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On well-quasi-ordering transfinite sequences

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Abstract. Let Q be a well-quasi-ordered set, i.e. a set on which a reflexive and transitive relation $\leqslant$ is defined and such that, for every infinite sequence $q_1, q_2, \ldots$ of elements of Q, there exist i and j such that $i < j$ and $q_i \leqslant q_j$. A *restricted transfinite sequence on* Q is a function from a well-ordered set onto a finite subset of Q. If f, g are restricted transfinite sequences on Q with domains A, B respectively and there exists a one-to-one order-preserving mapping μ of A into B such that $f(\alpha) \leqslant h(\mu(\alpha))$ for every $\alpha \in A$, we write $f \leqslant g$. It is proved that this rule well-quasi-orders the set of restricted transfinite sequences on Q. The proof uses the following subsidiary theorem, which is a generalization of a classical theorem of Ramsey (4). Let P be the set of positive integers, and $A(I)$ denote the set of ascending finite sequences of elements of a subset I of P. If $s, t \in A(P)$, write $s \prec t$ if, for some m, the terms of s are the first m terms of t. Let $T_1, \ldots, T_n$ be disjoint subsets of $A(P)$ whose union T does not include two distinct sequences s, t such that $s \prec t$. Then there exists an infinite subset I of P such that $T \cap A(I)$ is contained in a single T_j.

1. *Introduction.* To avoid confusion, we shall say that a set *includes* its elements and *contains* its subsets. P will denote the set of positive integers. A sequence $x_1, \ldots, x_n$ of positive integers is *ascending* if $x_1 < x_2 < \ldots < x_n$. The *length* of this sequence is n. The length of a sequence s will be denoted by $l(s)$. If $I \subset P$, $A(I)$ will denote the set of all ascending finite sequences of elements of I, including the sequence of length 0, which we shall denote by $\square$. If $m \leqslant n$, we shall say that the sequence $x_1, \ldots, x_m$ is a *left-segment* of the sequence $x_1, \ldots, x_n$. (We naturally count $\square$ as a left-segment of every finite sequence.) If s is a left-segment of t, we also call t an *extension* of s and write $s \prec t$. A subset of $A(P)$ is *thin* if it does not include two distinct sequences s, t such that $s \prec t$. The following theorem is proved in section 2, and constitutes an extension of a well-known theorem of Ramsey ((4), Theorem A).

Theorem 1. *Let $m \in P$ and I be an infinite subset of P and $\{T_1, \ldots, T_m\}$ be a partition of a thin subset T of $A(I)$ into m disjoint subsets. Then there exists an infinite subset K of I such that $T \cap A(K)$ is contained in a single T_j.*

(Ramsey's Theorem is in effect the special case of Theorem 1 in which all elements of T have the same length.)

A *quasi-ordered* (*qo*) set is a set on which a reflexive and transitive relation $\leqslant$ is defined. The letter Q will denote a qo set. An infinite sequence $q_1, q_2, \ldots$ of elements of Q is *good* if there exist positive integers i, j such that $i < j$ and $q_i \leqslant q_j$, and *bad* if not.

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A *well-quasi-ordered* (*wqo*) set is a qo set Q such that every infinite sequence of elements of Q is good. For any ordinal α, O_α will denote the set of ordinals less than α. A *transfinite sequence of type* α *on* Q is a function from O_α into Q. A transfinite sequence f of type α on Q is *restricted* if $f(O_\alpha)$ is a finite subset of Q. The set of restricted transfinite sequences on Q will be denoted by $\mathbf{R}(Q)$. If f, g are transfinite sequences on Q of types α, β, respectively, an (f,g)-*conveyance* is a one-to-one order-preserving function μ from O_α into O_β such that

$$f(\theta) \leqslant g(\mu(\theta))$$

for every $\theta \in O_\alpha$. We quasi-order $\mathbf{R}(Q)$ by writing $f \leqslant g$ if and only if there exists an (f,g)-conveyance. The main object of this paper is to prove (with the aid of Theorem 1) the following conjecture of Rado (3).

THEOREM 2. $\mathbf{R}(Q)$ *is wqo if* Q *is wqo.*

Theorem 2 was proved for finite sequences (i.e. transfinite sequences of type less than ω) by Higman (2). It was proved for transfinite sequences of type less than ω^3 by Rado (3), and for transfinite sequences of type less than ω^ω by J. B. Kruskal (in unpublished work) and by Erdős and Rado (1). The present paper uses an extension of the idea underlying Theorem 3 of (3).

I am indebted to Prof. R. Rado for some improvements in the presentation of this paper.

2. *Proof of Theorem* 1. The rth term of a sequence $s \in A(P)$ will be denoted by $s(r)$ and its last term by $\lambda(s)$. If $I \subset P$, $A_n(I)$ will denote the set of elements of $A(I)$ of length n. Elements of $A(P)$ will sometimes (when this assists clarity) be indicated by writing their terms in ascending order between square brackets: in particular, $[x]$ is the sequence with sole term x. If s, t denote the sequences $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ respectively, st will denote the sequence $x_1, \ldots, x_m, y_1, \ldots, y_n$. If $s \neq \square$, ${}_*s$ and s_* will denote the sequences obtained from s by omitting respectively its first and its last term. We note that, as a consequence of the definition in section 1, $s \prec s$ for every $s \in A(P)$. If $s \prec t$ and $s \neq t$, we call s a *strict left-segment* of t and t a *strict extension* of s. If $s \in A(P)$ and $l(s) \geqslant 2$, $\bar{s}$ will denote the left-segment $[s(1), s(2)]$ of s. If I denotes an infinite subset of P, i_n will denote the nth element of I in ascending order of magnitude and I_n will denote the subset $\{i_1, \ldots, i_n\}$ of I. If $s \in A(I) - \{\square\}$, I/s will denote $I - P_{\lambda(s)}$, i.e. the set of elements of I greater than $\lambda(s)$. We define $I/\square$ to be I. If $s \in A(I)$, a sequence of the form $s[x]$, where $x \in I/s$, will be called an I-*successor* of s. An *admissible class of subsets of* I is a class $\mathfrak{C}$ of infinite subsets of I such that (i) every infinite subset of I contains a member of $\mathfrak{C}$ and (ii) every infinite subset of a member of $\mathfrak{C}$ belongs to $\mathfrak{C}$. An I-*admissible function* is a function $\mathfrak{F}$ with domain $A(I)$ such that $\mathfrak{F}(s)$ is an admissible class of subsets of I/s for every $s \in A(I)$.

LEMMA 1. *Let* I, J *be infinite subsets of* P *such that* $J \subset I$, n *be a non-negative integer and* $\mathfrak{F}$ *be an* I*-admissible function such that* $J/s \in \mathfrak{F}(s)$ *for every* $s \in A(J_n)$. *Then there exists an infinite subset* K *of* J *such that* $K_{n+1} = J_{n+1}$ *and* $K/s \in \mathfrak{F}(s)$ *for every* $s \in A(K_{n+1})$.

Proof. Let $\bar{I}$ be the set of elements of I greater than j_{n+1}. Let the elements of $A(J_{n+1})$ with last term j_{n+1} be $s_1, \ldots, s_r$. Since $\mathfrak{F}(s_1)$ is an admissible class of subsets of $\bar{I}$, it

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includes an infinite subset H^1 of $J \cap \bar{I}$. Since $\mathfrak{F}(s_2)$ is an admissible class of subsets of $\bar{I}$, it includes an infinite subset H^2 of H^1. Since $\mathfrak{F}(s_3)$ is an admissible class of subsets of $\bar{I}$, it includes an infinite subset H^3 of H^2. Continue this construction until we reach H^r, and write $K = J_{n+1} \cup H^r$. Then $K_{n+1} = J_{n+1}$. Let $s \in A(K_{n+1}) = A(J_{n+1})$. If $s \in A(J_n)$, then $K/s \subset J/s$ which belongs to $\mathfrak{F}(s)$ by hypothesis, and therefore $K/s \in \mathfrak{F}(s)$ since $\mathfrak{F}(s)$ is admissible. If $s \notin A(J_n)$, then $\lambda(s) = j_{n+1}$ and therefore $s = s_p$ for some p. Then $K/s = H^r \subset H^p \in \mathfrak{F}(s)$ and hence $K/s \in \mathfrak{F}(s)$ since $\mathfrak{F}(s)$ is admissible.

LEMMA 2. *If I is an infinite subset of P and $\mathfrak{F}$ is an I-admissible function, there exists an infinite subset J of I such that $J/s \in \mathfrak{F}(s)$ for every $s \in A(J)$.*

Proof. Since $\mathfrak{F}(\square)$ is an admissible class of subsets of I, it includes an infinite subset I^0 of I. Since $A(I_0^0) = \{\square\}$, $I^0/s \in \mathfrak{F}(s)$ for every $s \in A(I_0^0)$. Therefore, by Lemma 1, there exists an infinite subset I^1 of I^0 such that $I_1^1 = I_1^0$ and $I^1/s \in \mathfrak{F}(s)$ for every $s \in A(I_1^1)$. By Lemma 1 again, there exists an infinite subset I^2 of I^1 such that $I_2^2 = I_2^1$ and $I^2/s \in \mathfrak{F}(s)$ for every $s \in A(I_2^2)$. By Lemma 1 again, there exists an infinite subset I^3 of I^2 such that $I_3^3 = I_3^2$ and $I^3/s \in \mathfrak{F}(s)$ for every $s \in A(I_3^3)$; and so on. Let

$$J = I^0 \cap I^1 \cap I^2 \cap \ldots.$$

Since $I_n^n = I_n^{n-1}$, J is an infinite set and $J_n = I_n^{n-1}$ for every $n \in P$. If $s \in A(J)$, then $s \in A(J_n) = A(I_n^n)$ for some n. Therefore $I^n/s \in \mathfrak{F}(s)$ and hence $J/s \in \mathfrak{F}(s)$ since $J/s \subset I^n/s$ and $\mathfrak{F}(s)$ is admissible.

Definitions. Let I be an infinite subset of P. Then an *extensor on* $A(I)$ is a function E with domain $A(I)$ such that, for every $s \in A(I)$, $E(s)$ is a set of extensions of s which belong to $A(I)$. Let E be an extensor on $A(I)$. Then, for any $s \in A(I)$, $\hat{E}(s)$ will denote $\{t \in A(I/s) \mid st \in E(s)\}$. If J is an infinite subset of I and $s \in A(J)$, then s will be called *E-extensible in J* or *E-inextensible in J* according as $E(s) \cap A(J)$ is non-empty or empty, respectively, and will be called *strongly E-extensible in J* if it is E-extensible in every infinite subset K of J such that $s \in A(K)$.

LEMMA 3. *If I is an infinite subset of P and E is an extensor on $A(I)$, there exists an infinite subset J of I such that every element of $A(J)$ is either E-inextensible in J or strongly E-extensible in J.*

Proof. If $s \in A(I)$, let $\mathfrak{F}(s)$ denote the class of all infinite subsets H of I/s such that either $\hat{E}(s) \cap A(K)$ is non-empty for every infinite subset K of H or $\hat{E}(s) \cap A(H) = \varnothing$. We show first that $\mathfrak{F}$ is an I-admissible function. Let $s \in A(I)$. If J is an infinite subset of I/s, then either $\hat{E}(s) \cap A(K)$ is non-empty for every infinite subset K of J, in which case $J \in \mathfrak{F}(s)$, or there is an infinite subset K of J such that $\hat{E}(s) \cap A(J) = \varnothing$, in which case $K \in \mathfrak{F}(s)$: in either case J contains a member of $\mathfrak{F}(s)$. Since obviously every infinite subset of a member of $\mathfrak{F}(s)$ belongs to $\mathfrak{F}(s)$, we may infer that $\mathfrak{F}(s)$ is an admissible class of subsets of I/s. Hence $\mathfrak{F}$ is an I-admissible function. Hence, by Lemma 2, there is an infinite subset J of I such that $J/s \in \mathfrak{F}(s)$ for every $s \in A(J)$. Thus, if $s \in A(J)$, either $\hat{E}(s) \cap A(J/s) = \varnothing$ and s is therefore E-inextensible in J or $\hat{E}(s) \cap A(K)$ is non-empty for every infinite $K \subset J/s$, in which case s is strongly E-extensible in J.

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LEMMA 4. *Let I be an infinite subset of P and $\{B, C\}$ be a partition of a thin subset T of $A(I)$ into two disjoint subsets. Then there exists an infinite subset K of I such that $T \cap A(K)$ is contained either in B or in C.*

Proof. Define an extensor E on $A(I)$ by the rule that, if $s \in A(I)$, $E(s)$ is the set of those extensions of s which belong to $A(I)$ and have a left-segment belonging to B. By Lemma 3, there exists an infinite subset J of I such that every element of $A(J)$ is either E-inextensible in J or strongly E-extensible in J. Hence, for each $s \in A(J)$, J/s can be partitioned into the set $X(s)$ of those $x \in J/s$ for which $s[x]$ is E-inextensible in J and the set $Y(s)$ of those $x \in J/s$ for which $s[x]$ is strongly E-extensible in J. Let $\mathfrak{G}(s)$ be the class of those infinite subsets of J/s which are contained either in $X(s)$ or in $Y(s)$: then $\mathfrak{G}(s)$ is an admissible class of subsets of J/s. Hence $\mathfrak{G}$ is a J-admissible function. Therefore, by Lemma 2, there exists an infinite subset K of J such that $K/s \in \mathfrak{G}(s)$ for every $s \in A(K)$. Hence for every $s \in A(K)$, either all K-successors of s are E-inextensible in J, in which case (since $K \subset J$) they are E-inextensible in K, or every K-successor t of s is strongly E-extensible in J, in which case (since K is an infinite subset of J such that $t \in A(K)$) it is E-extensible in K. Thus the K-successors of s are either all E-extensible in K or all E-inextensible in K. But it is clear from the definition of E that s is E-extensible in K if and only if it has a K-successor which is E-extensible in K. Hence either s and all its K-successors are E-inextensible in K or s and all its K-successors are E-extensible in K. It follows that, if all elements of $A_n(K)$ are E-extensible in K, then so are all elements of $A_{n+1}(K)$, whilst, if all elements of $A_n(K)$ are E-inextensible in K, then so are all elements of $A_{n+1}(K)$. Thus, if the sole element $\square$ of $A_0(K)$ is E-extensible in K, then by induction so are all elements of $A(K)$, whilst, if $\square$ is E-inextensible in K, then so are all elements of $A(K)$: thus either all elements of $A(K)$ are E-inextensible in K or they are all E-extensible in K. But any element of $B \cap A(K)$ is E-extensible in K. On the other hand, if $s \in C \cap A(K)$, then, since T is thin, no left-segment of an extension of s belongs to B, whence $E(s) = \varnothing$ and therefore s is E-inextensible in K. Hence either all elements of $A(K)$ are E-inextensible in K and $B \cap A(K) = \varnothing$ or all elements of $A(K)$ are E-extensible in K and $C \cap A(K) = \varnothing$; and the lemma is proved.

Note Prof. P. Erdős has sent me an alternative proof of Lemma 4 which appears to be somewhat simpler and more transparent than mine. It depends on the theorem (deducible from the Axiom of Choice) that there exists a 'measure' μ on the class of all subsets of any enumerably infinite set I such that $\mu(X)$ is 0 or 1 for every $X \subset I$, and is in particular 0 whenever X is finite and 1 whenever $I - X$ is finite, and such that $\mu(X \cup Y) = \mu(X) + \mu(Y)$ for every pair of disjoint subsets X, Y of I.

Proof of Theorem 1. We may suppose that $m > 1$ and make the inductive hypothesis that, for every infinite subset H of P, the theorem is true for partitions of thin subsets of $A(H)$ into $m-1$ subsets. Then, by Lemma 4, there exists an infinite subset H of I such that $T \cap A(H)$ is contained either in T_1 or in $T_2 \cup \ldots \cup T_m$. In the former case we may put $K = H$ and in the latter case, if S_i denotes $T_i \cap A(H)$, there exists by the inductive hypothesis an infinite $K \subset H$ such that $(S_2 \cup \ldots \cup S_m) \cap A(K)$ is contained in a single S_j, i.e. $T \cap A(K)$ is contained in a single T_j.

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3. *Further preliminary lemmas.*

LEMMA 5. *Any infinite sequence $q_1, q_2, \ldots$ of elements of a wqo set Q has an infinite subsequence $q'_1, q'_2, \ldots$ such that $q'_1 \leqslant q'_2 \leqslant \ldots$.*

Proof. Call q_m *terminal* if there is no $n > m$ such that $q_m \leqslant q_n$. There can only be finitely many terminal q_m, since otherwise they would form a bad subsequence of $q_1, q_2, \ldots$. Therefore there is an N such that no q_m $(m > N)$ is terminal. We can therefore select an integer $f(1) > N$, then an $f(2) > f(1)$ such that $q_{f(1)} \leqslant q_{f(2)}$, then an $f(3) > f(2)$ such that $q_{f(2)} \leqslant q_{f(3)}$ and so on, thus obtaining the required subsequence of (q_i).

Definitions. Let f, g be transfinite sequences on Q of types α, β, respectively. If $\gamma \leqslant \alpha$, f^γ will denote the restriction of f to O_γ, which is a transfinite sequence of type γ. Let the ordinal $\sigma(f,g)$ be defined by the property that $O_{\sigma(f,g)}$ is the set of those $\theta \in O_\alpha$ for which $f^{\theta+1} \leqslant g$. If $\theta < \sigma(f,g)$, let $\mu_{f,g}(\theta)$ or $\mu(f,g;\theta)$ denote the least ordinal ξ such that $\nu(\theta) = \xi$ for some $(f^{\theta+1}, g)$-conveyance ν.

LEMMA 6. *$\sigma(f,g) \leqslant \alpha$ and $\mu_{f,g}$ is an $(f^{\sigma(f,g)}, g)$-conveyance.*

Proof. $\sigma(f,g) \leqslant \alpha$ since $O_{\sigma(f,g)}$ is defined to be a subset of O_α. If $\theta < \phi < \sigma(f,g)$, then by the definition of $\mu_{f,g}(\phi)$, there exists an $(f^{\phi+1}, g)$-conveyance ν such that $\nu(\phi) = \mu_{f,g}(\phi)$. If ν' is the restriction of ν to $O_{\theta+1}$, then ν' is an $(f^{\theta+1}, g)$-conveyance and therefore

$$\mu_{f,g}(\theta) \leqslant \nu'(\theta) = \nu(\theta) < \nu(\phi) = \mu_{f,g}(\phi).$$

Hence $\mu_{f,g}$ is a one-to-one order-preserving map of $O_{\sigma(f,g)}$ into O_β. Furthermore, if $\psi < \sigma(f,g)$, there exists by the definition of $\mu_{f,g}(\psi)$ an $(f^{\psi+1}, g)$-conveyance ν such that $\nu(\psi) = \mu_{f,g}(\psi)$. Since $f(\psi) \leqslant g(\nu(\psi))$, it follows that $f^{\sigma(f,g)}(\psi) \leqslant g(\mu_{f,g}(\psi))$.

COROLLARY 6A. *If $f \not\leqslant g$, then $\sigma(f,g) < \alpha$.*

LEMMA 7. *If $\theta < \alpha$, $\xi < \beta$, $\theta \leqslant \sigma(f,g)$, $\mu_{f,g}(O_\theta) \subset O_\xi$ and $f(\theta) \leqslant g(\xi)$, then $\theta < \sigma(f,g)$ and $\mu_{f,g}(\theta) \leqslant \xi$.*

Proof. By Lemma 6 and the data of Lemma 7, we may construct an $(f^{\theta+1}, g)$-conveyance ν by writing $\nu(\theta') = \mu_{f,g}(\theta')$ if $\theta' < \theta$ and $\nu(\theta) = \xi$. Therefore $f^{\theta+1} \leqslant g$, which implies that $\theta < \sigma(f,g)$ and, since $\nu(\theta) = \xi$, it follows that $\mu_{f,g}(\theta) \leqslant \xi$.

4. *Proof of Theorem* 2. Let Q be wqo and (to exclude a trivial case) non-empty. We will suppose that there exists a bad infinite sequence $f_1, f_2, \ldots$ of elements of $\mathbf{R}(Q)$ and show that this leads to a contradiction.

Let f_i have type α_i. Write $\mu_{ij}(\theta)$ for $\mu(f_i, f_j; \theta)$. For any ordinal β and any $s \in A_2(P)$, let $U(s;\beta)$ be the set of all ordinals γ such that $\mu_{s(1),s(2)}(\gamma)$ is defined and is less than β. We may then inductively define an ordinal-number-valued function ϕ on $A(P) - \{\square\}$ by the rules that (i) $\phi([i]) = \alpha_i$ for every $i \in P$ and (ii) if $n \geqslant 1$ and $\phi(s)$ has been defined for every $s \in A_n(P)$, then, for every $t \in A_{n+1}(P)$, $\phi(t)$ is the least ordinal number not in $U(\bar{t}; \phi(*t))$.

LEMMA 8. *If $s \in A_2(P)$, then $\phi(s_*) > \phi(s) = \sigma(f_{s(1)}, f_{s(2)})$. If $\square \neq v \prec w \in A(P)$, then $\phi(v) \geqslant \phi(w)$.*

Proof. If $s \in A_2(P)$, $\phi(s)$ is the least ordinal not in $U(s;\alpha_{s(2)})$, i.e. the least γ such that $\mu_{s(1),s(2)}(\gamma)$ is undefined. Therefore $\phi(s) = \sigma(f_{s(1)}, f_{s(2)})$, which is less than $\alpha_{s(1)} = \phi(s_*)$

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by Corollary 6A since $f_{s(1)} \notin f_{s(2)}$ by hypothesis. Suppose now that we have proved that $\phi(t) \leqslant \phi(t_*)$ for every $t \in A_n(P)$, where $n \geqslant 2$. Then, if $u \in A_{n+1}(P)$, we have $\phi({}_*u) \leqslant \phi({}_*u_*)$ and therefore $U(\overline{u};\ \phi({}_*u)) \subset U(\overline{u};\ \phi({}_*u_*))$ and therefore $\phi(u) \leqslant \phi(u_*)$. Hence $\phi(t) \leqslant \phi(t_*)$ for every $t \in A(P)$ such that $l(t) \geqslant 2$. The second part of the lemma follows by induction from this result.

COROLLARY 8A. *$f_{s(1)}(\phi(s))$ is defined for every $s \in A(P)$ such that $l(s) \geqslant 2$.*

Proof. $f_{s(1)}(\phi(s))$ is defined since, by Lemma 8,

$$\phi(s) \leqslant \phi(\overline{s}) < \phi([s(1)]) = \alpha_{s(1)}.$$

Construct an extensor E on $A(P)$ by defining $E(\square)$ to be $A(P)$ and, if $s \in A(P) - \{\square\}$, $E(s)$ to be the set of those extensions t of s for which $\phi(t) < \phi(s)$. By Lemma 3, there exists an infinite subset I of P such that every element of $A(I)$ is either E-inextensible in I or strongly E-extensible in I. Let S be the set of those elements of $A(I)$ which are E-inextensible in I. Call an element of S *minimal* if it has no strict left-segment belonging to S. Let M be the set of minimal elements of S. (Then M is clearly thin.)

LEMMA 9. *$l(s) \geqslant 2$ for every $s \in S$ (and hence also for every $s \in M$).*

Proof. $\square \notin S$ since $E(\square) = A(P)$. If $x \in I$, then $[x] \notin S$ since

$$[x, y] \in E([x]) \cap A(I)$$

for every $y \in I/[x]$ by Lemma 8.

LEMMA 10. *There exists an infinite subset K of I such that $f_{s(1)}(\phi(s)) \leqslant f_{t(1)}(\phi(t))$ whenever $s, t \in M \cap A(K)$ and $s(1) \leqslant t(1)$.*

Proof. By Corollary 8A and Lemma 9, $f_{s(1)}(\phi(s))$ is defined for every $s \in M$. For any $x \in I$, let M_x be the set of those $t \in A(I/[x])$ for which $[x]t \in M$ and write $g_x(t) = f_x(\phi([x]t))$ for every $t \in M_x$. Then, since f_x is restricted, $g_x(M_x)$ is a finite subset of Q and so there exists a non-empty finite subset $\{q_x^1, \ldots, q_x^{n(x)}\}$ of Q which contains it. Let $\mathfrak{F}([x])$ be the class of those infinite subsets G of $I/[x]$ for which $M_x \cap A(G)$ is contained in one of the sets $g_x^{-1}(q_x^1), \ldots, g_x^{-1}(q_x^{n(x)})$. Since M is thin, so is any subset of M_x. Hence, if H is an infinite subset of $I/[x]$, application of Theorem 1 to the partition of $M_x \cap A(H)$ into its subsets $g_x^{-1}(q_x^i) \cap A(H)$ shows that H contains an infinite set G such that

$$M_x \cap A(G) \subset g_x^{-1}(q_x^i) \cap A(H)$$

for some i. Hence every infinite subset of $I/[x]$ contains a member of $\mathfrak{F}([x])$. Moreover, an infinite subset of a member of $\mathfrak{F}([x])$ clearly belongs to $\mathfrak{F}([x])$. Hence $\mathfrak{F}([x])$ is an admissible class of subsets of $I/[x]$. Therefore, if we extend the definition of $\mathfrak{F}$ to the whole of $A(I)$ by defining $\mathfrak{F}(s)$ to be the class of all infinite subsets of I/s when

$$s \in A(I) - A_1(I),$$

$\mathfrak{F}$ becomes an I-admissible function. There is therefore by Lemma 2 an infinite subset J of I such that $J/s \in \mathfrak{F}(s)$ for every $s \in A(J)$. If $x \in J$, then $J/[x] \in \mathfrak{F}([x])$ and therefore $M_x \cap A(J/[x]) \subset g_x^{-1}(q_x)$ for some $q_x \in Q$. Therefore $f_x(\phi(s)) = q_x$ for every $s \in M \cap A(J)$ such that $s(1) = x$. By Lemma 5, there exists an infinite subset K of J such that $q_x \leqslant q_y$ whenever $x, y \in K$ and $x \leqslant y$. If $s, t \in M \cap A(K)$ and $s(1) \leqslant t(1)$, then

$$f_{s(1)}(\phi(s)) = q_{s(1)} \leqslant q_{t(1)} = f_{t(1)}(\phi(t)).$$

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LEMMA 11. *Every element of* $A(K)-(S\cup\{\square\})$ *has a strict extension* $t\in M\cap A(K)$ *such that* $\phi(t_*)>\phi(t)$.

Proof. We observe first that, if $s\in A(K)-S$, then s is not E-inextensible in I and therefore s is strongly E-extensible in I and therefore s is E-extensible in K and therefore $E(s)\cap A(K)\neq\varnothing$. Consider any $s_0\in A(K)-(S\cup\{\square\})$. Let s_1 be an element of minimum length in $E(s_0)\cap A(K)$. If $s_1\notin S$, let s_2 be an element of minimum length in $E(s_1)\cap A(K)$. If $s_2\notin S$, let s_3 be an element of minimum length in $E(s_2)\cap A(K)$; and so on. If this process defines s_i where $i>0$, then $\phi(s_i)<\phi(s_{i-1})$ since $s_i\in E(s_{i-1})$. Since the $\phi(s_i)$ are ordinal numbers, we cannot have an infinite descending sequence

$$\phi(s_0)>\phi(s_1)>\ldots;$$

therefore our process terminates on reaching some $s_n\in S$ for which we will write $s_n=t$. Then $n\geqslant 1$. Since $\phi(s_n)<\phi(s_{n-1})$, s_n is a strict extension of s_{n-1} and therefore s_{n*} is an extension of s_{n-1}. But $s_{n*}\notin E(s_{n-1})$ since s_n is of minimum length in

$$E(s_{n-1})\cap A(K).$$

Therefore $\phi(s_{n*})$ is not less than $\phi(s_{n-1})$; hence $\phi(s_{n*})>\phi(s_n)$, i.e. $\phi(t_*)>\phi(t)$. It follows that, if $u\neq\square$ and u is a strict left-segment of t, then $\phi(t)<\phi(t_*)\leqslant\phi(u)$ by Lemma 8; therefore $t\in E(u)\cap A(I)$ and so $u\notin S$. Moreover, $\square\notin S$ by Lemma 9. Therefore no strict left-segment of t belongs to S and hence $t\in M$. Thus t is a strict extension of s_0 satisfying the conditions of Lemma 11.

By Lemma 9, any element of $A_1(K)$ belongs to $A(K)-(S\cup\{\square\})$ and so has by Lemma 11 an extension in $M\cap A(K)$. Therefore $M\cap A(K)\neq\varnothing$. Let s be an element of minimum length in $M\cap A(K)$. Then $l(s)\geqslant 2$ by Lemma 9. Furthermore, since s is of minimum length, no left-segment of ${}_*s$ can belong to M and therefore ${}_*s\notin S$. Therefore, by Lemma 11, ${}_*s$ has a strict extension $t\in M\cap A(K)$ such that $\phi(t_*)>\phi(t)$. Write $s(1)=x$, $s(2)=t(1)=y$. Then $x<y$ since $s(1)<s(2)$; therefore $f_x(\phi(s))\leqslant f_y(\phi(t))$ by Lemma 10. Write $u=[x]t$. Since t is a strict extension of ${}_*s$, u and u_* are extensions of s. But $u\notin E(s)$ and $u_*\notin E(s)$ since $s\in M\subset S$ and is therefore E-inextensible in I. Hence

$$\phi(s)=\phi(u_*)=\phi(u)$$

by Lemma 8. Since $\phi(s)=\phi(u)$, which is the least ordinal number not in $U([x,y],\phi(t))$, it follows that $\mu_{xy}(\theta)$ is defined and belongs to $O_{\phi(t)}$ for every $\theta\in O_{\phi(s)}$. Furthermore, by Lemma 8,

$$\phi(s)<\phi([x])=\alpha_x,\quad \phi(t)<\phi([y])=\alpha_y,$$
$$\phi(s)\leqslant\phi(\bar{s})=\sigma(f_x,f_y).$$

We have also noted that $f_x(\phi(s))\leqslant f_y(\phi(t))$. Therefore, by Lemma 7, $\phi(s)<\sigma(f_x,f_y)$ and $\mu_{xy}(\phi(s))\leqslant\phi(t)<\phi(t_*)$. Therefore $\phi(s)\in U([x,y];\phi(t_*))$. But $\phi(s)=\phi(u_*)$, which is defined to be the least ordinal number not in $U([x,y];\phi(t_*))$. This contradiction establishes Theorem 2.

REFERENCES

(1) ERDŐS, P. and RADO, R. A theorem on partial well-ordering of sets of vectors. *J. London Math. Soc.* **34** (1959), 222–224.

(2) HIGMAN, G. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc.* (3), **2** (1952), 326–336.

(3) RADO, R. Partial well-ordering of sets of vectors. *Mathematika* **1** (1954), 89–95.

(4) RAMSEY, F. P. On a problem of formal logic. *Proc. London Math. Soc.* (2), **30** (1930), 264–286.

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