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Dynamics of piecewise conformal automorphisms of the Riemann sphere

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Abstract. The basic goal of this paper is to describe the dynamics of an $m + 1$-parameter family of piecewise conformal automorphisms of the Riemann sphere. In the 'symmetric' figure shown later (Figure 1), we observe the regions of attraction of periodic attractors. In fact, we prove that the only ω-limit sets are periodic cycles. This shows that the conformality of the functions is stronger than the discontinuity.

1. *Introduction*

As is shown in Il'yashenko [**Ily**], the study of polynomial differential equations in the complex plane has grown since the beginning of the last century, motivated by the attempt to solve Hilbert's 16th problem which asks for the number and location of limit cycles of real planar polynomial vector fields of degree n. The complexification of these real vector fields determine a polynomial differential equation of the form

$$\frac{dw}{dz} = \frac{P(z, w)}{Q(z, w)}, \quad P, Q \in \mathbb{C}[z, w], \quad z, w \in \mathbb{C}.$$

In the study of these polynomial differential equations an important idea is to restrict $z \in \mathbb{C}$ to move only along a closed path γ not contained in the image of a singular point $P = Q = 0$. The solutions are then functions $w(\widetilde{z}), \widetilde{z} \in \widetilde{\gamma}$, where $\widetilde{\gamma}$ is a lifting of γ. The Poincaré first return map will be, in this case, a function which is holomorphic in some domains with smooth boundary $D_1, D_2, \ldots, D_n$, having continuous extension to ∂D_i but not coinciding on $\partial D_i \cap \partial D_j$, for $i \neq j$, giving origin to a discontinuity. One is interested in iterating this kind of piecewise conformal functions.

We say that $F : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is a *piecewise conformal* transformation if there exists a family of subsets of the sphere $\mathcal{D} = \{D_n\}_{n\geq 1}$, called the *domain of conformality* of F, such that:

(1) $\bigcup_{n\geq 1} \overline{D_n} = \widehat{\mathbb{C}}$;
(2) $F : D_n \longrightarrow F(D_n)$ is conformal for each n;

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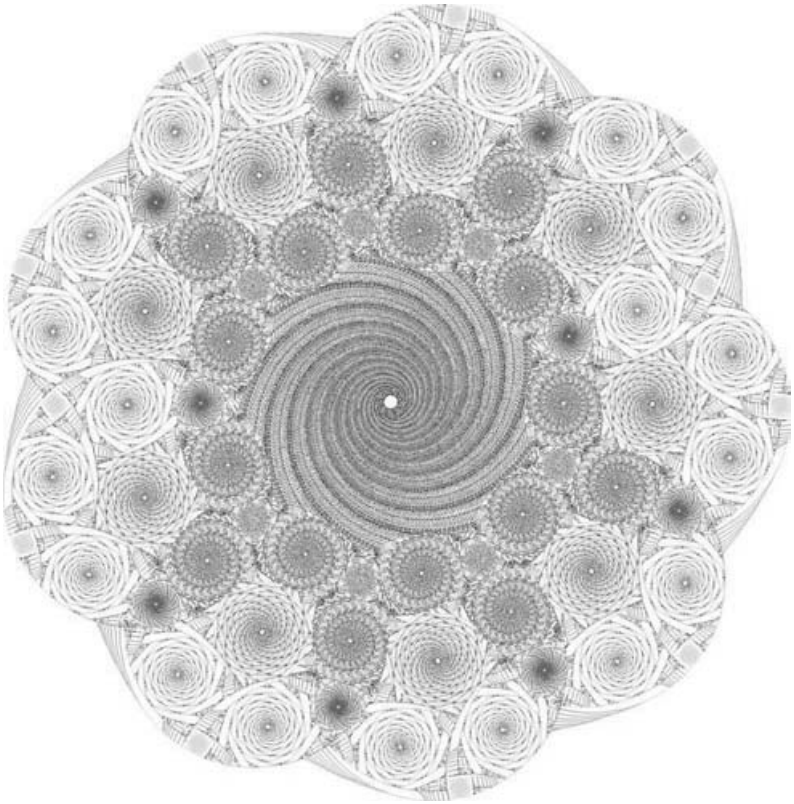


FIGURE 1. The dynamics of $F_{\alpha,\theta}$.

(3) for each $D_n \in \mathcal{D}$, F extends conformally in a neighborhood of $\overline{D_n}$ and $F : \overline{D_n} \longrightarrow \overline{F(D_n)}$ is bijective; and
(4) ∂D_n and $\partial(F(D_n))$ are piecewise smooth curves.

Let $F : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a piecewise conformal map. We say that $z_0 \in \widehat{\mathbb{C}}$ is a *periodic point* of period n for F if z_0 is in the domain of conformality of F and $F^n(z_0) = z_0$. In this case, we call the orbit of z_0 under F, $\{z_0, F(z_0), \ldots, F^{n-1}(z_0)\}$ a periodic cycle.

Let $\mathcal{D} := \{D_j\}_{j=1}^m$ be a finite collection of closed, pairwise disjoint simply connected regions with smooth boundaries $\partial\mathcal{D} := \{\gamma_j\}_{j=1}^m$. By the Riemann Mapping Theorem, D_j is conformally equivalent to a closed disc for each j. Suppose also that 0 does not belong to any of these discs. For each $1 \leq j \leq m$, define the piecewise conformal maps

$$F_j(z) := \begin{cases} f(z) & \text{if } z \notin D_j, \\ g_j(z) & \text{if } z \in D_j, \end{cases}$$

where

$$f(z) = \alpha z \quad \text{and} \quad g_j(z) = \alpha g_{\theta_j}(z)$$

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with $\alpha \in \mathbb{D} := \{\alpha \in \mathbb{C} : |\alpha| < 1\}$ and for each $\theta_j \in S^1$, $g_{\theta_j} : D_j \longrightarrow D_j$ is a conformal and bijective map. By definition, $F_j : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is conformal and bijective on $D_j \cup (\widehat{\mathbb{C}} \setminus D_j)$ but discontinuous along γ_j. Every map F_j first rotates the disc D_j by an angle θ_j and then contracts with multiplication by α.

Now define the general piecewise conformal automorphism $F_{\alpha,\theta} : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ by $F_{\alpha,\theta} = \{F_j\}_{j=1}^m$, where $\theta := (\theta_1, \ldots, \theta_m) \in \mathbb{T}^m$. Clearly, $\mathcal{D} = \{D_j\}_{j=1}^m$ is the domain of conformality of F. The dynamics of $F_{\alpha,\theta}$ with $j = 1$ is shown in Figure 1.

For each $1 \leq j \leq m$, we have associated to each map the following 'irregular' and 'regular' domains:

$$\mathrm{Spid}(F_j) = \overline{\bigcup_{n\geq 0} F_j^{-n}(\partial D_j)} \quad \text{and} \quad \mathcal{F}(F_j) = \widehat{\mathbb{C}} \setminus \mathrm{Spid}(F_j).$$

Definition 1.1. The *spider web* associated with F is

$$\mathrm{Spid}(F_{\alpha,\theta}) := \bigcup_{j=1}^{m} \mathrm{Spid}(F_j),$$

and the *regular* set

$$\mathcal{F}(F_{\alpha,\theta}) := \widehat{\mathbb{C}} \setminus \mathrm{Spid}(F_{\alpha,\theta}).$$

The first important thing to deal with is the study of the accumulation set of the spider web associated with $F_{\alpha,\theta}$. For this contracting case, it happens that this accumulation set consists only of the point at infinity. This is a consequence of the following.

THEOREM 3.7. *For every $\alpha \in \mathbb{D}$, the function $\theta \longrightarrow \mathrm{Spid}(F_{\alpha,\theta})$ is uniformly continuous from $\mathbb{T}^m$ to $H(\widehat{\mathbb{C}})$.*

Here, $H(\widehat{\mathbb{C}})$ is the compact metric space of closed subsets of the Riemann sphere with the Hausdorff topology. The main idea of the proof is that, if the contraction (expansion) is fixed, then a small variation of the rotation parameter does not produce any 'dramatic' change in the global behavior of the corresponding spider webs. This produces the continuous variation.

Definition 1.2. We say that the spider web associated with $F_{\alpha,\theta}$ is *locally finite* in $\mathbb{C}$ if ∞ is the only accumulation point of sequences $\{z_n\} \in F_{\alpha,\theta}^{-n}(\partial\mathcal{D})$.

THEOREM 3.8. *If $(\alpha, \theta) \in \mathbb{D} \times \mathbb{T}^m$, then $\mathrm{Spid}(F_{\alpha,\theta})$ is locally finite in $\mathbb{C}$.*

Similar to the classical Sullivan's No Wandering Domain Theorem for rational functions on the Riemann sphere (see [**Sul**]), we have, as a consequence of the locally finiteness theorem, the corresponding no wandering domain theorem for these kind of piecewise conformal transformations, and this implies the Orbit Classification Theorem as follows.

THEOREM 3.9. *Let $(\alpha, \theta) \in \mathbb{D} \times \mathbb{T}^m$. Then, for each $z \in \mathbb{C}$, there is a periodic cycle $\{z_0, \ldots, z_{N-1}\}$ such that $F_{\alpha,\theta}^n(z)$ converges to $\{z_j\}$ when n goes to ∞.*

Finally we want to point out that similar piecewise transformations defined in the complex plane were studied by Boshernitzan and Goetz [**BG**]. They consider a system of two rotations in the plane with common angle of rotation. This type of transformation

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generalize interval exchange transformations and appear in many different contexts, see [**BG**] and references therein for more information. The author does not know whether or not the methods developed in this paper can be adapted to the case of piecewise isometries.

In §2 we give the basic definitions and in §3 we prove the above theorems.

2. *Basic concepts*

2.1. *General definitions.* We are interested in iterating the piecewise conformal function $F_{\alpha,\theta}$ defined as follows. Let $\mathcal{D} := \{D_j\}_{j=1}^m$ be a finite collection of closed, pairwise disjoint simply-connected regions with smooth boundaries $\partial\mathcal{D} := \{\gamma_j\}_{j=1}^m$. Suppose also that 0 does not belong to any of these regions. For each $1 \le j \le m$, define the piecewise conformal maps

$$F_j(z) = \begin{cases} f(z) & \text{if } z \notin D_j, \\ g_j(z) & \text{if } z \in D_j, \end{cases}$$

where

$$f(z) = \alpha z \quad \text{and} \quad g_j(z) = \alpha g_{\theta_j}(z)$$

with $\alpha \in \mathbb{D}$ and for each $\theta_j \in S^1$, $g_{\theta_j} : D_j \longrightarrow D_j$ is a conformal and bijective map. By definition, $F_j : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is conformal and bijective on $D_j \cup (\widehat{\mathbb{C}} \setminus D_j)$ but discontinuous along γ_j. Every map F_j first rotates the disc D_j by an angle θ_j and then contracts with multiplication by α.

Now define the general piecewise conformal automorphism $F_{\alpha,\theta} : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ by $F_{\alpha,\theta} = \{F_j\}_{j=1}^m$, where $\theta := (\theta_1, \ldots, \theta_m) \in \mathbb{T}^m$. Clearly, $\mathcal{D} = \{D_j\}_{j=1}^m$ is the domain of conformality of F.

We have also defined, for each $1 \le j \le m$, the piecewise conformal inverse function of F_j:

$$F_j^{-1}(z) = \begin{cases} f^{-1}(z) & \text{if } z \notin F_j(D_j), \\ g_j^{-1}(z) & \text{if } z \in F_j(D_j), \end{cases}$$

where

$$f^{-1}(z) = \alpha^{-1} z \quad \text{and} \quad g_j^{-1}(z) = \alpha^{-1} g_{\theta_j}^{-1}.$$

2.2. *The spider web associated with $F_{\alpha,\theta}$.* We define, for each $1 \le j \le m$,

$$\mathrm{Spid}(F_j) = \overline{\bigcup_{n\ge 0} F_j^{-n}(\partial D_j)} \qquad \text{and} \qquad \mathcal{F}(F_j) = \widehat{\mathbb{C}} \setminus \mathrm{Spid}(F_j).$$

Definition 2.1. The *spider web* associated with $F_{\alpha,\theta}$ is

$$\mathrm{Spid}(F_{\alpha,\theta}) := \bigcup_{j=1}^m \mathrm{Spid}(F_j),$$

and the *regular* set

$$\mathcal{F}(F_{\alpha,\theta}) := \widehat{\mathbb{C}} \setminus \mathrm{Spid}(F_{\alpha,\theta}).$$

The following properties are immediate from the above definitions.

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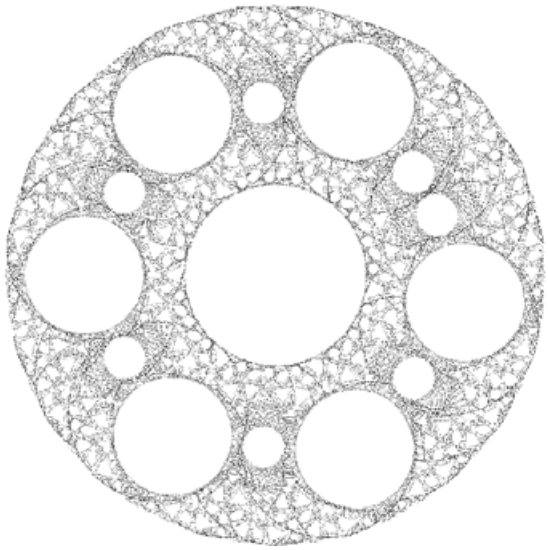


FIGURE 2. The spider web and the regular set.

(1) $\mathrm{Spid}(F_{\alpha,\theta})$ is backward invariant and $\mathcal{F}(F_{\alpha,\theta})$ is forward invariant, i.e. $F_{\alpha,\theta}^{-1}(\mathrm{Spid}(F_{\alpha,\theta})) \subset \mathrm{Spid}(F_{\alpha,\theta})$ and $F_{\alpha,\theta}(\mathcal{F}(F_{\alpha,\theta})) \subset \mathcal{F}(F_{\alpha,\theta})$. (See Figure 2.)

(2) The connected components of $\mathcal{F}(F_{\alpha,\theta})$ are domains of normality of $F_{\alpha,\theta}$. That is, $\{\{F_{\alpha,\theta}^n\}_{|U}\}$ is a normal family of holomorphic maps for any $U \in \mathcal{F}(F_{\alpha,\theta})$; in other words, the family contains uniformly convergent subsequences in the spherical metric on compact subsets of U (see Ahlfors [**Ahl**]).

3. *Results*

Remark 3.1. Let $(\alpha, \theta) \in \mathbb{D} \times \mathbb{T}^m$. If U_0 (respectively, U_∞) is any neighborhood of 0 (respectively, ∞) such that

$$U_0 \subset \widehat{\mathbb{C}} \setminus \bigcup_j D_j \quad \left(U_\infty \subset \widehat{\mathbb{C}} \setminus \bigcup_j D_j\right),$$

then 0 and ∞ are fixed points of $F_{\alpha,\theta}$ and the forward orbit of any $z \in U_0$ (respectively, the backward orbit of any $z \in U_\infty$) converges to 0 (respectively, ∞) under iteration of $F_{\alpha,\theta}$ (respectively, $F_{\alpha,\theta}^{-1}$).

Definition 3.2. We say that the spider web is *locally finite* in $\mathbb{C}$ if ∞ is the only accumulation point of sequences $\{z_n\} \in F_{\alpha,\theta}^{-n}(\partial\mathcal{D})$.

PROPOSITION 3.3. *If* $0 < |\alpha| < \epsilon$, *then* $\mathrm{Spid}(F_{\alpha,\theta})$ *is locally finite in* $\mathbb{C}$ *and* $F_{\alpha,\theta}^n(z)$ *converges to* 0 *when* $n \to \infty$ *for all* $z \in \mathbb{C}$.

Proof. Let V_∞ be a neighborhood of infinity of the form $\{|z| > M\}$, with $M \gg 1$. Since $|\alpha| < \epsilon$, then $F(D_j) \subset \mathbb{D}$ for each j, and $F_{\alpha,\theta}^{-1}\big(\bigcup_j D_j\big) \subset \widehat{\mathbb{C}} \setminus \overline{D(0, M)}$. Hence, it is possible to find a positive integer N such that $F_{\alpha,\theta}^{-n}\big(\bigcup_j \gamma_j\big) \subset V_\infty$ for all $n \geq N$.

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Since $F^{-1}_{\alpha,\theta|V_\infty} = \alpha^{-1}z$, we have that $\overline{\bigcup_{n\geq N} F^{-n}_{\alpha,\theta}\big(\bigcup_j \gamma_j\big)} \subset V_\infty$ and the only accumulation point of $\mathrm{Spid}(F_{\alpha,\theta})$ is ∞. That is, $\mathrm{Spid}(F_{\alpha,\theta})$ is locally finite in $\mathbb{C}$.

Now, the regular set consists of a countable union of disjoint simply connected regions $F^{-n}_{\alpha,\theta}(D_j)$ and the complement $\widehat{\mathbb{C}} \setminus \bigcup_{n\geq 0} \overline{F^{-n}_{\alpha,\theta}(D_j)}$. So, if U is any component of the regular set, then either U coincides with D_j (some j) after some iteration, or its images are disjoint with D_j. In both cases, $F^n_{\alpha,\theta}(U) \longrightarrow 0$ as n goes to ∞. □

PROPOSITION 3.4. *Let $(\alpha,\theta) \in \mathbb{D}\times\mathbb{T}^m$. If z_0, z_1 are any two points on the same connected component of $\mathcal{F}(F_{\alpha,\theta})$, then*

$$\lim_{n\to\infty} |F^n_{\alpha,\theta}(z_0) - F^n_{\alpha,\theta}(z_1)| = 0.$$

Proof. This follows by induction on n, since $|F'_{\alpha,\theta}| = |\alpha| < 1$. □

Now let $H(\widehat{\mathbb{C}})$ be the space which consists of all closed subsets of $\widehat{\mathbb{C}}$. In McMullen [**Mc**], the *Hausdorff topology* on $H(\widehat{\mathbb{C}})$ is characterized by saying that $S_k \longrightarrow S$ if:

(a) every neighborhood of a point $z \in S$ meets all but finitely many S_k; and

(b) if every neighborhood of z meets infinitely many S_k;

then $z \in S$.

So, given any arbitrary sequence of closed subsets $\{S_k\}$ of $\widehat{\mathbb{C}}$, it is possible to define $\liminf S_k$ as the largest set satisfying condition (a), and $\limsup S_k$ as the smallest set satisfying condition (b). Both $\liminf S_k$ and $\limsup S_k$ are closed and $\liminf S_k \subset \limsup S_k$. Then, $S_k \longrightarrow S$ if and only if $\liminf S_k = \limsup S_k$. It is also stated that $H(\widehat{\mathbb{C}})$ is a compact metric space in the Hausdorff topology (see also Nadler [**Nad**]).

Now define, for each $n \geq 1$, the sequence of functions $\beta_n : \mathbb{T}^m \longrightarrow H(\widehat{\mathbb{C}})$ given by

$$\beta_n(\theta) := \bigcup_{j=0}^{n} F^{-j}_{\alpha,\theta}(\partial\mathcal{D}),$$

and denote by Λ_n the value of β_n at θ, i.e. $\beta_n(\theta) := \Lambda_n$.

LEMMA 3.5. *For each $n \geq 1$, β_n is a uniformly continuous function.*

Proof. Fix any $n \geq 1$ and suppose that $\{\theta_k\}_{k\geq 1} \subset \mathbb{T}^m$ is a sequence of arguments which converges to $\theta_0 \in \mathbb{T}^m$. Denote by Λ^k_n the value of β_n at θ_k; i.e. $\beta_n(\theta_k) := \Lambda^k_n$. By definition of Λ^0_n, we have that $\Lambda^0_n = \liminf \Lambda^k_n \subset \limsup \Lambda^k_n$, so we only have to prove the other inclusion. If U is any neighborhood of $z \in \limsup \Lambda^k_n$, by convergence of θ_k to θ_0, there are infinitely many k such that θ_k is near to θ_0. This implies that, for infinitely many k there is an arc $A_k \in \Lambda^k_n$ such that

$$A_k \cap U \neq \emptyset.$$

In particular, $A_k \cap U \neq \emptyset$, except for finitely many k and therefore $\limsup \Lambda^k_n \subset \liminf \Lambda^k_n$. Hence, $\Lambda^k_n \longrightarrow \Lambda^0_n$ when $k \to \infty$ and we conclude that β is continuous. Since $\mathbb{T}^m$ is compact, $H(\widehat{\mathbb{C}})$ is a complete metric space and β is continuous, we conclude that β is uniformly continuous (see Dieudonné [**Die**, III.16]). □

LEMMA 3.6. *For each $\theta \in \mathbb{T}^m$, the sequence $\{\beta_n(\theta)\}$ converges to $\beta_\infty(\theta) : \mathbb{T}^m \longrightarrow H(\widehat{\mathbb{C}})$, given by $\beta_\infty(\theta) := \mathrm{Spid}(F_{\alpha,\theta})$.*

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Proof. If $\theta \in \mathbb{T}^m$ is fixed, then, by the definition of β_n, $\{\Lambda_n\}$ is an increasing sequence of closed subsets in $\widehat{\mathbb{C}}$. By the same argument as in the proof of the above lemma, we conclude that the sequence $\{\Lambda_n\}$ converges to $\mathrm{Spid}(F_{\alpha,\theta})$ in the Hausdorff topology of $H(\widehat{\mathbb{C}})$. □

THEOREM 3.7. *For every $\alpha \in \mathbb{D}$, the function $\theta \longrightarrow \mathrm{Spid}(F_{\alpha,\theta})$ is uniformly continuous from $\mathbb{T}^m$ to $H(\widehat{\mathbb{C}})$.*

Proof. By the above lemmas, $\{\beta_n\}$ is a sequence of uniformly continuous functions which converges pointwise to β_∞. Again (see Dieudonné [**Die**, VII.5]), $\{\beta_n\}$ converges uniformly to β_∞ and from here we conclude that β_∞ is continuous and, therefore, uniformly continuous. □

This theorem has, as a consequence, the following important theorem.

THEOREM 3.8. *If $(\alpha, \theta) \in \mathbb{D} \times \mathbb{T}^m$, then $\mathrm{Spid}(F_{\alpha,\theta})$ is locally finite in $\mathbb{C}$.*

Proof. Fix any $\alpha \in \mathbb{D}$. If $\theta = 0$, then $F_{\alpha,0}^{-1} \equiv \alpha^{-1}z$, and the spider web associated with $F_{\alpha,0}$ is a union of closed curves which go to ∞ under iteration of $F_{\alpha,0}^{-1}$. Hence, $\mathrm{Spid}(F_{\alpha,0})$ is locally finite in $\mathbb{C}$.

Let

$$A = \{\theta \in \mathbb{T}^m : \mathrm{Spid}(F_{\alpha,\theta}) \text{ is locally finite in } \mathbb{C}\}.$$

Clearly, $A \neq \emptyset$ since $0 \in A$. We now prove that A is a closed subset of $\mathbb{T}^m$. Let θ_0 be an accumulation point of A and suppose that the spider web associated with F_{α,θ_0} is not locally finite in $\mathbb{C}$. This means that there exists at least one point $z \neq \infty$ which is an accumulation point of $\mathrm{Spid}(F_{\alpha,\theta_0})$. That is, there exist an $N \gg 0$ and a sequence of closed arcs $\{a_n\} \subset \Lambda_n^0$ such that $z \in a_n$ for each $n \geq N$.

Let $\{\theta_k\} \subset A$ be a sequence of arguments such that $\theta_k \longrightarrow \theta_0$. Since $\theta_k \in A$, the corresponding spider web $\mathrm{Spid}(F_{\alpha,\theta_k})$ is locally finite in $\mathbb{C}$ for each k.

By continuity, the sequence $\mathrm{Spid}(F_{\alpha,\theta_k})$ converges to $\mathrm{Spid}(F_{\alpha,\theta_0})$. So, given any neighborhood U of z, for each $n \geq 1$, there exists a sequence of arcs $A_k \subset \Lambda_n^k$ such that

$$A_k \cap U \neq \emptyset,$$

for infinitely many k. Then, for each k arbitrarily large and for each $n \geq N$, we can choose an arc $A_{k,n} \subset \Lambda_n^k$ such that $A_{k,n} \cap U \neq \emptyset$. This means that z is also an accumulation point of $\mathrm{Spid}(F_{\alpha,\theta_k})$, which is impossible since $\mathrm{Spid}(F_{\alpha,\theta_k})$ is locally finite in $\mathbb{C}$. Hence, $\mathrm{Spid}(F_{\alpha,\theta_0})$ is locally finite in $\mathbb{C}$ and A is closed.

Again by continuity, A is open and therefore $A = \mathbb{T}^m$.

Finally, since $\alpha \in \mathbb{D}$ was arbitrary we have that the spider web is locally finite in $\mathbb{C}$ for every $(\alpha, \theta) \in \mathbb{D} \times \mathbb{T}^m$. □

We arrive at the Orbit Classification Theorem.

THEOREM 3.9. *Let $(\alpha, \theta) \in \mathbb{D} \times \mathbb{T}^m$. Then, for each $z \in \mathbb{C}$, there is a periodic cycle $\{z_0, \ldots, z_{N-1}\}$ such that $F_{\alpha,\theta}^n(z)$ converges to $\{z_j\}$ when $n \to \infty$.*

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Proof. Let U_0 be any bounded component of the regular set and define

$$U_n = F^n_{\alpha,\theta}(U_0), \quad n \geq 1.$$

Then, $\operatorname{diam}(U_n) \to 0$ and $|(F^n_{\alpha,\theta})'| = |\alpha|^n \to 0$ when $n \to \infty$.

If $U_n \cap U_m = \emptyset$, for any $n \neq m$, then $\{U_n\}$ has an accumulation point in $\operatorname{Spid}(F_{\alpha,\theta}) \cap (\widehat{\mathbb{C}} \setminus V_\infty)$, where V_∞ is a connected component of ∞ different from the U_n. The locally finiteness of $\operatorname{Spid}(F_{\alpha,\theta})$ implies that this is not possible. Hence, $U_n \cap U_m \neq \emptyset$, for some $n \neq m$.

If necessary, after renaming the components, we have that there exists an N such that $U_N \subset U_n$. Then, if $k \geq N$, by Proposition 3.4, $\{F_k = F^k_{\alpha,\theta|U_N}\}$ is a normal family and the only limit functions are constants. They correspond to periodic points of $F_{\alpha,\theta}$. □

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