|
| *hyperbolic* | | an arc having $x$ in its interior, |
| a *singularity* | according to | a non-degenerate tree with only one node at $x$, |
| an *épine* | whether $CW^u_\varepsilon(x)$ is | an arc having $x$ at one of its endpoints, |
| a *sink* | | $x$, and $CW^s_\varepsilon(x)$ is a neighborhood of $x$ in $X$, |
| a *source* | | is a neighborhood of $x$, |

for some suitable $\varepsilon > 0$, depending on $x$. By a *tree* with only one node at $x$ we mean here a finite (but $\geq 3$) union of arcs, all verifying that exactly one of its endpoints is $x$. An *arc* is a homeomorphic image of the interval.

All definitions above are based on the set

$$CW^u_\varepsilon(x)$$

that will be called the *$\varepsilon$-unstable component of $x$* and is the connected component of $x$ in the $\varepsilon$-local unstable set of $x$, $W^u_\varepsilon(x)$. The $\varepsilon$-unstable set of $x$ is, as usual, the set of points $y \in X$ verifying

$$d(f^n(x), f^n(y)) \leq \varepsilon \quad \text{for all } n \leq 0.$$

The set $CW^s_\varepsilon(x)$ is defined analogously. It follows from the proof that $CW^u_\varepsilon(x)$ is a segment for all save a finite number of periodic points. Proving this fact was the starting point in the classification of expansive homeomorphisms on surfaces [**H, L**], the case where the attractor is the whole manifold (in this case the set of épines $E$ is always empty).

Let us mention that we do not require sinks to be periodic, although *a fortiori* it is always the case. Also note that, for non-empty $E$, $W^u(E)$ accumulates on $pH$. The set $pH$ is therefore non-empty in the absence of sinks. We present examples of expansive attractors having all kinds of points defined above.

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We also prove the existence of local product structure except at the set of épines and singularities. In fact, what we have is a global structure (lamination) in $pH$ similar to that in pseudo-Anosov dynamics.

THEOREM 1.2. *There is local product structure on $A \setminus (S \cup E \cup P)$, where $P$ is the set of singularities.*

Observe that periodic points are not necessarily dense in $pH$; however, it is easy to see that they are dense in the non-wandering set of $f|_A$. This results suggest the following conjecture.

CONJECTURE. *All transitive expansive attractors are conjugate to either a hyperbolic attractor or a derived from pseudo-Anosov attractor.*

We believe that the following questions should give hints about this aspect.

*Question 1.3.* Is $\bigcup_{x \in A} W^s_\varepsilon(x)$ a neighborhood of $A$ for some $\varepsilon > 0$?

As a straightforward corollary of Theorem 1.1 it follows that there are no non-trivial expansive attractors having the disk as a basin.

*Question 1.4.* Are there expansive attractors admitting the 3-punctured sphere $S_3$ as a basin? Note that are no such hyperbolic attractors.

*Question 1.5.* Are there expansive attractors admitting the annulus $S_2$ as a basin?

## 2. *Sketch of the proof*

The proof is based on a topological description of the pieces $CW^u_\varepsilon(x)$.

(1) Sinks and sources of $A$ are periodic points, whose basins contain uniformly sized balls. This implies there are no sources on $A$, and that the set of sinks is finite and isolated from the rest of $A$ (see §3.1).
(2) Unstable components are non-trivial on $\Lambda = A \setminus S$. Stable components are non-trivial on a neighborhood of $\Lambda$ (see §3.2).
(3) $CW^u_\varepsilon(x)$ is locally connected for $x \in \Lambda$. As $CW^u_\varepsilon(x)$ cannot enclose any region, it is uniquely arcwise connected, i.e. a dendrite.
(4) The number of *unstable prongs* of a point (i.e. the number of components of $CW^u_\varepsilon(x) \setminus \{x\}$) is bounded on $\Lambda$. Singularities cannot accumulate, hence they are finite and periodic (see §4).
(5) Épines are periodic. It follows also that they are finite (see §5).

Once this is done it is easy to see that on any compact set not containing sinks, singularities or épines there is local product structure.

## 3. *Local stable and unstable sets*

The main result in this section is the following theorem.

THEOREM 3.1. *Let $A$ be an expansive attractor on a surface. Then:*
(a) *$S$, the set of sinks of $A$, is finite, periodic and isolated from $\Lambda = A \setminus S$;*

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(b) *for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\inf_{x\in\Lambda} \operatorname{diam} CW^u_\varepsilon(x) \geq \delta, \qquad \inf_{x\in B_\delta(\Lambda)} \operatorname{diam} CW^s_\varepsilon(x) \geq \delta;$$

(c) $CW^u_\varepsilon(x)$ *is a dendrite for all* $x \in \Lambda$ *and* $0 < \varepsilon < \alpha/2$.

In (a) it is proved that sinks and sources are recurrent. This implies that they are periodic. However, sources of $A$ form an open set in $M$, so no sources are allowed. On the other hand, it is proved that basins of attraction of sinks of $A$ contain uniformly sized balls. Therefore sinks must be finite.

Part (b) follows from (a): the lack of sinks in $\Lambda$ implies that for each $x \in \Lambda$ there is a continuum containing $f^n(x)$ that 'swells' in the past, reaching a diameter greater than $\alpha$ for large $n > 0$. Because of the local connectedness of $M$, a continuum cannot shrink once it has swollen. Using this property, we obtain, for each $n > 0$, a non-trivial continuum containing $x$, whose first $n$ iterates are close to the respective $n$ iterates of $x$. These continua are uniformly sized, and hence taking the Hausdorff limit we obtain a non-trivial stable component of $x$.

In part (c), it is shown that $CW^u_\varepsilon(x)$ are locally connected. If it were not the case, there would exist disjoint subcontinua $u_i$ of $CW^u_\varepsilon(x)$, $i = 1, 2, 3$, very close in the Hausdorff topology, so that $u_2$ is 'between' $u_1$ and $u_3$. Stable and unstable components cannot be Hausdorff-close, so stable components of some points of $u_2$ should intersect either $u_1$ or $u_3$, a situation not allowed in an expansive setting. See §3.3.

3.1. *Finiteness of sinks (Theorem 3.1(a)).* This follows from the following propositions.

PROPOSITION 3.2. *Sinks and sources of A are periodic.*

PROPOSITION 3.3. *All sinks of A verify* $B_\delta(x) \subset W^s_\alpha(x)$, *for some fixed* $\delta > 0$.

The unstable component of a point of $A$ is contained in $A$. This implies that the set of sources of $A$ is an open set of $M$. From Proposition 3.2 it follows that there are no sources in $A$.

Also, Proposition 3.3 states that basins of attraction of sinks of $A$ contain uniformly sized balls. Then sinks must be finite, for otherwise there would be two periodic points in the same basin of attraction.

Let us denote

$$W^s_{\varepsilon,N}(x) = \{y \in M : d(f^n(x), f^n(y)) \leq \varepsilon,\ n = 0, \ldots, N\},$$
$$W^u_{\varepsilon,N}(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon,\ n = 0, \ldots, N\}.$$

Properties (3.1)–(3.5) below are easily deduced from expansiveness and compactness of $A$. Properties (3.6) and (3.7) prove Proposition 3.2; properties (3.8) and (3.9) prove Proposition 3.3.

(3.1) For all $\varepsilon > 0$ there exists $N_\varepsilon > 0$ and $\eta > 0$ such that

$$W^s_{\alpha,N_\varepsilon}(x) \cap W^u_{\alpha,N_\varepsilon}(x) \subset B_\varepsilon(x) \quad \text{for all } x \in B_\eta(A).$$

(3.2) $f^m(W^s_\alpha(x)) \subset W^s_\varepsilon(f^m(x))$ for all $m \geq N_\varepsilon$ and $x \in B_\eta(A)$.

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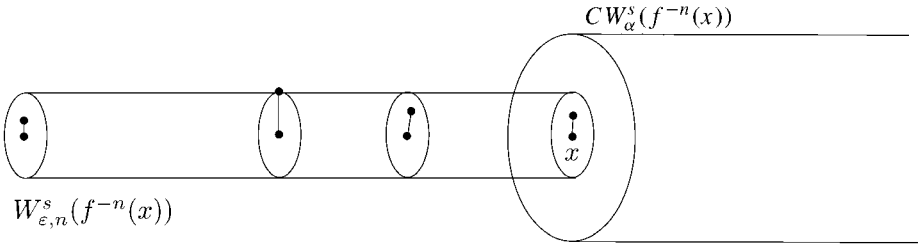



FIGURE 1. Sinks are recurrent—statement (3.7).

(3.3) $f^{-m}(W^u_\alpha(x)) \subset W^u_\varepsilon(f^{-m}(x))$ for all $m \geq N_\varepsilon$ and $x \in A$.

(3.4) $\lim_{m\to\infty} \operatorname{diam} f^m(W^s_\alpha(x)) = 0$ for all $x \in B_\eta(A)$.

(3.5) $\lim_{m\to\infty} \operatorname{diam} f^{-m}(W^u_\alpha(x)) = 0$ for all $x \in A$.

(3.6) A recurrent sink or source is always periodic. Choose $\varepsilon > 0$ and $k \geq N_\varepsilon$ so that $B_\varepsilon(f^k(x)) \subset W^s_\alpha(x)$. It follows from (3.2) that $f^k(W^s_\alpha(x)) \subset W^s_\alpha(x)$. Recurrence of $x$ and (3.4) imply $x$ is periodic.

(3.7) Sinks and sources of $A$ are recurrent. Let $x \in A$ be a sink and let $\varepsilon > 0$ satisfy $B_\varepsilon(x) \subset W^s_\alpha(x)$. Then $W^s_{\varepsilon,n}(f^{-n}(x)) \subset W^s_\alpha(f^{-n}(x))$ for all $n \geq 0$, see Figure 1. Take $\delta > 0$ such that

$$B_\delta(y) \subset W^s_{\varepsilon,N_\varepsilon}(y) \cap W^u_{\varepsilon,N_\varepsilon}(y) \quad \text{for all } y \in M.$$

Then $B_\delta(f^{-n}(x)) \subset W^s_\alpha(f^{-n}(x))$ for all $n \geq 0$. Indeed, if this were not the case, there would be a point $f^{-n}(y)$ in $B_\delta(f^{-n}(x))$ belonging to $\partial W^s_{\varepsilon,n}(f^{-n}(x))$. This would imply that $d(f^{-n+k}(x), f^{-n+k}(y)) = \varepsilon$ for some $k > 0$. But, $k$ being greater than $N_\varepsilon$ (due to choice of $\delta$), this would imply $f^{-n+k}(y) \in W^u_{\alpha,N_\varepsilon}(f^{-n+k}(x)) \cap W^s_\alpha(f^{-n+k}(x))$. In that case, one should have $d(f^{-n+k}(x), f^{-n+k}(y)) < \varepsilon$ because of (3.1), which is absurd.

Now, take $m, n > 0$ such that $m > N_\varepsilon$ and $f^{-n}(x) \in B_\delta(f^{-m}(x))$. Then $f^{m-n}(x) \subset f^m(W^s_\alpha(f^{-m}(x))) \subset W^s_\varepsilon(x)$.

(3.8) All sinks $x \in A$ satisfy $W^s_\alpha(x) = W^s_{\alpha,K}(x)$ for some $K = K_x > 0$. Let $m$ be the period of $x$, and consider $k > 0$ so that $km > N_\rho$, where $B_\rho(x)$ is any ball contained in $W^s_\alpha(x)$. Then $K = 2km$ satisfies $W^s_{\alpha,K}(x) = W^s_\alpha(x)$ since, for any $y \in W^s_{\alpha,K}(x)$, we shall have $f^{km}(y) \in W^s_{\alpha,km}(x) \cap W^u_{\alpha,km}(x)$. Property (3.1) and the choice of $k$ guarantees $f^{km}(y) \in B_\rho(x) \subset W^s_\alpha(x)$.

(3.9) There exists $\delta > 0$ such that $B_\delta(x) \subset W^s_\alpha(x)$ for all sinks $x \in A$. Let $0 < \varepsilon < \alpha$, and take $\delta > 0$ so that

$$B_\delta(x) \subset W^s_{\varepsilon,N_\varepsilon}(x) \cap W^u_{\varepsilon,N_\varepsilon}(x) \quad \text{for all } x \in M.$$

If $x$ were a sink for which $B_\delta(x) \not\subset W^s_\alpha(x)$, then, due to property (3.8), there would be a point $y$ in $B_\delta(x)$ belonging to $\partial W^s_{\alpha,K_x}(x) \subset W^s_\alpha(x)$. This would imply there exists $n > N_\varepsilon$ such that $d(f^n(x), f^n(y)) = \alpha$. But $f^n(y)$ would be in $W^u_{\alpha,N_\varepsilon}(f^n(x)) \cap W^s_\alpha(f^n(x))$, contradicting (3.1).

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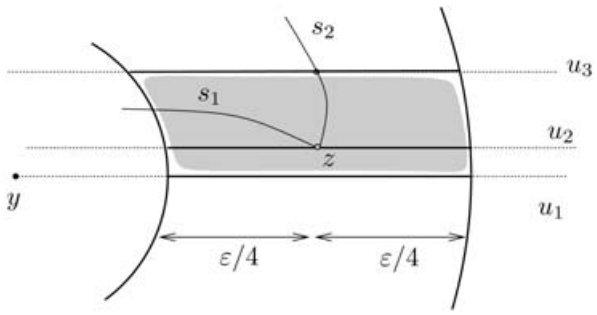



FIGURE 2. $u_1$, $u_2$ and $u_3$ are disjoint continua of $CW^u_\varepsilon(x)$.

3.2. *Non-trivial stable and unstable components (Theorem 3.1(b)).* Take $0 < \varepsilon < \alpha$, and choose $\delta > 0$ so that for $N$ as in property (3.1) the following holds:

$$B_\delta(x) \subset W^s_{\varepsilon,N_\varepsilon}(x) \cap W^u_{\varepsilon,N_\varepsilon}(x) \quad \text{for all } x \in M.$$

Denote $CW^\sigma_{\varepsilon,n}(x)$ the connected component of $W^\sigma_{\varepsilon,n}(x)$ containing $x$, where $\sigma = s, u$. The proof consists in seeing that $CW^\sigma_{\varepsilon,n}(x) \cap \partial B_\delta(x) \neq \emptyset$ for sufficiently large $n > 0$. The result follows immediately, since $CW^\sigma_\varepsilon(x) = \lim_H CW^\sigma_{\varepsilon,n}(x)$.

Consider $x \in B_\delta(\Lambda)$ (it is assumed that $0 < \delta < \eta$, where $\eta$ is as in property (3.1)). Then there exists $K > 0$ such that $B_\delta(f^n(x))$ is not included in $CW^u_{\varepsilon,n}(f^n(x))$ for all $n \geq K$; otherwise there would be a source in $\omega(x) \subset A$. So, for some $f^n(y) \in B_\delta(f^n(x)) \cap W^u_{\varepsilon,n}(f^n(x))$, there exists $0 \leq k \leq n$ such that $d(f^k(x), f^k(y)) = \varepsilon$. By the same arguments as in property (3.7), we get $k < N_\varepsilon$, therefore the choice of $\delta > 0$ implies $d(x, y) \geq \delta$, where $y \in CW^s_{\varepsilon,n}(x)$, since $f^n(y) \in CW^u_{\varepsilon,n}(f^n(x)) = f^n(CW^s_{\varepsilon,n}(x))$.

3.3. *Unstable components are dendrites (Theorem 3.1(c)).* In this section, it assumed that $B_\alpha(x)$ is homeomorphic to $\mathbb{R}^2$. For simplicity, let us call a continuum $s$ a *stable continuum* if $s \subset W^s_\alpha(x)$ for some $x$ in $A$. A continuum $u \subset W^u_\alpha(x)$ is analogously called an *unstable continuum*. Let us call $u$ an *$\varepsilon$-unstable continuum* if $u$ is an unstable continuum satisfying $\operatorname{diam} u \geq \varepsilon$. $\varepsilon$-stable continua are defined analogously.

LEMMA 3.4. *For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $s$ is an $\varepsilon$-stable continuum, then $s \not\subset B_\delta(u)$ for all unstable subcontinua $u$.*

*Proof.* Otherwise one would have, for some $\varepsilon > 0$, a sequence of $\varepsilon$-stable continua $s_n$, such that $s_n \subset B_{1/n}(u_n)$ for some unstable continuum $u_n$. We may assume $s_n$ and $u_n$ are convergent in the Hausdorff topology to some pair $s, u$, which would satisfy $s \subset u$. The non-trivial continuum $s$ would be both stable and unstable, contradicting expansiveness. □

PROPOSITION 3.5. *For each $\varepsilon > 0$ there exist at most $K_\varepsilon > 0$ disjoint $\varepsilon$-unstable continua in $CW^u_{\alpha/2}(x)$, if $x \in A$.*

COROLLARY 3.6. *For all $x \in A$, and $0 < \varepsilon \leq \alpha/2$, $CW^u_\varepsilon(x)$ is locally connected.*

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*Proof of Proposition 3.5.* Let $\varepsilon > 0$ be given and let $\delta > 0$ be taken as in Lemma 3.4 for $\varepsilon/4$. We lose no generality in assuming $\varepsilon/4 < \inf_{x\in A} \operatorname{diam} CW^u_{\alpha/2}(x)$. Due to compactness of the family of continua with respect to the Hausdorff metric, there exists $K > 0$ such that all collections $u_1, \ldots, u_K$ of continua verify that three of them are $\delta$-close with respect to the metric $d_H$.

So, let $u_1, \ldots, u_K$ be a disjoint collection of unstable continua of $CW^u_\varepsilon(x)$, with $x \in A$, such that $\operatorname{diam} u_i \geq \varepsilon$; and assume that $d_H(u_i, u_j) < \delta$ for $i, j = 1, 2, 3$. Let $y \in u_1$, and consider the closed annulus delimited by $\partial B_{\varepsilon/4}(y)$ and $\partial B_{3\varepsilon/4}(y)$ (see Figure 2).

By renaming the $u_i$ if necessary, it follows there exists a connected set $R$ such that:

- $\partial R \subset u_1 \cup u_3 \cup \partial B_{\varepsilon/4}(y) \cup \partial B_{3\varepsilon/4}(y)$;
- $R \subset B_\delta(u_1)$;
- $u_2 \cap \operatorname{int}(R) \cap \partial B_{\varepsilon/2}(y)$ is non-empty.

A continuum $R$ as above can be obtained, for instance, as follows: let $U$ be the sum of all connected components of $B_\alpha(x) \setminus (u_1 \cup u_2 \cup u_3 \cup \partial B_{\varepsilon/4}(y) \cup \partial_{3\varepsilon/4}(y))$ contained in $B_\delta(u_1)$. Then take $R$ as the closure of $U$. See, for instance, [**K**, §54, p. 355].

The stable component of any point $z \in u_2 \cap \partial B_{\varepsilon/2}(y) \cap \operatorname{int}(R)$, should be in either one of two situations: it behaves like $s_1$ in the picture, that is, $s_1 \cap u_i$ is void for $i = 1, 2$, in which case it would contradict Lemma 3.4; or else it behaves like $s_2$, in that there is a stable component intersecting an unstable set twice, which would contradict expansiveness. □

The following lemma finishes the proof that $CW^u_\varepsilon(x)$ are dendrites.

LEMMA 3.7. *Unstable continua do not contain Jordan curves.*

*Proof.* This is because any bounded region enclosed by a Jordan curve contained in an unstable continuum must be also part of the unstable continuum, due to continuity of $f$. In that case the region consists of sources. □

## 4. *Singularities*

In this section we shall prove that there are only finitely many singular points and hence all of them are periodic points. The absence of singularities shall imply the presence of a product structure.

THEOREM 4.1. *There exist at most finitely many singularities, so they are periodic. Also, the number of unstable prongs of a point is globally bounded in $M$.*

Let us introduce the following definitions. An *unstable prong* is the closure of any connected component of $CW^u_\varepsilon(x) \setminus \{x\}$. An *$\varepsilon$-unstable prong* is any unstable prong of diameter greater that $\varepsilon$. We shall call $x$ a *singular point* if $CW^u_\alpha(x)$ contains three $\varepsilon$-unstable prongs for some $\varepsilon > 0$.

Let $0 < \varepsilon < \alpha/2$ and let $0 < \delta < \frac{1}{2} \inf_{x\in A} \operatorname{diam} CW^\sigma_\varepsilon(x)$, $\sigma = s, u$.

LEMMA 4.2. *If $u_1$ and $u_2$ are two different $\varepsilon$-unstable prongs of $x \in A$, and $I$ is a connected component of $\partial B_\delta(x) \setminus (u_1 \cup u_2)$ such that $\partial I \cap u_i \neq \emptyset$ for $i = 1, 2$ then $CW^s_{\alpha/2}(x) \cap I \neq \emptyset$.*

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*Proof.* Call $R$ the closed (small) component bounded by $u_1, u_2$ and $I$. Then there exists an open neighborhood $U$ of $CW^s_\varepsilon(x)$ such that $CW^s_\varepsilon(y) \cap I = \emptyset$ for all $y \in U$, or else upper semi-continuity would imply the thesis. Indeed, one could find a sequence $y_n \to z \in CW^s_\varepsilon(x)$, such that $CW^s_\varepsilon(y_n) \cap I \neq \emptyset$, so $CW^s_\varepsilon(z) \cap I \neq \emptyset$ and hence $CW^s_{\alpha/2}(x) \cap I \neq \emptyset$.

Take a continuous arc $\gamma$ in $U \cap R$, not meeting $CW^s_\varepsilon(x)$, and such that $\gamma \cap u_i \neq \emptyset$ for $i = 1, 2$. Call $\alpha_i = \{y \in \gamma : CW^s_\varepsilon(y) \cap u_i \neq \emptyset\}$ for $i = 1, 2$. It easily follows that $\gamma_i$ are closed, non-empty, disjoint arcs whose union is $\gamma$, and that is absurd. □

Let us call a point $x$ an *$\varepsilon$-singularity* if it has more than two $\varepsilon$-unstable prongs. Recall that, due to Proposition 3.5, there are at most $K$ unstable prongs of diameter greater than $\delta$. This $K$ is uniform over $A$, and depends only on $\delta$. Let us prove the following lemma.

LEMMA 4.3. *For each fixed $\varepsilon > 0$, the number of $\varepsilon$-singularities is finite.*

*Proof.* Let $\varepsilon > 0$ be given, and take $\delta > 0$ and $K$ as at the beginning of the proof of Proposition 3.5. Let $x_1, x_2$ be $\varepsilon$-singularities, such that $d(x_1, x_2) \leq \varepsilon/4$, and suppose $d_H(CW^u_{\alpha/4}(x_1), CW^u_{\alpha/4}(x_2)) \leq \delta$. Then there are two $\varepsilon$-unstable prongs $u_1, u_2$ in, say $CW^u_{\alpha/4}(x_1)$, verifying $d_H(u_1, u_2) < \delta$. But this is impossible by Lemmas 4.2 and 3.4. This implies there are at most $K$ singularities in $B_{\varepsilon/4}(x_1)$. □

LEMMA 4.4. *There exists $\varepsilon_0 > 0$ such that all unstable continua $u$ satisfying, for some $N > 0$, that* $\inf_{n \geq N} \operatorname{diam} f^n(u) \geq \varepsilon_0$.

*Proof.* Let $\varepsilon_0 > 0$ satisfy that $B_{\varepsilon_0}(x) \subset W^u_{\alpha, N_{\alpha/2}}(x)$ for all $x \in M$; and let $u \subset W^u_\alpha(x)$, be a non-trivial continuum. Then there is an $N > 0$ such that $\operatorname{diam} f^N(u) > \alpha$. This implies $f^n(u) \cap \partial W^s_{\alpha,n}(f^n(x)) \neq \emptyset$ for all $n \geq N$. Choice of $\varepsilon_0$ implies the claim. See the proof of property (3.7). □

So, the number of $\varepsilon$-singularities does not depend on $\varepsilon > 0$: call $S_\varepsilon$ the set of $\varepsilon$-singularities. Then, by Lemma 4.4 there is an $N > 0$ such that $f^N(S_\varepsilon) \subset S_{\varepsilon_0}$. As $S_{\varepsilon_0}$ is a finite set by Lemma 4.3, we get that $S_\varepsilon = S_{\varepsilon_0}$ for all $\varepsilon$; singularities are finite and hence periodic.

## 5. *Épines*

An *épine* is a point $x$ such that $CW^u_{\alpha/2}(x)$ has only one unstable prong. The main result in this section is the following theorem.

THEOREM 5.1. *There exist at most finitely many épines, so they are periodic. Also, there is a neighborhood $U$ of the set $E$ of épines such that $A' = A \setminus \bigcup_{n \geq 0} f^n(U)$ is an expansive attractor without épines, $A = A' \cup W^u(E)$ and $A \cap \Omega(f) = A' \cup E$.*

Let us prove first the following lemmas.

LEMMA 5.2. *For $x \in \Lambda$, let $\varepsilon > 0$ be such that $CW^s_{\alpha/2}(x) \cup CW^u_{\alpha/2}(x)$ separates $B_\varepsilon(x)$. Then, for each arc $I \subset \partial B_\varepsilon(x)$ such that $\partial I \cap CW^\sigma_{\alpha/2}(x) \neq \emptyset$ for both $\sigma = s, u$, we have $CW^s_{\alpha/2}(y) \cap I \neq \emptyset$ for all $y \in CW^u_{\alpha/2}(x) \cap B_\varepsilon(x)$.*

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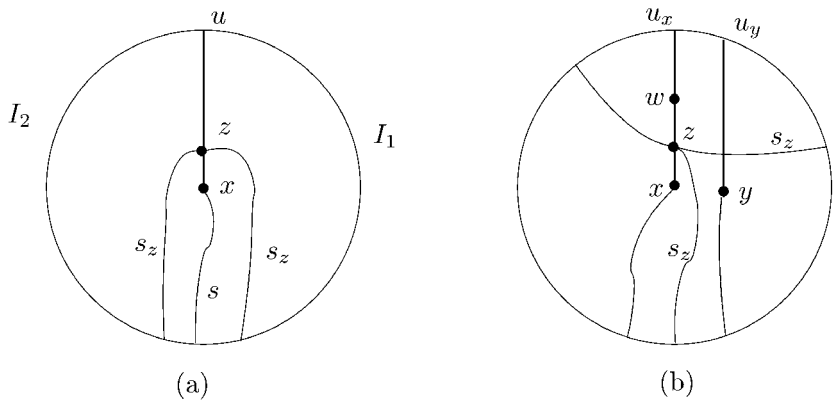



FIGURE 3. Lemmas 5.2 and 5.3.

The proof of the lemma above was the same technique as that of Lemma 4.2. See Figure 3.

PROPOSITION 5.3. *For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x$ and $y$ are épines satisfying $d_H(CW^u_\varepsilon(x), CW^u_\varepsilon(y)) \leq \delta$ then one of the following holds where $s_x$, $u_x$ denote, respectively, $CW^s_\varepsilon(x)$, $CW^u_\varepsilon(x)$:*

(1) *$s_x \cap u_y \neq \emptyset$;*
(2) *$s_y \cap u_x \neq \emptyset$;*
(3) *there is a singularity $p$ in the sector delimited by $s_x \cup u_x \cup s_y \cup u_y$ and two suitable small segments—see Figure 4(b).*

*Proof.* Let $\varepsilon > 0$ be given, and take $\delta > 0$ as in the proof of Proposition 3.5 (finiteness of disjoint $\varepsilon$-unstable continua in each $CW^u_{\alpha/2}(x)$). Using Lemma 5.2 and arguing as in Proposition 3.5 (see Figure 3), we obtain that $s_w \cap u_y \neq \emptyset$ for some $w \in u_x$. Call $[x, w]^u$ the segment going from $x$ to $w$ in $u_x$, and let $z$ be the minimum satisfying $s_z \cap u_y \neq \emptyset$ ($z$ is indeed a minimum, due to upper semi-continuity).

Three situations appear: case (1) $x = z$; case (2) $z \in s_y$; case (3) none of the above. The first two cases imply the thesis. In case (3) $z$ is the supremum of the points of $[x, w]^u$ whose stable components do not intersect $u_y$, which leads us to Figure 4. Note that these are the same cases as in the statement of Proposition 5.3.

CLAIM 5.4. *In case (3) (Figure 4), $\omega(z)$ is a singular orbit.*

*Proof of Claim 5.4.* Let $A, B, C, D$ be as in Figure 4. Due to Lemma 4.4 (growth of unstable continua), there exists $N > 0$ such that diam $f^n(X) \geq \varepsilon_0$ for $X = A, B, C, D$ and $n \geq N$. Moreover, Lemma 5.2 (Figure 3(a)) implies that $f^n(A)$, $f^n(B)$ and $f^n(D)$ cannot collapse, so they converge to three different unstable prongs. Hence, $\omega(z)$ consists of singularities and, due to their finitude, it is a singular orbit. This ends the Proof of Claim 5.4. □

Consider $N > 0$ as in the claim above, and observe that, for all $n \geq N$, there exists only one region $R_n$ delimited by $\partial B_{\varepsilon_0}(f^n(z))$, $f^n(u_x)$ and $f^n(u_y)$, such that $\partial R_n$ intersects all

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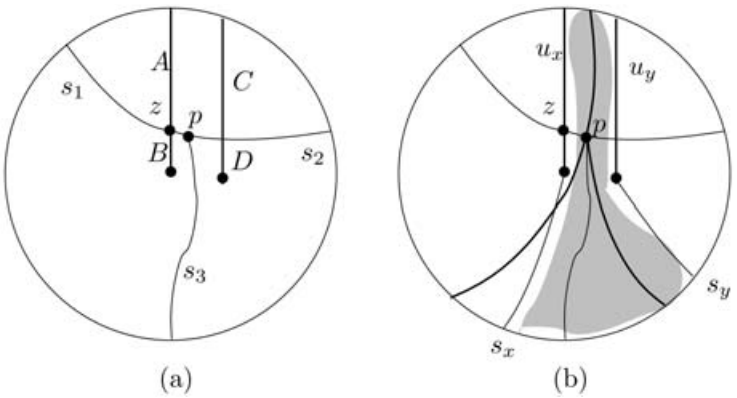



FIGURE 4. Case (3): $x \neq z$ and $z \notin s_y$.

these continua. Since $\omega(z)$ consists of singularities, the fact that unstable prongs cannot accumulate on each other implies that there must be a singularity $p_n$ in $R_n$. Indeed, this is the only possible situation since $f^n(X)$, with $X = A, B, D$, accumulate on different unstable prongs of $p_n$.

A quick analysis of the position of unstable prongs shows that $p_n$ must belong, in fact, to $s_{f^n(z)}$, so applying $f^{-n}$ we obtain Proposition 5.3. □

CLAIM 5.5. *If $y$ is not an épine, then $\omega(y) \cap E = \emptyset$.*

The claim is a consequence of Lemma 5.2, its proof is as in Claim 5.4.

Let us postpone the proofs of the following two claims.

CLAIM 5.6. *If $x$ is an epine, then either $\omega(x) \cap E = \emptyset$ or $\omega(x)$ is a periodic orbit.*

CLAIM 5.7. *For any $x \in E$, $\alpha(x) \subset E$.*

*Proof of Theorem 5.1.* Let $x$ be an épine. Using Claims 5.6 and 5.7 we get that there is a periodic épine $p \in \alpha(x)$. We are going to prove that the orbit of $p$ equals the orbit of $x$. Take a sequence $n_i \to +\infty$ such that $f^{-n_i}(x) \to p$ and assume that $f^k(p) = p$. Let us denote by $u_i$ the set $u_{f^{-n_i}(x)}$, and by $s_i$ the set $s_{f^{-n_i}(x)}$. Using Proposition 5.3 we get that for all $i$ either:

(1) $s_p \cap u_i \neq \emptyset$; or

(2) $s_i \cap u_p \neq \emptyset$.

Assume that for some $i$ case (1) holds, but not case (2). Then we get a point $z \in s_p \cap u_i$ such that $z \neq f^{-n_i}(x)$. This implies $z$ is not an épine, and $p$ is in the omega limit set of $z$. So, $p$ cannot be an épine, due to Claim 5.5, which is false. So let us deal with case (2). Call $z_i = s_i \cap u_p$. As $z_i \in s_i$, we have that $d(f^{n_i}(z_i), x) \to 0$. Now we have that either diam $f^{n_i}([z_i, p]^u) \to 0$, or there is a subsequence where this diameter is bigger than some fixed $\delta$. If the diameter goes to zero, then the distance between $f^{n_i}(p)$ and $x$ goes to zero, and, as $p$ is periodic, we get that they have the same orbit, and we are done. If not, $f^{n_i}(z_i)$ has two unstable prongs of uniform size, by Lemma 5.2 and an argument as used in Claim 5.4, we get that $x$ must have two unstable prongs which is a contradiction. □

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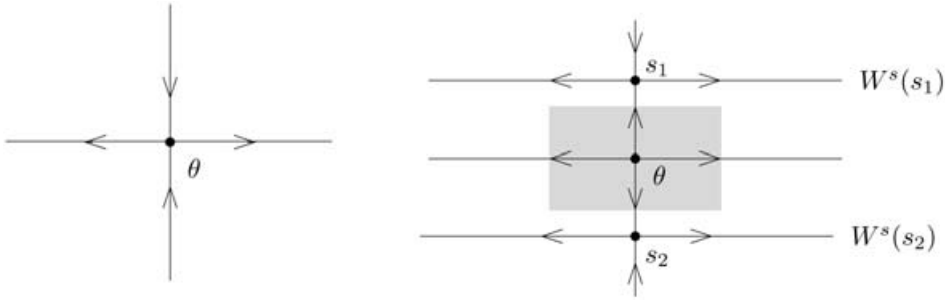



FIGURE 5. Construction of $A$ derived from a pseudo-Anosov attractor.

*Proof of Claim 5.6.* Take $x$ an épine and $z$ in $\omega(x)$. Let us assume first that $z$ is not a singularity. Take $f^{n_1}(x)$ and $f^{n_2}(x)$ two positive iterates of $x$ close to $z$ and such that $u_{f^{n_1}(x)}$ and $u_{f^{n_2}(x)}$ are $\delta$ Hausdorff close where $\delta$ is as in Proposition 5.3. As $z$ is not a singularity, we may assume that Proposition 5.3(3) does not hold. Let us assume that $f^{n_1}(x) \notin s_{f^{n_2}(x)}$, otherwise $\omega(x)$ will be a periodic orbit. Renumbering we may assume that $s_{f^{n_2}(x)} \cap u_{f^{n_1}(x)} = \{y\} \neq \emptyset$ and $y \neq f^{n_1}(x)$. Hence, $y$ cuts $u_{f^{n_1}(x)}$ into two arcs and so $y$ is not an épine. As $\omega(y) = \omega(x)$ we get the claim if there is a $z$ in $\omega(x)$ that is not a singularity. But if any point in $\omega(x)$ is a singularity, we get that $\omega(x)$ is a periodic singularity. □

*Proof of Claim 5.7.* Take $y \in \alpha(x)$ and assume that $y$ is not an épine. Take $f^{-n_i}(x) \to y$. Using a similar argument to Proposition 5.3 we can prove that $s_{f^{-n_i}(x)} \cap u_y = \{z_i\} \neq \emptyset$ for $i$ large enough. As $z_i \in s_{f^{-n_i}(x)}$, we have that $d(f^{n_i}(z_i), x) \to 0$. We have assumed that $y$ is not an épine, so we have that $f^{n_i}(z_i)$ cuts $u_{f^{n_i}(z_i)}$ into two unstable prongs of uniform size. Using Lemma 5.2 and arguing as in Claim 5.4, we get that $x$ must have two unstable prongs which is a contradiction. □

## 6. *Derived from pseudo-Anosov attractors*

Let $f$ be a pseudo-Anosov homeomorphism on a surface $M$, and assume, by eventually taking a suitable iterate, that $\theta \in M$ is a hyperbolic fixed point of $f$. A standard procedure (see, for instance, [**W1**]) gives us a perturbation $g$ of $f$, such that $\Omega(g) = \{\theta\} \cup A$, where $A$ is a one-dimensional attractor, and the saddle $\theta$ unfolds into a source and two new saddles $s_1, s_2$, as in Figure 5. We shall call $A$ a *derived from pseudo-Anosov attractor*. We observe that in this case $A$ is transitive.

It is not difficult to see that $A$ is expansive. Indeed, there is a semiconjugacy $h : M \to M$ satisfying $f \circ h = h \circ g$, such that $h^{-1}(x)$ is a closed interval $[a, b]$ for all $x \in M$ with $a, b \in A$. Moreover, in the case when $a \neq b$, we have that $a$ and $b$ belong to different connected components of $\partial A$. Here, a point $x \in A$ is called a *boundary point of* $A$, and denoted $x \in \partial A$, if there is a stable prong of $x$, $P$, such that $P \cap A = \{x\}$. Observe that, in the case when $A$ is hyperbolic, the usual definition of $\partial A$ coincides with this.

Also, if $a \neq b$, we have $(a, b) \cap A = \emptyset$. So, any two points belonging to different semiconjugacy classes will obviously get separate under the action of $g$. But also, if two

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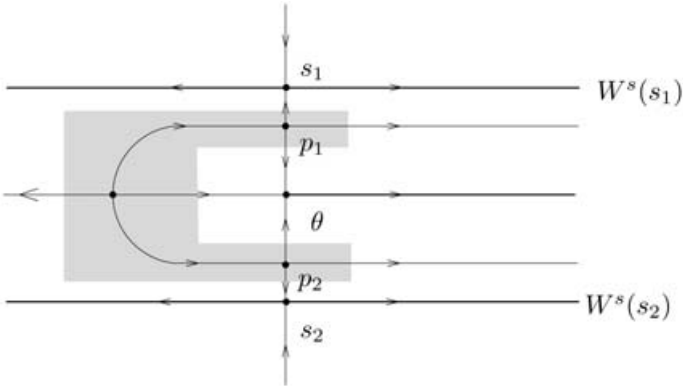



FIGURE 6. Construction of an expansive attractor with epines.

different points belong to the same semiconjugacy class, the only possibility is that they are in different connected components of $\partial A$. Now $\partial A = W^u(s_1) \cup W^u(s_2)$, hence there is a (negative) iterate of $g$ for which they are at a distance $\approx d(s_1, s_2)$.

6.1. *An expansive attractor with épines.* Let us consider $g$ as in the previous example, and consider a new perturbation $k$ of $g$ as in Figure 6. Here $\theta$ splits into a source and a saddle-node fixed point $\theta$, provoking the appearance of two new saddle-node periodic points $p_1$ and $p_2$. $k$ is semiconjugate to $g$, and hence to $f$. Considering an open set as depicted in gray in the figure, one obtains that the maximal invariant set $B$ of its complement consists of a transitive set $C$ on which $k$ is conjugate to $g|_A$ and $W^u(\theta)$ the unstable manifold of $\theta$. In a very similar way as in the previous example, it is easily seen that $B$ is a non-transitive expansive attractor.

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