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A Two-Grid Finite Element Method for Time-Dependent Incompressible Navier-Stokes Equations with Non-Smooth Initial Data

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Abstract. We analyze here, a two-grid finite element method for the two dimensional time-dependent incompressible Navier-Stokes equations with non-smooth initial data. It involves solving the non-linear Navier-Stokes problem on a coarse grid of size H and solving a Stokes problem on a fine grid of size h, $h << H$. This method gives optimal convergence for velocity in H^1-norm and for pressure in L^2-norm. The analysis mainly focuses on the loss of regularity of the solution at $t = 0$ of the Navier-Stokes equations.

AMS subject classifications: 65M60; 65M12; 65M15

Key words: Two-grid, finite element, Navier-Stokes equations, non-smooth initial data, error estimates.

1. Introduction

We consider a two-grid semi-discrete finite element approximation to the two dimensional time-dependent incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(x,t), \quad x \in \Omega,\ t > 0 \tag{1.1}$$

with incompressibility condition

$$\nabla \cdot \mathbf{u} = 0, \quad x \in \Omega,\ t > 0, \tag{1.2}$$

and initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \ \text{ in } \Omega, \quad \mathbf{u} = 0, \ \text{ on } \partial\Omega,\ t \geq 0. \tag{1.3}$$

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Here, Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\nu > 0$ is the viscosity. $\mathbf{u}$ and p are the velocity field and pressure, respectively. And $\mathbf{f}$ is a given force field.

Two-grid methods are well-established and efficient methods for solving non-linear partial differential equations. Due to high computational cost of solving a non-linear problem on a fine grid, we solve the original problem on a coarse grid and update the solution by solving a linearized problem on a fine grid. In other words, in the first step, we discretize the non-linear PDE on a coarse mesh, of mesh-size H and compute an approximate solution, say, $\mathbf{u}_H$. Then, in the second step, we solve a linear problem on a fine mesh, of mesh-size $h,\ h \ll H$, thereby, compute an approximate solution, say, $\mathbf{u}^h$. With appropriate h, H, we obtain comparable solution to that of Galerkin approximation on the fine grid, although, in this case, with far less computational cost, since, instead of solving a large non-linear system, we solve a small non-linear system and a large linear system.

In this article, we study the following two-grid finite element approximation for the problem (1.1)-(1.3): Compute a semi-discrete Galerkin finite element approximations $(\mathbf{u}_H, p_H)$, over a coarse mesh of mesh-size H. Then, refine this rough approximation $\mathbf{u}_H$ by solving the following Stokes problem:

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}_H \cdot \nabla\mathbf{u}_H + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega \tag{1.4}$$

over a fine mesh of mesh-size $h \ll H$.

The above algorithm is nothing new and in fact, this and similar algorithms have been studied on numerous occasions, see [1,2], [5]- [7], [9]- [11], [13]- [15] and [18]- [25]. But to the best of our knowledge, no study has been done for non-smooth initial data. The importance of this, in the context of Navier-Stokes equations, has been pointed out by Heywood and Rannacher in their paper [16]. It has been noted that higher-order regularity estimates remain valid (as $t \to 0$), only in the presence of non-local compatibility conditions of various order, for the given data, at time $t = 0$. Since these conditions are not natural and are difficult to verify, we have to allow the possibility that regularity estimates are no longer bounded as $t \to 0$. These singularities are more prominent in case of non-smooth initial data. Since error analysis is based on various regularity assumptions, this is of vital importance. Hence the task at hand is to carry out the necessary analysis under realistically assumed data.

Two-grid method was first introduced by Xu [29, 30] for semi-linear elliptic problems and by Layton *et.al* [18–20] for steady Navier-Stokes equations. It was carried out for time dependent Navier-Stokes by Girault and Lions [11] for semi-discrete case. The method may vary depending on the algorithm, by formulating different linearized problem, to solve in the second step; as in the case of Navier-Stokes, one can chose a Stokes problem or an Oseen problem or a Newton step to solve on the fine mesh. Several works in this direction, involving both semi-discrete and fully discrete analysis, can be found in [1, 2, 5, 6, 9, 13–15, 21–23, 25] and references therein.

The two-grid methods are similar to non-linear Galerkin methods, post-processed and dynamical post-processed methods, in the sense that, all these method try to control the computational cost or efficiency by controlling the non-linear term, implement-

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ing similar ideas: solve the non-linear problem on a coarse grid and use that solution to solve a linearized problem on a fine grid. A discussion on this can be found out in [9].

But in all these methods, including two-grid, very few articles can be found which have carried out the analysis, taking into account the loss of regularity of the solution of the Navier-Stokes equations at the initial time. Recently, two articles [9,10] have taken this into account. For example, both these articles assume no more than second-order spatial derivative of the velocity bounded in $\mathbf{L}^2$, up to initial time $t = 0$. Under realistic assumptions on the given data, the solution of the Navier-Stokes equations suffers from lack of regularity at $t = 0$. For us to assume regularity at $t = 0$, we need that the data satisfy some non-local compatibility conditions at $t = 0$ (see [16]). These conditions are very hard to verify even for simple models and hardly appear in physical context. We have avoided these conditions in our analysis and more. We have worked here with non-smooth initial data, that is, initial velocity $\mathbf{u}_0$ belong to $\mathbf{H}_0^1$. Now, even the second-order spatial derivative of the velocity is not bounded in $\mathbf{L}^2$, up to initial time $t = 0$. Lemma 2.1 tells us that $\|\mathbf{u}(t)\|_2 \sim O(t^{-1/2})$. So, our assumption on the initial data forces us to have only the first-order spatial derivative of the velocity bounded in $\mathbf{L}^2$ at $t = 0$. This proves to be a bottle-neck in our error analysis.

Our work here closely resembles the works of [1,9,13]. In [1], Abboud *et al.* have studied a second-order two-grid fully discrete scheme, whereas, in [9], Frutos *et al.* have considered two-grid mixed finite elements for spatial discretization. And in [13], Y. He has considered the fully discrete two-level finite element Galerkin method. We present here similar results that have been obtained in [13], but with weaker condition on $\mathbf{u}_0$. In that sense, this work is devoted to the technicalities required to deal with the non-smooth initial data.

Apart from the error analysis of this two-grid method, we have also discussed the error analysis of finite element Galerkin method for (1.1)-(1.3). Although, these later results are discussed in [16], the results here vary slightly, due to non-smooth initial data. Also, the pressure estimate has been improved for conforming finite element, in the sense that, it now reads as $\sim O(t^{-1/2})$ rather than $\sim O(t^{-1})$. For smooth initial data, it can be shown to have no singularity at $t = 0$ (using similar argument presented here, e.g., using Lemma 4.1 for pressure error), rather that the estimate $\sim O(t^{-1/2})$ in [16]. Similar improvement is also realized in the two-grid method. Theorem 5.3 in [13] states that the error in velocity is $\sim O(H^2)$ and the error in pressure is $\sim O(t^{-1/2}H^2)$, when the initial data is smooth. Note that whereas the velocity error remains bounded at $t = 0$, it is not so for the pressure error. In our case, we have shown that velocity and pressure errors are $\sim O(t^{-1}H^2)$ (see Lemmas 6.3 and 6.4), when initial data is non-smooth. By following similar procedure, for smooth initial data, we can show that the errors, in fact, are $\sim O(H^2)$. This improves the pressure error presented in [13]. We would like to mention here that this improvement of error bound for the pressure has already been observed in [7, Corollary 4.19, Remark 4.3] for higher-order Taylor-Hood elements, but for smooth initial data. Similar observations for first-order mixed finite elements (more precisely, for the mini-element) can also be found in [7, Remark 4.2] and [8, (24),(25)]; that the errors for both velocity and pressure are bounded at $t = 0$

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for smooth initial data.

Finally, we observe that, it is of practical importance to discretize in time and to implement the fully discretize scheme to verify the usefulness of the present method. In fact, computational results are shown in [15] to support the fact that this two-grid method is better that the standard Galerkin finite element method. But the analysis presented there, for a fully discrete two-grid method is for smooth initial data, that is, the initial velocity $\mathbf{u}_0$ is in $\mathbf{H}_0^1 \cap \mathbf{H}^2$. We will deal with non-smooth initial data for a fully discrete two-grid method somewhere else.

In the following sections, we will assume C and K to be generic positive constants, where K generally depends on the given data, that is, $\mathbf{u}_0, \mathbf{f}$. And for the sake of convenience, we would write $\mathbf{v}(t)$ as $\mathbf{v}$ when there arises no confusion.

The article is organized as follows. In Section 2, we introduce some notations and preliminaries, and present *a priori* estimates and regularity results for the solution of (1.1)-(1.3), when the initial data is non-smooth. Section 3 deals with finite element Galerkin approximations, whereas Section 4 deals with the error analysis of the Galerkin approximation. The two-grid method is briefly discussed in Section 5 and its error analysis is carried out in Section 6.

2. Preliminaries

For our subsequent use, we denote by bold face letters the $\mathbb{R}^2$-valued function space such as

$$\mathbf{H}_0^1 = [H_0^1(\Omega)]^2, \quad \mathbf{L}^2 = [L^2(\Omega)]^2.$$

Note that $\mathbf{H}_0^1$ is equipped with a norm

$$\|\nabla \mathbf{v}\| = \Big(\sum_{i,j=1}^{2} (\partial_j v_i, \partial_j v_i) \Big)^{1/2} = \Big(\sum_{i=1}^{2} (\nabla v_i, \nabla v_i) \Big)^{1/2}.$$

Further, we introduce some divergence free function spaces:

$$\mathbf{J}_1 = \Big\{ \boldsymbol{\phi} \in \mathbf{H}_0^1 : \nabla \cdot \boldsymbol{\phi} = 0 \Big\}$$
$$\mathbf{J} = \Big\{ \boldsymbol{\phi} \in \mathbf{L}^2 : \nabla \cdot \boldsymbol{\phi} = 0 \ \text{ in } \ \Omega, \boldsymbol{\phi} \cdot \mathbf{n}|_{\partial\Omega} = 0 \ \text{ holds weakly} \Big\},$$

where $\mathbf{n}$ is the outward normal to the boundary $\partial\Omega$ and $\boldsymbol{\phi} \cdot \mathbf{n}|_{\partial\Omega} = 0$ should be understood in the sense of trace in $\mathbf{H}^{-1/2}(\partial\Omega)$, see [26]. For any Banach space X, let $L^p(0,T;X)$ denote the space of measurable X -valued functions $\boldsymbol{\phi}$ on $(0,T)$ such that

$$\int_0^T \|\boldsymbol{\phi}(t)\|_X^p \, dt < \infty \quad \text{if } \ 1 \le p < \infty,$$

and for $p = \infty$

$$ess \sup_{0<t<T} \|\boldsymbol{\phi}(t)\|_X < \infty \quad \text{if } \ p = \infty.$$

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When there arises no confusion, we simply denote these spaces by $L^p(X)$.
Through out this paper, we make the following assumptions:
(A1). For $\mathbf{g} \in \mathbf{L}^2$, let the unique pair of solutions $\{\mathbf{v} \in \mathbf{J}_1, q \in L^2/\mathbb{R}\}$ for the steady state Stokes problem

$$\begin{aligned} &-\Delta \mathbf{v} + \nabla q = \mathbf{g}, \\ &\nabla \cdot \mathbf{v} = 0 \quad \text{in} \quad \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0, \end{aligned}$$

satisfy the following regularity result

$$\|\mathbf{v}\|_2 + \|q\|_{H^1/\mathbb{R}} \leq C\|\mathbf{g}\|.$$

(A2). The initial velocity $\mathbf{u}_0$ and the external force $\mathbf{f}$ satisfy for positive constant M_0, and for T with $0 < T \leq \infty$

$$\mathbf{u}_0 \in \mathbf{J}_1,\ \mathbf{f}, \mathbf{f}_t, \mathbf{f}_{tt} \in L^\infty(0,T;\mathbf{L}^2) \quad \text{with} \quad \|\mathbf{u}_0\|_1 \leq M_0, \quad \sup_{0<t<T} \big\{\|\mathbf{f}\|, \|\mathbf{f}_t\|, \|\mathbf{f}_{tt}\|\big\} \leq M_0.$$

With $P : \mathbf{L}^2 \to \mathbf{J}$ as orthogonal projection and $\tilde{\Delta} = P(-\Delta) : \mathbf{J}_1 \cap \mathbf{H}^2 \subset \mathbf{J} \to \mathbf{J}$ as Stokes operator, we first note that $(\mathbf{A1})$ implies

$$\|\mathbf{v}\|_2 \leq C\|\tilde{\Delta}\mathbf{v}\|, \quad \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2.$$

And (see [16, (2.4)])

$$\|\mathbf{v}\| \leq \lambda_1^{-1/2}\|\mathbf{v}\|_1 \leq \lambda_1^{-1}\|\mathbf{v}\|_2, \quad \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2, \tag{2.1}$$

where $\lambda_1 > 0$ to be the least eigenvalue of the Stokes operator.
With these assumptions, let us now introduce the weak formulation of (1.1)-(1.3). Find a pair of functions $\{\mathbf{u}(t), p(t)\},\ t > 0$, such that

$$\begin{aligned} &(\mathbf{u}_t, \boldsymbol{\phi}) + \nu(\nabla\mathbf{u}, \nabla\boldsymbol{\phi}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}) = (p, \nabla\cdot\boldsymbol{\phi}) + (\mathbf{f}, \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \mathbf{H}_0^1, \\ &(\nabla\cdot\mathbf{u}, \chi) = 0 \quad \forall \chi \in L^2. \end{aligned} \tag{2.2}$$

Equivalently, find $\mathbf{u}(t) \in \mathbf{J}_1,\ t > 0$ such that

$$(\mathbf{u}_t, \boldsymbol{\phi}) + \nu(\nabla\mathbf{u}, \nabla\boldsymbol{\phi}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}) = (\mathbf{f}, \boldsymbol{\phi}),\ \forall \boldsymbol{\phi} \in \mathbf{J}_1. \tag{2.3}$$

For existence and uniqueness, and the regularity of the solution of the problem (2.2) or (2.3), we refer to [16, Theorem 2.1], [12, Ch. V, Theorem 1.4, 1.5], [26, Ch. III, Theorem 3.1, 3.2] and [27, Part I., Theorem 3.1, Remark 3.1, Sec. 6.2].

Theorem 2.1. *Let* $(\mathbf{A1})$ *hold and let* $u_0 \in \mathbf{J}_1$ *and* $\mathbf{f} \in L^\infty(0,T;L^2),\ T > 0$. *Then, there exists a unique weak solution* $\mathbf{u}$ *of (2.3) such that*

$$\mathbf{u} \in L^\infty(0,T;\mathbf{J}) \cap L^2(0,T;\mathbf{J}_1).$$

In fact, the solution has further regularity.

$$\mathbf{u} \in L^\infty(0,T;\mathbf{J}_1) \cap L^2(0,T;\mathbf{J}_1 \cap \mathbf{H}^2).$$

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Remark 2.1. Since the two weak formulations (2.2) are (2.3) equivalent, Theorem 2.1 provides us with a unique solution $\mathbf{u}$ of (2.2) and we recover pressure p, unique up to a constant. In fact, $p \in L^2(0,T;H^1/\mathbb{R})$.

For the sake of completeness, we present below the *a priori* estimates and higher-order estimates, which will be used in our error analysis. These vary slightly with the results presented in [16] due to non-smooth initial data. And hence, we sketch a proof here.

Lemma 2.1. *Assume that* $(\mathbf{A1})$ *and* $(\mathbf{A2})$ *hold and let* $0 < \alpha < \nu\lambda_1$. *Then, for some constant* $K > 0$, *which depends only on the given data, the weak solution pair* $(\mathbf{u}, p)$ *of* (1.1)-(1.3) *satisfies the following estimates:*

$$\|\mathbf{u}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}(s)\|_1^2 ds \le K, \tag{2.4}$$

$$\|\mathbf{u}(t)\|_1^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\big\{\|\mathbf{u}(s)\|_2^2 + \|\mathbf{u}_s(s)\|^2\big\}\, ds \le K, \tag{2.5}$$

$$\tau^*(t)\big\{\|\mathbf{u}_t(t)\|^2 + \|\mathbf{u}(t)\|_2^2 + \|p(t)\|_1^2\big\} + e^{-2\alpha t}\int_0^t \sigma(s)\|\mathbf{u}_s(s)\|_1^2 ds \le K, \tag{2.6}$$

$$(\tau^*(t))^2\|\mathbf{u}_t(t)\|_1^2 + e^{-2\alpha t}\int_0^t \sigma_1(s)\big\{\|\mathbf{u}_s(s)\|_2^2 + \|\mathbf{u}_{ss}(s)\|^2 + \|p_s(s)\|_1^2\big\}\, ds \le K, \tag{2.7}$$

$$(\tau^*(t))^{3/2}\big\{\|\mathbf{u}_{tt}(t)\| + \|\mathbf{u}_t(t)\|_2 + \|p_t(t)\|_1\big\} \le K, \tag{2.8}$$

where $\tau^*(t) = \min\{1,t\}$, $\sigma(t) = \tau^*(t)e^{2\alpha t}$ *and* $\sigma_1(t) = (\tau^*(t))^2 e^{2\alpha t}$.

Proof. We choose $\boldsymbol{\phi} = e^{2\alpha t}\mathbf{u} = e^{\alpha t}\hat{\mathbf{u}}$ in (2.3) and use (2.1) to find

$$\frac{1}{2}\frac{d}{dt}\|\hat{\mathbf{u}}\|^2 + \Big(\nu - \frac{\alpha}{\lambda_1}\Big)\|\hat{\mathbf{u}}\|_1^2 \le \frac{1}{\sqrt{\lambda_1}}\|\hat{\mathbf{f}}\|\|\hat{\mathbf{u}}\|_1. \tag{2.9}$$

Use kickback argument, then integrate and multiply by $e^{-2\alpha t}$ to obtain

$$\|\mathbf{u}\|^2 + \Big(\nu - \frac{\alpha}{\lambda_1}\Big)e^{-2\alpha t}\int_0^t \|\hat{\mathbf{u}}(s)\|_1^2 ds \le e^{-2\alpha t}\|\mathbf{u}_0\|^2 + \frac{(1-e^{-2\alpha t})}{2\alpha(\nu\lambda_1 - \alpha)}\|\mathbf{f}\|_\infty^2 =: K,$$

where $\|\mathbf{f}\|_\infty = \|\mathbf{f}\|_{L^\infty(\mathbf{L}^2)}$. Now choose $\boldsymbol{\phi} = \mathbf{u}$ in (2.3) and integrate from t to $t + t_0$ for some fix $t_0 > 0$.

$$\|\mathbf{u}(t+t_0)\|^2 + \int_t^{t+t_0}\|\mathbf{u}(s)\|_1^2 ds \le \|\mathbf{u}(t)\|^2 + \frac{\lambda_1 t_0}{\nu}\|\mathbf{f}\|_\infty^2 \le K(t_0). \tag{2.10}$$

Next, we choose $\boldsymbol{\phi} = \tilde{\Delta}\mathbf{u}$ in (2.3) to find (see [16, (2.11)])

$$\frac{d}{dt}\|\mathbf{u}\|_1^2 + \nu\|\tilde{\Delta}\mathbf{u}\|^2 \le C\|\mathbf{f}\|^2 + C\|\mathbf{u}\|^2\|\mathbf{u}\|_1^4. \tag{2.11}$$

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Keeping in mind (2.10), we apply uniform Gronwall lemma (see [28, Lemma 1.1, pp. 91]) to observe that for fixed $t_0 > 0$

$$\|\mathbf{u}(t+t_0)\|_1 \leq K, \;\; t > 0.$$

Multiply (2.11) by $e^{2\alpha t}$, integrate with respect to time, use the above estimate to conclude the first part of (2.5). Choose $\boldsymbol{\phi} = e^{2\alpha t}\mathbf{u}_t$ for the rest of (2.5).
For next estimate, we differentiate (2.3) with respect to time and put $\boldsymbol{\phi} = \sigma(t)\mathbf{u}_t$ to get

$$\frac{d}{dt}\{\sigma(t)\|\mathbf{u}_t\|^2\} + 2\nu\sigma(t)\|\mathbf{u}_t\|_1^2 = \sigma_t(t)\|\mathbf{u}_t\|^2 - \sigma(t)(\mathbf{u}_t\cdot\nabla\mathbf{u},\mathbf{u}_t) + \sigma(t)(\mathbf{f}_t,\mathbf{u}_t).$$

Observe that

$$(\mathbf{u}_t\cdot\nabla\mathbf{u},\mathbf{u}_t) \leq \|\mathbf{u}_t\|^2_{\mathbf{L}^4(\Omega)}\|\mathbf{u}\|_1 \leq C\|\mathbf{u}_t\|\|\mathbf{u}_t\|_1\|\mathbf{u}\|_1.$$

And therefore, we have

$$\frac{d}{dt}\{\sigma(t)\|\mathbf{u}_t\|^2\} + \nu\sigma(t)\|\mathbf{u}_t\|_1^2 \leq Ce^{2\alpha t}\|\mathbf{u}_t\|^2(1+\|\mathbf{u}\|_1^2) + \sigma(t)\|\mathbf{f}_t\|^2.$$

Integrate with respect to time and use (2.5) to conclude part of (2.6). The rest of it can be proved by using the Eqs. (2.3) and (2.2).
Put $\boldsymbol{\phi} = \sigma_1(t)\tilde{\Delta}\mathbf{u}_t$ and $\boldsymbol{\phi} = \sigma_1(t)\mathbf{u}_{tt}$, respectively, after differentiating (2.3) and proceed in similar fashion as above to find ($\sigma_1(t) = (\tau^*(t))^2e^{2\alpha t}$)

$$(\tau^*(t))^2\|\mathbf{u}_t(t)\|_1^2 + e^{-2\alpha t}\int_0^t \sigma_1(s)\{\|\mathbf{u}_s(s)\|_2^2 + \|\mathbf{u}_{ss}(s)\|^2\}\,ds \leq K. \tag{2.12}$$

Differentiate the Eq. (2.2) and use (2.12) to obtain the pressure estimate of (2.7).
Finally, we differentiate (2.3) twice, with respect to time. First, put $\boldsymbol{\phi} = \sigma_2(t)\mathbf{u}_{tt}$, $\sigma_2(t) = (\tau^*(t))^3e^{2\alpha t}$ and use (2.7) to obtain

$$(\tau^*(t))^3\|\mathbf{u}_{tt}(t)\|_1^2 + e^{-2\alpha t}\int_0^t \sigma_2(s)\|\mathbf{u}_{ss}(s)\|_1^2 ds \leq K.$$

And then use the double differentiated equation and the Eq. (2.2) (after double differentiation) with the above obtained estimate to conclude (2.8) and this completes the rest of the proof. □

We present below Gronwall's Lemma, which will be used subsequently.

Lemma 2.2 (Gronwall's Lemma). *Let g, h, y be three locally integrable non-negative functions on the time interval $[0,\infty)$ such that for all $t \geq 0$*

$$y(t) + G(t) \leq C + \int_0^t h(s)\,ds + \int_0^t g(s)y(s)\,ds,$$

where $G(t)$ is a non-negative function on $[0,\infty)$ and $C \geq 0$ is a constant. Then,

$$y(t) + G(t) \leq \Big(C + \int_0^t h(s)\,ds\Big)exp\Big(\int_0^t g(s)\,ds\Big).$$

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3. Classical Galerkin method

From now on, we denote h with $0 < h < 1$ to be a real positive discretization parameter tending to zero. Let $\mathbf{H}_h$ and L_h, $0 < h < 1$ be two conforming finite element spaces; subspaces of $\mathbf{H}_0^1$ and $L^2/\mathbb{R}$, respectively, approximating velocity vector and the pressure. Assume that the following approximation properties are satisfied for the spaces $\mathbf{H}_h$ and L_h:
(**B1**) For each $\mathbf{w} \in \mathbf{H}_0^1 \cap \mathbf{H}^2$ and $q \in H^1/\mathbb{R}$ there exist approximations $i_h w \in \mathbf{H}_h$ and $j_h q \in L_h$ such that

$$\|\mathbf{w} - i_h\mathbf{w}\| + h\|\nabla(\mathbf{w} - i_h\mathbf{w})\| \leq Ch^2\|\mathbf{w}\|_2, \quad \|q - j_h q\| \leq Ch\|q\|_1.$$

Further, suppose that the following inverse hypothesis holds for $\mathbf{w}_h \in \mathbf{H}_h$:

$$\|\nabla \mathbf{w}_h\| \leq Ch^{-1}\|\mathbf{w}_h\|. \tag{3.1}$$

To define the Galerkin approximations, we set for $\mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_0^1$,

$$a(\mathbf{v}, \boldsymbol{\phi}) = (\nabla \mathbf{v}, \nabla \boldsymbol{\phi}),$$
$$b(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}) = \frac{1}{2}(\mathbf{v} \cdot \nabla \mathbf{w}, \boldsymbol{\phi}) - \frac{1}{2}(\mathbf{v} \cdot \nabla \boldsymbol{\phi}, \mathbf{w}).$$

Note that the operator $b(\cdot, \cdot, \cdot)$ preserves the antisymmetric property of the original nonlinear term, that is,

$$b(\mathbf{v}_h, \mathbf{w}_h, \mathbf{w}_h) = 0 \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{H}_h.$$

The discrete analogue of the weak formulation (2.2) now reads as: Find $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$\begin{aligned} &(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - (p_h, \nabla \cdot \boldsymbol{\phi}_h) = (\mathbf{f}, \boldsymbol{\phi}_h), \\ &(\nabla \cdot \mathbf{u}_h, \chi_h) = 0, \end{aligned} \tag{3.2}$$

for $\boldsymbol{\phi}_h \in \mathbf{H}_h$, $\chi_h \in L_h$. Here $\mathbf{u}_{0h} \in \mathbf{H}_h$ is a suitable approximation of $\mathbf{u}_0 \in \mathbf{J}_1$.
In order to consider a discrete space analogous to $\mathbf{J}_1$, we impose the discrete incompressibility condition on $\mathbf{H}_h$ and call it as $\mathbf{J}_h$. Thus, we define $\mathbf{J}_h$, as

$$\mathbf{J}_h = \Big\{ v_h \in \mathbf{H}_h : (\chi_h, \nabla \cdot v_h) = 0 \quad \forall \chi_h \in L_h \Big\}.$$

Note that $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$. With $\mathbf{J}_h$ as above, we now introduce an equivalent Galerkin formulation as: Find $\mathbf{u}_h(t) \in \mathbf{J}_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h, \boldsymbol{\phi}_h) = -b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) + (\mathbf{f}, \boldsymbol{\phi}_h) \ \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{3.3}$$

Since $\mathbf{J}_h$ is finite dimensional, the problem (3.3) leads to a system of nonlinear ordinary differential equations. For global existence of a unique solution of (3.3) (or unique solution pair of (3.2)), we again refer to [16, pp. 289].

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For continuous dependence of the discrete pressure $p_h(t) \in L_h$ on the discrete velocity $u_h(t) \in \mathbf{J}_h$, we assume the following discrete inf-sup (LBB) condition on the finite dimensional spaces $\mathbf{H}_h$ and L_h:

($\mathbf{B2}'$) For every $q_h \in L_h$, there exists a non-trivial function $\boldsymbol{\phi}_h \in \mathbf{H}_h$ and a positive constant K_0, independent of h, such that

$$|(q_h, \nabla \cdot \boldsymbol{\phi}_h)| \geq C\|\nabla \boldsymbol{\phi}_h\|\|q_h\|_{L^2/\mathbb{R}}.$$

Moreover, we also assume that the following approximation property holds true for $\mathbf{J}_h$.

($\mathbf{B2}$) For every $\mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2$, there exists an approximation $r_h\mathbf{w} \in \mathbf{J_h}$ such that

$$\|\mathbf{w} - r_h\mathbf{w}\| + h\|\nabla(\mathbf{w} - r_h\mathbf{w})\| \leq Ch^2\|\mathbf{w}\|_2.$$

The L^2 projection $P_h : \mathbf{L}^2 \mapsto \mathbf{J}_h$ satisfies the following properties (see [16, Lemma 4.3]): for $\boldsymbol{\phi} \in \mathbf{J}_h$,

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla P_h\boldsymbol{\phi}\| \leq Ch\|\nabla\boldsymbol{\phi}\|, \tag{3.4}$$

and for $\boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2$,

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla(\boldsymbol{\phi} - P_h\boldsymbol{\phi})\| \leq Ch^2\|\tilde{\Delta}\boldsymbol{\phi}\|. \tag{3.5}$$

We now define the discrete operator $\Delta_h : \mathbf{H}_h \mapsto \mathbf{H}_h$ through the bilinear form $a(\cdot,\cdot)$ as

$$a(\mathbf{v}_h, \boldsymbol{\phi}_h) = (-\Delta_h\mathbf{v}_h, \boldsymbol{\phi}) \quad \forall \mathbf{v}_h, \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{3.6}$$

Set the discrete analogue of the Stokes operator $\tilde{\Delta} = P(-\Delta)$ as $\tilde{\Delta}_h = P_h(-\Delta_h)$. Note that the $\tilde{\Delta}_h$ restricted to $\mathbf{J}_h$ is invertible and we denote its inverse by $(\tilde{\Delta}_h)^{-1}$; for details, see [16, 17]. Following [17], we define *discrete* Sobolev norm as:

$$\|\mathbf{v}_h\|_r = \|(-\tilde{\Delta}_h)^{r/2}\mathbf{v}_h\|, \quad \text{for} \quad \mathbf{v}_h \in \mathbf{J}_h,\ r \in \mathbb{R}.$$

Using Sobolev imbedding and Sobolev inequality, it is easy to prove the following Lemma (similar to [17, (3.4)]).

Lemma 3.1. *Suppose conditions (**A1**), (**B1**) and (**B2**) are satisfied. Then there exists a positive constant K such that for $\mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_h$, the following holds:*

$$|(\mathbf{v}\cdot\nabla\mathbf{w}, \boldsymbol{\phi})| \leq K \begin{cases} \|\mathbf{v}\|^{1/2}\|\nabla\mathbf{v}\|^{1/2}\|\nabla\mathbf{w}\|^{1/2}\|\Delta_h\mathbf{w}\|^{1/2}\|\boldsymbol{\phi}\|, \\ \|\mathbf{v}\|^{1/2}\|\Delta_h\mathbf{v}\|^{1/2}\|\nabla\mathbf{w}\|\|\boldsymbol{\phi}\|, \\ \|\mathbf{v}\|^{1/2}\|\nabla\mathbf{v}\|^{1/2}\|\nabla\mathbf{w}\|\|\boldsymbol{\phi}\|^{1/2}\|\nabla\boldsymbol{\phi}\|^{1/2}, \\ \|\mathbf{v}\|\|\nabla\mathbf{w}\|\|\boldsymbol{\phi}\|^{1/2}\|\Delta_h\boldsymbol{\phi}\|^{1/2}, \\ \|\mathbf{v}\|\|\nabla\mathbf{w}\|^{1/2}\|\Delta_h\mathbf{w}\|^{1/2}\|\boldsymbol{\phi}\|^{1/2}\|\nabla\boldsymbol{\phi}\|^{1/2} \end{cases} \tag{3.7}$$

Examples of subspaces $\mathbf{H}_h$ and L_h satisfying assumptions ($\mathbf{B1}$) and ($\mathbf{B2}'$) are abundant in literature, for example, see [3, 4, 12]. We present below a couple of lemmas, one dealing with *a priori* and regularity estimates of $\mathbf{u}_h$ and the other, with higher-order regularity results. The proof is similar to that of Lemma 2.1.

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Lemma 3.2. *Under the assumptions of Lemmas* 2.1 *and* 3.1*, the semi-discrete Galerkin approximation* $\mathbf{u}_h$ *of the velocity* $\mathbf{u}$ *satisfies, for* $t > 0$,

$$\|\mathbf{u}_h(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_1^2\, ds \le K, \tag{3.8}$$

$$\|\mathbf{u}_h(t)\|_1^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\big\{\|\mathbf{u}_h(s)\|_2^2 + \|\mathbf{u}_{h,s}(s)\|^2\big\}\, ds \le K, \tag{3.9}$$

$$(\tau^*(t))^{1/2}\big\{\|\mathbf{u}_h(t)\|_2 + \|p\|_{H^1/\mathbb{R}}\big\} \le K, \tag{3.10}$$

where $\tau^*(t) = \min\{1, t\}$ *and* K *depends only on the given data. In particular,* K *is independent of* h *and* t.

Lemma 3.3. *Under the assumptions of Lemma* 3.2*, the semi-discrete Galerkin approximation* $\mathbf{u}_h$ *of the velocity* $\mathbf{u}$ *satisfies, for* $t > 0$ *and for* $r \in \{0, 1\},\ i \in \{1, 2\},\ r + i \le 2$*,*

$$(\tau^*(t))^{r+2i-1}\|D_t^i\mathbf{u}_h(t)\|_r^2 + e^{-2\alpha t}\int_0^t (\tau^*(s))^{r+2i-1}e^{2\alpha s}\|D_s^i\mathbf{u}_h(s)\|_{r+1}^2\, ds \le K, \tag{3.11}$$

where $D_t^i = \frac{\partial^i}{\partial t^i}$. *And*

$$e^{-2\alpha t}\int_0^t (\tau^*(s))^2 e^{2\alpha s}\{\|\mathbf{u}_{h,ss}(s)\|^2 + \|p_{h,s}(s)\|_{H^1/\mathbb{R}}^2\}\, ds \le K, \tag{3.12}$$

$$(\tau^*(t))^{3/2}\big\{\|\mathbf{u}_{ht}\|_2 + \|p_{ht}(t)\|_1\big\} \le K. \tag{3.13}$$

Here K *depends only on the given data. In particular,* K *is independent of* h *and* t.

4. Error analysis: Galerkin method

In this section, we briefly present a convergence analysis for Galerkin approximation. The analysis for smooth initial data, that is, $\mathbf{u}_0 \in \mathbf{J}_1 \cap \mathbf{H}^2$ can be found in [16]. The analysis below is for non-smooth initial data, that is, $\mathbf{u}_0 \in \mathbf{J}_1$. Although the proofs are along the similar lines as those for smooth data (see [16]), they are more involved due to the presence of singularities at $t = 0$.

Let us denote the finite element Galerkin approximation error as $\mathbf{E} = \mathbf{u} - \mathbf{u}_h$. We split the error in two parts, one due to non-linearity and the other due to linearization;

$$\mathbf{E} = (\mathbf{u} - \mathbf{v}_h) + (\mathbf{v}_h - \mathbf{u}_h) =: \boldsymbol{\xi} + \boldsymbol{\eta}.$$

Here $\mathbf{v}_h$ satisfies the linearized Eq. (5.3) from [16]. And then we have (see [16, Lemma 5.1]; same proof will go through)

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|^2 ds \le Kh^4. \tag{4.1}$$

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For $L^\infty(\mathbf{L}^2)$ estimate of $\boldsymbol{\xi}$, we now split $\boldsymbol{\xi}$ as follows:

$$\boldsymbol{\xi} = (\mathbf{u} - \mathbf{v}_h) = (\mathbf{u} - S_h\mathbf{u}) + (S_h\mathbf{u} - \mathbf{v}_h) =: \boldsymbol{\zeta} + \boldsymbol{\theta},$$

where S_h is given by (4.52), [16]. Lemma 4.7, [16] tells us that

$$\|\boldsymbol{\zeta}\| + h\|\boldsymbol{\zeta}\|_1 \le Ch^2\Big((1 + \|\mathbf{u}\|_1)\|\mathbf{u}\|_2 + \|p\|_1\Big), \tag{4.2}$$

$$\|\boldsymbol{\zeta}_t\| + h\|\boldsymbol{\zeta}_t\|_1 \le Ch^2\Big((1 + \|\mathbf{u}\|_1)\|\mathbf{u}_t\|_2 + \|p_t\|_1\Big). \tag{4.3}$$

We note here that the above estimates of $\boldsymbol{\zeta}$ are not quite the same as in [16], since $\|\mathbf{u}(t)\|_2$ is no longer bounded near $t = 0$; rather it behaves like $t^{-1/2}$, due to non-smooth initial data.

In order to complete the estimate for $\boldsymbol{\xi}$, we now need to estimate $\boldsymbol{\theta}$. The equation in $\boldsymbol{\theta}$ reads as

$$(\boldsymbol{\theta}_t, \boldsymbol{\phi}_h) + \nu a(\boldsymbol{\theta}, \boldsymbol{\phi}_h) = -(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h), \quad \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{4.4}$$

Simply putting $\boldsymbol{\phi}_h = \boldsymbol{\theta}$ would not work, since after integrating, we need to estimate $\int_0^t \|\boldsymbol{\zeta}_s(s)\|^2 ds$. Using (4.3), we find a term $\int_0^t \|\mathbf{u}_s(s)\|_2^2 ds$, which does not make sense, again due to singularity at $t = 0$. In order to take care of this, we introduce a weight $\sigma(t) = \tau^*(t)e^{2\alpha t}$ as in Lemma 2.1.

We put $\boldsymbol{\phi}_h = \sigma(t)\boldsymbol{\theta}$ in (4.4) to find

$$\frac{d}{dt}\{\sigma(t)\|\boldsymbol{\theta}\|^2\} - \sigma_t(t)\|\boldsymbol{\theta}\|^2 + 2\nu\sigma(t)\|\boldsymbol{\theta}\|_1^2 = -2\sigma(t)(\boldsymbol{\zeta}_t, \boldsymbol{\theta}).$$

Using Cauchy-Schwarz and Young's inequality, we obtain

$$\begin{aligned}&\frac{d}{dt}\{\sigma(t)\|\boldsymbol{\theta}\|^2\} + 2\nu\sigma(t)\|\boldsymbol{\theta}\|_1^2\\ &\le Ce^{2\alpha t}\|\boldsymbol{\theta}\|^2 + \sigma_1(t)\|\boldsymbol{\zeta}_t\|^2 \le Ce^{2\alpha t}\Big(\|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\zeta}\|^2\Big) + \sigma_1(t)\|\boldsymbol{\zeta}_t\|^2.\end{aligned}$$

Integrate with respect to time and use (4.1), (4.2) and (4.3).

$$\begin{aligned}&\sigma(t)\|\boldsymbol{\theta}(t)\|^2 + \nu\int_0^t \sigma(s)\|\boldsymbol{\theta}(s)\|_1^2 ds\\ &\le Ke^{2\alpha t}h^4 + Ch^4\int_0^t e^{2\alpha s}\Big(\|\mathbf{u}(s)\|_2^2 + \|p(s)\|_1^2\Big)\, ds\\ &\qquad + Ch^4\int_0^t \sigma_1(s)\Big(\|\mathbf{u}_s(s)\|_2^2 + \|p_s(s)\|_1^2\Big)\, ds.\end{aligned}$$

Use (2.5) and (2.7) and then multiply by $e^{-2\alpha t}$ to get

$$\tau^*(t)\|\boldsymbol{\theta}(t)\|^2 + e^{-2\alpha t}\int_0^t \sigma(s)\|\boldsymbol{\theta}(s)\|_1^2 ds \le Kh^4.$$

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An use of triangle inequality along with inverse hypothesis (3.1) results in

$$\|\boldsymbol{\xi}(t)\| + h\|\boldsymbol{\xi}(t)\|_1 \leq Kt^{-1/2}h^2. \tag{4.5}$$

We pause here and note again that $\|\boldsymbol{\xi}\|$ is no longer bounded for all $t > 0$ (as in [16]). It now remains to estimate $\boldsymbol{\eta}$. The equation in $\boldsymbol{\eta}$ is

$$(\boldsymbol{\eta}_t, \boldsymbol{\phi}_h) + \nu a(\boldsymbol{\eta}, \boldsymbol{\phi}_h) = \Lambda_{1,h}(\boldsymbol{\phi}_h), \quad \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{4.6}$$

where

$$\Lambda_{1,h}(\boldsymbol{\phi}_h) = b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) = -b(\mathbf{u}_h, \mathbf{E}, \boldsymbol{\phi}_h) - b(\mathbf{E}, \mathbf{u}, \boldsymbol{\phi}_h). \tag{4.7}$$

Since $\boldsymbol{\eta}$ satisfies a non-linear equations, the complexity due to the singularity at $t = 0$ is much more telling. We can not expect to handle it just by introducing a weight function. In our case, we need to handle the non-linear term $\Lambda_{1,h}$. The analysis says that we need an additional estimate of $\boldsymbol{\eta}$ in $L^2(\mathbf{L}^2)$-norm, before we proceed for $L^\infty(\mathbf{L}^2)$ estimate of $\boldsymbol{\eta}$. For that, choose $\boldsymbol{\phi}_h = e^{2\alpha t}(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}$ in (4.6) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\hat{\boldsymbol{\eta}}\|_{-1}^2 - \alpha\|\hat{\boldsymbol{\eta}}\|_{-1}^2 + \nu\|\hat{\boldsymbol{\eta}}\|^2 = e^{2\alpha t}\Lambda_{1,h}((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}). \tag{4.8}$$

Here, $\hat{\boldsymbol{\eta}} = e^{\alpha t}\boldsymbol{\eta}$. Using (4.7) and the definition of $b(\cdot,\cdot,\cdot)$, we observe that

$$\begin{aligned}
&\Lambda_{1,h}((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}) \\
&= -b(\mathbf{u}_h, \boldsymbol{\xi}+\boldsymbol{\eta}, (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}) - b(\boldsymbol{\xi}+\boldsymbol{\eta}, \mathbf{u}, (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}) \\
&= -\frac{1}{2}((\mathbf{u}_h\cdot\nabla)(\boldsymbol{\xi}+\boldsymbol{\eta}), (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}) - \frac{1}{2}((\mathbf{u}_h\cdot\nabla)(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}, \boldsymbol{\xi}+\boldsymbol{\eta}) \\
&\quad - \frac{1}{2}\Big(((\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\nabla)\mathbf{u}, (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}\Big) - \frac{1}{2}\Big(((\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\nabla)(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}, \mathbf{u}\Big).
\end{aligned} \tag{4.9}$$

Use Lemma 3.1 to arrive at the following:

$$-\frac{1}{2}((\mathbf{u}_h\cdot\nabla)(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}, \boldsymbol{\xi}+\boldsymbol{\eta}) \leq C\|\mathbf{u}_h\|^{1/2}\|\mathbf{u}_h\|_1^{1/2}\|\boldsymbol{\eta}\|_{-1}^{1/2}\|\boldsymbol{\eta}\|^{1/2}\big(\|\boldsymbol{\xi}\| + \|\boldsymbol{\eta}\|\big), \tag{4.10}$$

$$-\frac{1}{2}\Big(((\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\nabla)\mathbf{u}, (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}\Big) \leq C\big(\|\boldsymbol{\xi}\| + \|\boldsymbol{\eta}\|\big)\|\mathbf{u}\|_1\|\boldsymbol{\eta}\|_{-2}^{1/2}\|\boldsymbol{\eta}\|^{1/2}, \tag{4.11}$$

$$-\frac{1}{2}\Big(((\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\nabla)(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}, \mathbf{u}\Big) \leq C\big(\|\boldsymbol{\xi}\| + \|\boldsymbol{\eta}\|\big)\|\boldsymbol{\eta}\|_{-1}^{1/2}\|\boldsymbol{\eta}\|^{1/2}\|\mathbf{u}\|^{1/2}\|\mathbf{u}\|_1^{1/2}. \tag{4.12}$$

For the first term on the right-hand side of (4.9), we have (with the notations $D_i = \frac{\partial}{\partial x_i}$

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and $\mathbf{v} = (v_1, v_2)$)

$$
\begin{aligned}
&((\mathbf{u}_h \cdot \nabla)(\boldsymbol{\xi} + \boldsymbol{\eta}), (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}) \qquad (4.13)\\
=&\sum_{i,j=1}^{2} \int_\Omega u_{h,i} D_i\{(\xi+\eta)_j\}((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta})_j \, d\mathbf{x}\\
=&-\sum_{i,j=1}^{2} \int_\Omega D_i u_{h,i}(\xi+\eta)_j((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta})_j \, d\mathbf{x} - \sum_{i,j=1}^{2} \int_\Omega u_{h,i} D_i\{((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta})_j\}(\xi+\eta)_j\\
=&-((\nabla \cdot \mathbf{u}_h)(\boldsymbol{\xi}+\boldsymbol{\eta}), (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}) - ((\mathbf{u}_h \cdot \nabla)(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}, (\boldsymbol{\xi}+\boldsymbol{\eta}))\\
\le& C\|\mathbf{u}\|_1\Big(\|\boldsymbol{\xi}\| + \|\boldsymbol{\eta}\|\Big)\|\boldsymbol{\eta}\|_{-2}^{1/2}\|\boldsymbol{\eta}\|^{1/2} + C\|\mathbf{u}_h\|^{1/2}\|\mathbf{u}_h\|_1^{1/2}\|\boldsymbol{\eta}\|_{-1}^{1/2}\|\boldsymbol{\eta}\|^{1/2}\Big(\|\boldsymbol{\xi}\| + \|\boldsymbol{\eta}\|\Big)
\end{aligned}
$$

in view of Lemma 3.1. Incorporate (4.10)-(4.13) in (4.9) and use Lemma 2.1 to find

$$
\begin{aligned}
&\Lambda_{1,h}((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta})\\
\le& C\|\boldsymbol{\xi}\|\|\boldsymbol{\eta}\|_{-1}^{1/2}\|\boldsymbol{\eta}\|^{1/2}\Big(\|\mathbf{u}_h\|_1 + \|\mathbf{u}\|_1\Big) + C\|\boldsymbol{\eta}\|_{-1}^{1/2}\|\boldsymbol{\eta}\|^{3/2}\|\mathbf{u}\|_1\\
&\le \varepsilon\|\boldsymbol{\eta}\|^2 + K\Big(\|\boldsymbol{\eta}\|_{-1}^2 + \|\boldsymbol{\xi}\|^2\Big). \qquad (4.14)
\end{aligned}
$$

Hence, with appropriate ε, we obtain from (4.8)

$$
\frac{d}{dt}\|\hat{\boldsymbol{\eta}}\|_{-1}^2 + \nu\|\hat{\boldsymbol{\eta}}\|^2 = K(\|\hat{\boldsymbol{\eta}}\|_{-1}^2 + \|\hat{\boldsymbol{\xi}}\|^2).
$$

Integrate and use (4.1). Apply Gronwall's lemma to coclude that

$$
\|\boldsymbol{\eta}(t)\|_{-1}^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\eta}(s)\|^2 ds \le K e^{Kt} h^4. \qquad (4.15)
$$

We are now in a position for the final estimate. Choose $\boldsymbol{\phi}_h = \sigma(t)\boldsymbol{\eta}$ in (4.6) to obtain

$$
\frac{d}{dt}\{\sigma(t)\|\boldsymbol{\eta}\|^2\} + 2\nu\sigma(t)\|\boldsymbol{\eta}\|_1^2 = \sigma_t(t)\|\boldsymbol{\eta}\|^2 + 2\sigma(t)\Lambda_{1,h}(\boldsymbol{\eta}).
$$

As in (4.9), we have

$$
\begin{aligned}
\Lambda_{1,h}(\boldsymbol{\eta}) =& -b(\mathbf{u}_h, \boldsymbol{\xi}, \boldsymbol{\eta}) - b(\boldsymbol{\xi}+\boldsymbol{\eta}, \mathbf{u}, \boldsymbol{\eta})\\
=& -\frac{1}{2}((\mathbf{u}_h \cdot \nabla)\boldsymbol{\xi}, \boldsymbol{\eta}) - \frac{1}{2}((\mathbf{u}_h \cdot \nabla)\boldsymbol{\eta}, \boldsymbol{\xi})\\
&-\frac{1}{2}\Big(((\boldsymbol{\xi}+\boldsymbol{\eta}) \cdot \nabla)\mathbf{u}, \boldsymbol{\eta}\Big) - \frac{1}{2}\Big(((\boldsymbol{\xi}+\boldsymbol{\eta}) \cdot \nabla)\boldsymbol{\eta}, \mathbf{u}\Big). \qquad (4.16)
\end{aligned}
$$

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Using Lemma 3.1, we find

$$\begin{aligned}
&-\frac{1}{2}\Big(((\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\nabla)\mathbf{u},\boldsymbol{\eta}\Big)-\frac{1}{2}\Big(((\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\nabla)\boldsymbol{\eta},\mathbf{u}\Big)\\
&=-\frac{1}{2}((\boldsymbol{\xi}\cdot\nabla)\mathbf{u},\boldsymbol{\eta})-\frac{1}{2}((\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\eta},\mathbf{u})-\frac{1}{2}((\boldsymbol{\eta}\cdot\nabla)\mathbf{u},\boldsymbol{\eta})-\frac{1}{2}((\boldsymbol{\eta}\cdot\nabla)\boldsymbol{\eta},\mathbf{u})\\
&\le C\|\boldsymbol{\xi}\|\Big\{\|\mathbf{u}\|_1^{1/2}\|\mathbf{u}\|_2^{1/2}\|\boldsymbol{\eta}\|^{1/2}\|\boldsymbol{\eta}\|_1^{1/2}+\|\boldsymbol{\eta}\|_1\|\mathbf{u}\|^{1/2}\|\mathbf{u}\|_2^{1/2}\Big\}\\
&\qquad+C\|\boldsymbol{\eta}\|\|\boldsymbol{\eta}\|_1\|\mathbf{u}\|_1+C\|\boldsymbol{\eta}\|^{1/2}\|\boldsymbol{\eta}\|_1^{3/2}\|\mathbf{u}\|^{1/2}\|\mathbf{u}\|_1^{1/2}. \qquad (4.17)
\end{aligned}$$

and

$$-\frac{1}{2}((\mathbf{u}_h\cdot\nabla)\boldsymbol{\eta},\boldsymbol{\xi})\le C\|\mathbf{u}_h\|^{1/2}\|\mathbf{u}_h\|_2^{1/2}\|\boldsymbol{\eta}\|_1\|\boldsymbol{\xi}\|. \qquad (4.18)$$

And as for the first term on the right-hand side of (4.16), we observe, as in (4.13),

$$\begin{aligned}
&-\frac{1}{2}((\mathbf{u}_h\cdot\nabla)\boldsymbol{\xi},\boldsymbol{\eta})=-\frac{1}{2}((\nabla\cdot\mathbf{u}_h)\boldsymbol{\xi},\boldsymbol{\eta})-\frac{1}{2}((\mathbf{u}_h\cdot\nabla)\boldsymbol{\eta},\boldsymbol{\xi})\\
&\le C\|\mathbf{u}_h\|_1^{1/2}\|\mathbf{u}_h\|_2^{1/2}\|\boldsymbol{\xi}\|\|\boldsymbol{\eta}\|^{1/2}\|\boldsymbol{\eta}\|_1^{1/2}+C\|\mathbf{u}_h\|^{1/2}\|\mathbf{u}_h\|_2^{1/2}\|\boldsymbol{\eta}\|_1\|\boldsymbol{\xi}\|. \qquad (4.19)
\end{aligned}$$

Incorporating (4.17)-(4.19) in (4.16) and using Lemmas 2.1 and 3.2, we obtain

$$\begin{aligned}
\Lambda_{1,h}(\boldsymbol{\eta})&\le K\|\boldsymbol{\xi}\|\|\boldsymbol{\eta}\|_1(\tau^*(t))^{-1/4}+K(\|\boldsymbol{\eta}\|\|\boldsymbol{\eta}\|_1+\|\boldsymbol{\eta}\|^{1/2}\|\boldsymbol{\eta}\|_1^{3/2})\\
&\le K\|\boldsymbol{\xi}\|\|\boldsymbol{\eta}\|_1(\tau^*(t))^{-1/4}+\varepsilon\|\boldsymbol{\eta}\|_1^2+K\|\boldsymbol{\eta}\|^2, \qquad (4.20)
\end{aligned}$$

for some $\varepsilon>0$. Therefore,

$$2\sigma(t)\Lambda_{1,h}(\boldsymbol{\eta})\le 4\varepsilon\sigma(t)\|\boldsymbol{\eta}\|_1^2+K\sigma(t)\|\boldsymbol{\eta}\|^2+Ke^{2\alpha t}\|\boldsymbol{\xi}\|^2.$$

With appropriate ε, we now obtain

$$\frac{d}{dt}\{\sigma(t)\|\boldsymbol{\eta}\|^2\}+\nu\sigma(t)\|\boldsymbol{\eta}\|_1^2\le K\{\|\hat{\boldsymbol{\xi}}\|^2+\|\hat{\boldsymbol{\eta}}\|^2\}.$$

Integrate and use (4.1) and (4.15) to find that

$$\tau^*(t)\|\boldsymbol{\eta}(t)\|^2+e^{-2\alpha t}\int_0^t\sigma(s)\|\boldsymbol{\eta}(s)\|_1^2ds\le Ke^{Kt}h^4. \qquad (4.21)$$

With inverse hypothesis (3.1), we conclude that

$$\|\boldsymbol{\eta}(t)\|+h\|\boldsymbol{\eta}(t)\|_1\le Ke^{Kt}t^{-1/2}h^2. \qquad (4.22)$$

This along with (4.5) gives us the following error estimate.

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Theorem 4.1. *Let Ω be a convex polygon and let the conditions (**A1**)-(**A2**) and (**B1**)-(**B2**) be satisfied. Further, let the discrete initial velocity $\mathbf{u}_{0h} \in \mathbf{J}_h$ with $\mathbf{u}_{0h} = P_h\mathbf{u}_0$, where $\mathbf{u}_0 \in \mathbf{J}_1$. Then, there exists a positive constant C, that depends only on the given data and the domain Ω, such that for $0 < T < \infty$ with $t \in (0, T]$*

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\| + h\|\nabla(\mathbf{u} - \mathbf{u}_h)(t)\| \leq Ke^{Kt}t^{-1/2}h^2.$$

Theorem 4.2. *Under the assumptions of Theorem* 4.1 *and with an additional assumption of* ($\mathbf{B2}'$)*, we have*

$$\|(p - p_h)(t)\| \leq Ke^{Kt}t^{-1/2}h.$$

Proof. Following Lemma 6.1, [16], and using ($\mathbf{B1}$), (2.6) and ($\mathbf{B2}'$) along with the observation that

$$\Lambda_{1,h}(\boldsymbol{\phi}_h) \leq K\|\mathbf{E}\|_1\|\boldsymbol{\phi}_h\|_1$$

we obtain

$$\|(p - p_h)(t)\|_{L^2/\mathbb{R}} \leq Kt^{-1/2}h + K\|\mathbf{E}(t)\|_1 + C\|\mathbf{E}_t(t)\|_{-1,h}, \tag{4.23}$$

where

$$\|\mathbf{E}_t\|_{-1,h} = \sup_{0\neq\boldsymbol{\phi}_h\in\mathbf{H}_h} \frac{<\mathbf{E}_t, \boldsymbol{\phi}_h>}{\|\boldsymbol{\phi}_h\|_1}.$$

Taking supremum over a bigger set, we find that

$$\|\mathbf{E}_t\|_{-1,h} \leq \|\mathbf{E}_t\|_{-1} = \sup_{0\neq\boldsymbol{\phi}\in\mathbf{H}_0^1} \frac{<\mathbf{E}_t, \boldsymbol{\phi}>}{\|\boldsymbol{\phi}\|_1}. \tag{4.24}$$

This is possible due to conforming finite elements. We now have from (4.23), using Theorem 4.1,

$$\|(p - p_h)(t)\|_{L^2/\mathbb{R}} \leq Ke^{Kt}t^{-1/2}h + C\|\mathbf{E}_t(t)\|_{-1}.$$

We complete the proof by proving the following Lemma. □

Lemma 4.1. *The error $\mathbf{E} = \mathbf{u} - \mathbf{u}_h$ due to Galerkin approximation satisfies, for $t > 0$*

$$\|\mathbf{E}_t(t)\|_{-1} \leq Ke^{Kt}t^{-1/2}h.$$

Proof. The equation in $\mathbf{E}$ reads as

$$(\mathbf{E}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{E}, \boldsymbol{\phi}_h) = \Lambda_{1,h}(\boldsymbol{\phi}_h) + (p, \nabla\cdot\boldsymbol{\phi}_h), \quad \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{4.25}$$

Now, for any $\boldsymbol{\phi} \in \mathbf{H}_0^1$, we use (4.25) to find

$$\begin{aligned}(\mathbf{E}_t, \boldsymbol{\phi}) &= (\mathbf{E}_t, P_h\boldsymbol{\phi}) + (\mathbf{E}_t, \boldsymbol{\phi} - P_h\boldsymbol{\phi}) \\ &= -\nu a(\mathbf{E}, P_h\boldsymbol{\phi}) + \Lambda_{1,h}(P_h\boldsymbol{\phi}) + (p, \nabla\cdot P_h\boldsymbol{\phi}) + (\mathbf{E}_t, \boldsymbol{\phi} - P_h\boldsymbol{\phi}).\end{aligned} \tag{4.26}$$

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Using discrete incompressibility condition, H^1-stability of P_h and $(\mathbf{B1})$, we obtain

$$\begin{cases} -\nu a(\mathbf{E}, P_h\boldsymbol{\phi}) \leq C\|\mathbf{E}\|_1\|\boldsymbol{\phi}\|_1 \\ (p, \nabla\cdot P_h\boldsymbol{\phi}) = (p - j_h p, \nabla\cdot P_h\boldsymbol{\phi}) \leq Ch\|p\|_1\|\boldsymbol{\phi}\|_1 \\ \Lambda_{1,h}(P_h\boldsymbol{\phi}) \leq C\|\mathbf{E}\|_1\|\boldsymbol{\phi}\|_1\Big(\|\mathbf{u}\|_1 + \|\mathbf{u}_h\|_1\Big) \\ (\mathbf{E}_t, \boldsymbol{\phi} - P_h\boldsymbol{\phi}) = (\mathbf{u}_t, \boldsymbol{\phi} - P_h\boldsymbol{\phi}) \leq Ch\|\mathbf{u}_t\|\|\boldsymbol{\phi}\|_1. \end{cases} \tag{4.27}$$

From (4.26), we now have, using the definition of negative norm

$$\|\mathbf{E}_t\|_{-1} \leq C\|\mathbf{E}\|_1 + Ch\|p\|_1 + K\|\mathbf{E}\|_1 + Ch\|\mathbf{u}_t\|. \tag{4.28}$$

Use Lemma 2.1 and Theorem 4.1 to complete the rest of the proof. □

Remark 4.1. Instead of using (4.24), if we had used a crude estimate, like $\|\mathbf{E}_t\|_{-1,h} \leq C\|\mathbf{E}_t\|$ (as in [16, (6.2)]), we had obtained

$$\|(p - p_h)(t)\| \leq Ke^{Kt}t^{-1}h.$$

In that sense, we have improved the pressure error estimate for conforming finite elements.

Remark 4.2. In the process, we observe the following estimate, combining (4.1) and (4.15):

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{E}(s)\|^2 ds \leq Ke^{Kt}h^4. \tag{4.29}$$

Below we present an estimate for $\mathbf{E}_t$, which will be needed in our error analysis for two-grid method.

Theorem 4.3. *Under the assumptions of Theorem* 4.1*, we have*

$$e^{-2\alpha t}\int_0^t \sigma_1(s)\|\mathbf{E}_s(s)\|^2 ds \leq Ke^{Kt}h^4. \tag{4.30}$$

Proof. The proof is similar to that of Theorem 4.1 and so, we will only sketch a proof here. First we split the error:

$$\mathbf{E}_t = \boldsymbol{\xi}_t + \boldsymbol{\eta}_t.$$

The equation in $\boldsymbol{\xi}_t$ is given by

$$(\boldsymbol{\xi}_{tt}, \boldsymbol{\phi}_h) + \nu a(\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) = (p_t, \nabla\cdot\boldsymbol{\phi}_h). \tag{4.31}$$

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With $P_h : \mathbf{L}^2(\Omega) \to \mathbf{J}_h$ as L^2-projection, we choose $\boldsymbol{\phi}_h = \sigma_1(t)P_h\boldsymbol{\xi}_t = \sigma_1(t)(\boldsymbol{\xi}_t - (\mathbf{u} - P_h\mathbf{u})_t)$ in (4.31) to find (recall $\sigma_1(t) = (\tau^*(t))^2 e^{2\alpha t},\ \tau^*(t) = \min\{1, t\}$)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\sigma_1(t)\|\boldsymbol{\xi}_t\|^2\} + \nu\sigma_1(t)\|\boldsymbol{\xi}_t\|_1^2 \\
&= \frac{1}{2}\sigma_{1,t}(t)\|\boldsymbol{\xi}_t\|^2 + \sigma_1(t)(\boldsymbol{\xi}_{tt}, (\mathbf{u} - P_h\mathbf{u})_t) \\
&\quad + \nu\sigma_1(t)a(\boldsymbol{\xi}_t, (\mathbf{u} - P_h\mathbf{u})_t) + \sigma_1(t)(p_t - j_h p_t, \nabla\cdot P_h\boldsymbol{\xi}_t).
\end{aligned}$$

Use projection properties $(\mathbf{B1})$ and Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\sigma_1(t)\|\boldsymbol{\xi}_t\|^2\} + \nu\sigma_1(t)\|\boldsymbol{\xi}_t\|_1^2 \\
&\le C\sigma(t)\|\boldsymbol{\xi}_t\|^2 + \frac{\sigma_1(t)}{2}\frac{d}{dt}\|(\mathbf{u} - P_h\mathbf{u})_t\|^2 + Ch.\sigma_1(t)\|\boldsymbol{\xi}_t\|_1\{\|\mathbf{u}_t\|_2 + \|p_t\|_1\}
\end{aligned} \tag{4.32}$$

(recall that $\sigma(t) = \tau^*(t)e^{2\alpha t},\ \tau^*(t) = \min\{1, t\}$). Use Young's inequality and kickback argument. Integrate the resulting inequality to find

$$\begin{aligned}
&\sigma_1(t)\|\boldsymbol{\xi}_t\|^2 + \nu\int_0^t \sigma_1(s)\|\boldsymbol{\xi}_s(s)\|_1^2 ds \\
&\le C\int_0^t \sigma(s)\|\boldsymbol{\xi}_s(s)\|^2 ds + Ch.\sigma_1(t)\|\mathbf{u}_t\|_1^2 \\
&\quad + Ch\int_0^t \sigma(s)\|\mathbf{u}_s(s)\|_1^2 ds + Ch^2\int_0^t \sigma_1(s)\{\|\mathbf{u}_t\|_2^2 + \|p_t\|_1^2\}\, ds.
\end{aligned}$$

Use Lemma 2.1 to observe that

$$\begin{aligned}
&(\tau^*(t))^2\|\boldsymbol{\xi}_t\|^2 + \nu e^{-2\alpha t}\int_0^t \sigma_1(s)\|\boldsymbol{\xi}_s(s)\|_1^2 ds \\
&\le Ce^{-2\alpha t}\int_0^t \sigma(s)\|\boldsymbol{\xi}_s(s)\|^2 ds + Kh^2.
\end{aligned} \tag{4.33}$$

To estimate the first term on the right hand-side of (4.33), we first recall the equation in $\boldsymbol{\xi}$:

$$(\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) + \nu a(\boldsymbol{\xi}, \boldsymbol{\phi}_h) = (p, \nabla\cdot\boldsymbol{\phi}_h),\ \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h.$$

Choose $\boldsymbol{\phi}_h = \sigma(t)P_h\boldsymbol{\xi}_t$ above.

$$\begin{aligned}
&\sigma(t)\|\boldsymbol{\xi}_t\|^2 + \frac{\nu}{2}\frac{d}{dt}\{\sigma(t)\|\boldsymbol{\xi}\|_1^2\} \\
&\le Ce^{2\alpha t}\|\boldsymbol{\xi}\|_1^2 + Ch.\sigma(t)\Big(\|\boldsymbol{\xi}_t\|\|\mathbf{u}_t\|_1 + \|\boldsymbol{\xi}\|_1\|\mathbf{u}_t\|_2 + \|p\|_1\|\boldsymbol{\xi}_t\|_1\Big).
\end{aligned}$$

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Use kickback argument and then integrate with respect to time. Use Lemma 2.1 to obtain

$$\int_0^t \sigma(s)\|\boldsymbol{\xi}_s(s)\|^2 ds + \sigma(t)\|\boldsymbol{\xi}(t)\|_1^2$$
$$\leq C\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|_1^2 ds + Ke^{2\alpha t}h^2 + \frac{\nu}{2C}\int_0^t \sigma_1(s)\|\boldsymbol{\xi}_s(s)\|_1^2 ds.$$

Use (5.7) from [16], that is,

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|_1^2 ds \leq Kh^2 \tag{4.34}$$

to find

$$e^{-2\alpha t}\int_0^t \sigma(s)\|\boldsymbol{\xi}_s(s)\|^2 ds + \tau^*(t)\|\boldsymbol{\xi}\|_1^2 \leq Kh^2 + \frac{\nu}{2C}e^{-2\alpha t}\int_0^t \sigma_1(s)\|\boldsymbol{\xi}_s(s)\|_1^2 ds. \tag{4.35}$$

Using (4.35) in (4.33) leads us to

$$(\tau^*(t))^2\|\boldsymbol{\xi}_t\|^2 + e^{-2\alpha t}\int_0^t \sigma_1(s)\|\boldsymbol{\xi}_s(s)\|_1^2 ds \leq Kh^2. \tag{4.36}$$

Next, we use parabolic duality argument to establish (as in Lemma 5.1 from [16], except that the right hand-side of (5.8), [16] should now read $\sigma_1(t)\boldsymbol{\xi}_t$)

$$e^{-2\alpha t}\int_0^t \sigma_1(s)\|\boldsymbol{\xi}_s(s)\|^2 ds \leq Kh^4. \tag{4.37}$$

An estimate of $\boldsymbol{\eta}_t$ would now complete the proof. By definition, we can easily deduce the equation satisfied by $\boldsymbol{\eta}_t$.

$$(\boldsymbol{\eta}_{tt}, \boldsymbol{\phi}_h) + \nu a(\boldsymbol{\eta}_t, \boldsymbol{\phi}_h) = \Lambda_{1,h,t}(\boldsymbol{\phi}_h), \tag{4.38}$$

where $\boldsymbol{\phi}_{1,h,t}$ is given by

$$\Lambda_{1,h,t}(\boldsymbol{\phi}_h) = -b(\mathbf{u}_{h,t}, \mathbf{E}, \boldsymbol{\phi}_h) - b(\mathbf{u}_h, \mathbf{E}_t, \boldsymbol{\phi}_h) - b(\mathbf{E}_t, \mathbf{u}, \boldsymbol{\phi}_h) - b(\mathbf{E}, \mathbf{u}_t, \boldsymbol{\phi}_h). \tag{4.39}$$

Choose $\boldsymbol{\phi}_h = \sigma_1(t)(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_t$ to find

$$\frac{1}{2}\frac{d}{dt}\{\sigma_1(t)\|\boldsymbol{\eta}_t\|_{-1}^2\} + \nu\sigma_1(t)\|\boldsymbol{\eta}_t\|^2$$
$$\leq C\sigma(t)\|\boldsymbol{\eta}_t\|_{-1}^2 + \sigma_1(t)\Lambda_{1,h,t}((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_t). \tag{4.40}$$

Using similar proof technique of (4.14), we estimate the non-linear term $\Lambda_{1,h,t}$ as follows:

$$\sigma_1(t)b(\mathbf{u}_{h,t}, \mathbf{E}, (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_t) + \sigma_1(t)b(\mathbf{E}, \mathbf{u}_t, (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_t)$$
$$\leq C\sigma_1(t)\|\mathbf{E}\|\|\boldsymbol{\eta}_t\|_{-1}^{1/2}\|\boldsymbol{\eta}_t\|^{1/2}(\|\mathbf{u}_{h,t}\|_1 + \|\mathbf{u}_t\|_1)$$
$$\leq \varepsilon\sigma_1(t)\|\boldsymbol{\eta}_t\|^2 + C\sigma(t)\|\boldsymbol{\eta}_t\|_{-1}^2 + Ke^{2\alpha t}\|\mathbf{E}\|^2 \tag{4.41}$$

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and

$$\begin{aligned}
&\sigma_1(t)b(\mathbf{u}_h,\mathbf{E}_t,(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_t)+\sigma_1(t)b(\mathbf{E}_t,\mathbf{u},(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_t)\\
&\le C\sigma_1(t)(\|\boldsymbol{\xi}_t\|+\|\boldsymbol{\eta}_t\|)\|\boldsymbol{\eta}_t\|_{-1}^{1/2}\|\boldsymbol{\eta}_t\|^{1/2}(\|\mathbf{u}_h\|_1+\|\mathbf{u}\|_1)\\
&\le K\sigma_1(t)\|\boldsymbol{\xi}_t\|\|\boldsymbol{\eta}_t\|_{-1}^{1/2}\|\boldsymbol{\eta}_t\|^{1/2}+K\sigma_1(t)\|\boldsymbol{\eta}_t\|_{-1}^{1/2}\|\boldsymbol{\eta}_t\|^{3/2}\\
&\le \varepsilon\sigma_1(t)\|\boldsymbol{\eta}_t\|^2+K\sigma(t)\|\boldsymbol{\eta}_t\|_{-1}^2+K\sigma_1(t)\|\boldsymbol{\xi}_t\|^2. \qquad (4.42)
\end{aligned}$$

Incorporate (4.41) and (4.42) in (4.40). Use Young's inequality and appropriate $\varepsilon>0$ and finally, integrate with respect to time to find

$$\begin{aligned}
&\sigma_1(t)\|\boldsymbol{\eta}_t(t)\|_{-1}^2+\nu\int_0^t\sigma_1(s)\|\boldsymbol{\eta}_s(s)\|^2ds\\
&\le K\int_0^t\sigma(s)\|\boldsymbol{\eta}_s(s)\|_{-1}^2ds+K\int_0^t e^{2\alpha s}\|\mathbf{E}(s)\|^2ds\\
&\quad+K\int_0^t\sigma_1(s)\|\boldsymbol{\xi}_s(s)\|^2ds.
\end{aligned}$$

Use (4.29) and (4.37) to obtain

$$\begin{aligned}
&\sigma_1(t)\|\boldsymbol{\eta}_t(t)\|_{-1}^2+\nu\int_0^t\sigma_1(s)\|\boldsymbol{\eta}_s(s)\|^2ds\\
&\le K(t)e^{Kt}h^4+K\int_0^t\sigma(s)\|\boldsymbol{\eta}_s(s)\|_{-1}^2ds. \qquad (4.43)
\end{aligned}$$

To estimate the last term of (4.43), we recall the equation in $\boldsymbol{\eta}$ (4.6):

$$(\boldsymbol{\eta}_t,\boldsymbol{\phi}_h)+\nu a(\boldsymbol{\eta},\boldsymbol{\phi}_h)=\Lambda_{1,h}(\boldsymbol{\phi}_h).$$

Choose $\boldsymbol{\phi}_h=\sigma(t)(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_t$ to find

$$\sigma(t)\|\boldsymbol{\eta}_t\|_{-1}^2+\frac{\nu}{2}\frac{d}{dt}\{\sigma(t)\|\boldsymbol{\eta}\|^2\}\le\frac{\nu}{2}\sigma_t(t)\|\boldsymbol{\eta}\|^2+\sigma(t)\Lambda_{1,h}((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_t). \qquad (4.44)$$

Similar to (4.14), we estimate the non-linear term as follows:

$$\begin{aligned}
\sigma(t)\Lambda_{1,h}((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_t)&=-\sigma(t)b(\mathbf{E},\mathbf{u}_h,(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_t)-\sigma(t)b(\mathbf{u},\mathbf{E},(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_t)\\
&\le C\sigma(t)\|\mathbf{E}\|\|\boldsymbol{\eta}_t\|_{-1}\{\|\mathbf{u}_h\|_2+\|\mathbf{u}\|_2\}
\end{aligned}$$

Incorporate this in (4.44), use kickback argument and then integrate with respect to time. Next, use Lemmas 2.1 and 3.2.

$$\begin{aligned}
&\int_0^t\sigma(s)\|\boldsymbol{\eta}_s(s)\|_{-1}^2ds+\sigma(t)\|\boldsymbol{\eta}(t)\|^2\\
&\le C\int_0^t e^{2\alpha s}\|\boldsymbol{\eta}(s)\|^2ds+K\int_0^t e^{2\alpha s}\|\mathbf{E}(s)\|^2ds. \qquad (4.45)
\end{aligned}$$

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Apply (4.15) and (4.29) in (4.45). Use the resulting estimate in (4.43) to obtain

$$\sigma_1(t)\|\boldsymbol{\eta}_t(t)\|_{-1}^2+\nu\int_0^t\sigma_1(s)\|\boldsymbol{\eta}_s(s)\|^2ds\leq K(t)e^{Kt}h^4. \tag{4.46}$$

This along with (4.37) now completes the rest of the proof. □

The pressure error estimate for the two-grid method requires another estimate.

Lemma 4.2. *Under the assumptions of Theorem* 4.1*, we have*

$$\int_0^t\sigma_1(s)\|\mathbf{E}_s(s)\|_1^2ds\leq Ke^{Kt}h^2. \tag{4.47}$$

Proof. Keeping in mind (4.36), we only need the estimate in $\boldsymbol{\eta}$. Choosing $\boldsymbol{\phi}_h=\sigma_1(t)\boldsymbol{\eta}_t$ in (4.39) and estimating non-linear as

$$\Lambda_{1,h,t}(\boldsymbol{\eta}_t)\leq\varepsilon\|\boldsymbol{\eta}_t\|_1^2+C\|\mathbf{E}\|_1^2\Big(\|\mathbf{u}_{h,t}\|_1+\|\mathbf{u}_t\|_1\Big)+C\|\boldsymbol{\xi}_t\|_1^2\Big(\|\mathbf{u}_h\|_1^2+\|\mathbf{u}\|_1^2\Big)+K\|\boldsymbol{\eta}_t\|^2,$$

we find

$$\begin{aligned}&\sigma_1(t)\|\boldsymbol{\eta}_t(t)\|^2+\nu\int_0^t\sigma_1(s)\|\boldsymbol{\eta}_s(s)\|_1^2ds\\&\leq K\int_0^t\sigma(s)\|\boldsymbol{\eta}_s(s)\|^2ds+K\int_0^t\sigma_1(s)\|\boldsymbol{\xi}_s(s)\|_1^2ds\\&\quad+Ke^{Kt}h^2\int_0^t\sigma(s)\big\{\|\mathbf{u}_{h,s}(s)\|_1^2+\|\mathbf{u}_s(s)\|_1^2\big\}ds\end{aligned}$$

Using (4.36) and Lemmas 2.1 and 3.3, we have

$$\sigma_1(t)\|\boldsymbol{\eta}_t(t)\|^2+\nu\int_0^t\sigma_1(s)\|\boldsymbol{\eta}_s(s)\|_1^2ds\leq K\int_0^t\sigma(s)\|\boldsymbol{\eta}_s(s)\|^2ds+Ke^{Kt}h^2. \tag{4.48}$$

Now, for the first term on the right-hand side, we take $\boldsymbol{\phi}_h=\sigma(t)\boldsymbol{\eta}_t$ in the $\boldsymbol{\eta}$ equation, to obtain, as in (4.44):

$$\sigma(t)\|\boldsymbol{\eta}_t\|^2+\frac{\nu}{2}\frac{d}{dt}\{\sigma(t)\|\boldsymbol{\eta}\|_1^2\}\leq\frac{\nu}{2}\sigma_t(t)\|\boldsymbol{\eta}\|_1^2+\sigma(t)\Lambda_{1,h}(\boldsymbol{\eta}_t).$$

We estimate the non-linear term as:

$$\Lambda_{1,h}(\boldsymbol{\eta}_t)\leq K\|\boldsymbol{\eta}_t\|\|\mathbf{E}\|_1(\|\mathbf{u}_h\|_2^{1/2}+\|\mathbf{u}\|_2^{1/2})$$

to find

$$\sigma(t)\|\boldsymbol{\eta}(t)\|_1^2+\int_0^t\sigma(s)\|\boldsymbol{\eta}_s(s)\|^2ds\leq Ke^{Kt}h^2+C\int_0^te^{2\alpha s}\|\boldsymbol{\eta}(s)\|_1^2ds. \tag{4.49}$$

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To estimate the last term, choose $\phi_h = e^{2\alpha t}\boldsymbol{\eta}$ in the $\boldsymbol{\eta}$ equation and estimate the non-linear term as:

$$\Lambda_{1,h}(\boldsymbol{\eta}) \leq \varepsilon\|\boldsymbol{\eta}\|_1 + K(\|\boldsymbol{\xi}\|_1^2 + \|\boldsymbol{\eta}\|^2)$$

to obtain

$$\frac{d}{dt}\|\hat{\boldsymbol{\eta}}\|^2 + \nu\|\hat{\boldsymbol{\eta}}\|_1^2 \leq K(\|\hat{\boldsymbol{\xi}}\|_1^2 + \|\hat{\boldsymbol{\eta}}\|^2)$$

Integrate with respect to time and the use (5.7), [16], that is

$$\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|_1^2 ds \leq Ke^{2\alpha t}h^2, \tag{4.50}$$

to observe

$$\|\hat{\boldsymbol{\eta}}(t)\|^2 + \int_0^t e^{2\alpha s}\|\boldsymbol{\eta}(s)\|_1^2 ds \leq Ke^{2\alpha t}h^2 + \int_0^t e^{2\alpha s}\|\boldsymbol{\eta}(S)\|^2 ds.$$

Use Gronwall's lemma.

$$\|\hat{\boldsymbol{\eta}}(t)\|^2 + \int_0^t e^{2\alpha s}\|\boldsymbol{\eta}(s)\|_1^2 ds \leq Ke^{Kt}h^2. \tag{4.51}$$

Combining (4.50) and (4.51), we conclude

$$\int_0^t e^{2\alpha s}\|\mathbf{E}(s)\|_1^2 ds \leq Ke^{Kt}h^2. \tag{4.52}$$

Now from (4.49), we have

$$\sigma(t)\|\boldsymbol{\eta}(t)\|_1^2 + \int_0^t \sigma(s)\|\boldsymbol{\eta}_s(s)\|^2 ds \leq Ke^{Kt}h^2. \tag{4.53}$$

And from (4.48), we have

$$\sigma_1(t)\|\boldsymbol{\eta}_t(t)\|^2 + \nu\int_0^t \sigma_1(s)\|\boldsymbol{\eta}_s(s)\|_1^2 ds \leq Ke^{Kt}h^2. \tag{4.54}$$

Combining with (4.36), we complete the rest of the proof. □

5. Two-Grid

In this section, we work with an additional space discretizing parameter H, that corresponds to a coarse mesh. In other words, $0 < h < H$ and both h and H tend to 0. The associated conforming finite element spaces are $(\mathbf{H}_H, L_H)$ and $(\mathbf{H}_h, L_h)$ such that $(\mathbf{H}_H, L_H) \subset (\mathbf{H}_h, L_h)$. Also the assumptions of Section 3 are still valid for these two pair of spaces.

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In the first step of our two-grid method, we compute the mixed finite element approximation $(\mathbf{u}_H, p_H) \in (\mathbf{H}_H, L_H)$ of $(\mathbf{u}, p)$ of (2.2). In other words, we solve the nonlinear problem in a coarse mesh: Find $(\mathbf{u}_H, p_H) \in (H_H, L_H)$ satisfying

$$\begin{aligned}&(\mathbf{u}_{Ht}, \boldsymbol{\phi}_H) + \nu a(\mathbf{u}_H, \boldsymbol{\phi}_H) + b(\mathbf{u}_H, \mathbf{u}_H, \boldsymbol{\phi}_H) - (p_H, \nabla \cdot \boldsymbol{\phi}_H) = (\mathbf{f}, \boldsymbol{\phi}_H),\\ &(\nabla \cdot \mathbf{u}_H, \chi_H) = 0,\end{aligned} \tag{5.1}$$

for $(\boldsymbol{\phi}_H, \chi_H) \in (\mathbf{H}_H, L_H)$.
In the second step, we solve a linearized problem on a fine mesh, that is to say, we solve a Stokes problem: Find $(\mathbf{u}^h, p^h) \in (\mathbf{H}_h, L_h)$ satisfying

$$\begin{aligned}&(\mathbf{u}^h_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}^h, \boldsymbol{\phi}_h) - (p^h, \nabla \cdot \boldsymbol{\phi}_h) = (\mathbf{f}, \boldsymbol{\phi}_h) - b(\mathbf{u}_H, \mathbf{u}_H, \boldsymbol{\phi}_h),\\ &(\nabla \cdot \mathbf{u}^h, \chi_h) = 0,\end{aligned} \tag{5.2}$$

for $(\boldsymbol{\phi}_h, \chi_h) \in (\mathbf{H}_h, L_h)$.
An equivalent way is to look for solution in a weekly divergent free space.

First step: Find $\mathbf{u}_H \in \mathbf{J}_H$ satisfying

$$(\mathbf{u}_{Ht}, \boldsymbol{\phi}_H) + \nu a(\mathbf{u}_H, \boldsymbol{\phi}_H) + b(\mathbf{u}_H, \mathbf{u}_H, \boldsymbol{\phi}_H) = (\mathbf{f}, \boldsymbol{\phi}_H), \quad \forall\, \boldsymbol{\phi}_H \in \mathbf{J}_H. \tag{5.3}$$

Second step: With $\mathbf{u}_H$ as the solution of (5.3), find $\mathbf{u}^h \in \mathbf{J}_h$ satisfying

$$(\mathbf{u}^h_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}^h, \boldsymbol{\phi}_h) = (\mathbf{f}, \boldsymbol{\phi}_h) - b(\mathbf{u}_H, \mathbf{u}_H, \boldsymbol{\phi}_h), \quad \forall\, \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{5.4}$$

Remark 5.1. We do not actually solve/implement for a weakly divergent free space. This is just for the ease of analysis.

Remark 5.2. The well-posedness of the above systems can be seen from the facts that (5.1) or (5.3) is classical Galerkin approximation and hence is well-posed as is stated in Section 3. And (5.2) or (5.4) represent linearized version. Given $\mathbf{u}_H$ and with suitable $\mathbf{u}^h(0)$, it is therefore an easy task to show the existence of a unique solution pair $(\mathbf{u}^h, p^h)$ (or an unique solution $\mathbf{u}^h$) for the linearized problem following the foot-steps of the non-linear problem.

Following Lemma 3.2, we can easily obtain the *a priori* estimates of $\mathbf{u}_H$.

Lemma 5.1. *Under the assumptions of Lemma* 3.2 *and for* $\mathbf{u}_H(0) = P_H\mathbf{u}_0$*, the solution* $\mathbf{u}_H$ *of* (5.3) *satisfies, for* $t > 0$,

$$\|\mathbf{u}_H(t)\| + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_H(t)\|_1^2\, ds \le K, \tag{5.5}$$

$$\|\mathbf{u}_H(t)\|_1 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_H(t)\|_2^2\, ds \le K, \tag{5.6}$$

$$(\tau^*(t))^{1/2}\|\mathbf{u}_H(t)\|_2 \le K, \tag{5.7}$$

where $\tau^*(t) = \min\{1, t\}$ *and* K *depends only on the given data. In particular,* K *is independent of* H *and* t.

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The following higher-order estimate of $\mathbf{u}_H$ is required for error analysis. The proof is similar to that of Lemma 3.3.

Lemma 5.2. *Under the assumptions of Lemma* 5.1*, the solution* $\mathbf{u}_H$ *of* (5.3) *satisfies, for* $t > 0$,

$$(\tau^*)^2(t)\|\mathbf{u}_{H,t}(t)\|_1^2 + (\tau^*)^3(t)\|\mathbf{u}_{H,t}(t)\|_2^2 + e^{-2\alpha t}\int_0^t \tau^*(s)e^{2\alpha s}\|\mathbf{u}_{H,s}(s)\|^2\, ds \le K. \quad (5.8)$$

For the *a priori* estimates of $\mathbf{u}^h$, we present another lemma.

Lemma 5.3. *Under the assumptions of Lemma* 5.1*, the solution* $\mathbf{u}^h$ *of* (5.4) *satisfies, for* $t > 0$,

$$\|\mathbf{u}^h(t)\| + e^{-2\alpha t}\textstyle\int_0^t e^{2\alpha s}\|\mathbf{u}^h(t)\|_1^2\, ds \le K, \quad (5.9)$$

$$\|\mathbf{u}^h(t)\|_1 + e^{-2\alpha t}\textstyle\int_0^t e^{2\alpha s}\|\mathbf{u}^h(t)\|_2^2\, ds \le K. \quad (5.10)$$

Proof. Given $\mathbf{u}_H$ along with the estimates of Lemma (5.1), we choose $\boldsymbol{\phi}_h = e^{2\alpha t}\mathbf{u}^h(t) = e^{\alpha t}\hat{\mathbf{u}}^h(t)$ in (5.4) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\hat{\mathbf{u}}^h\|^2 - \alpha\|\hat{\mathbf{u}}^h\|^2 + \nu\|\hat{\mathbf{u}}^h\|_1^2 = (\hat{\mathbf{f}}, \hat{\mathbf{u}}^h) - e^{2\alpha t}b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{u}^h). \quad (5.11)$$

Use Cauchy-Schwarz inequality, Poincaré inequality with first eigenvalue of Stokes operator as the constant and Young's inequality, we have

$$(\hat{\mathbf{f}}, \hat{\mathbf{u}}^h) \le \|\hat{\mathbf{f}}\|\|\hat{\mathbf{u}}^h\| \le \frac{1}{\lambda_1^{1/2}}\|\hat{\mathbf{f}}\|\|\hat{\mathbf{u}}^h\|_1 \le \frac{\nu}{4}\|\hat{\mathbf{u}}^h\|_1^2 + \frac{1}{\nu\lambda_1}\|\hat{\mathbf{f}}\|^2. \quad (5.12)$$

From Lemmas 3.1 and 5.1, we find that

$$\begin{aligned} &b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{u}^h) \\ &\le \|\mathbf{u}_H\|^{1/2}\|\mathbf{u}_H\|_1^{3/2}\|\mathbf{u}^h\|^{1/2}\|\mathbf{u}^h\|_1^{1/2} + \|\mathbf{u}_H\|\|\mathbf{u}_H\|_1\|\mathbf{u}^h\|_1 \\ &\le K + \frac{\nu}{4}\|\mathbf{u}^h\|_1^2. \end{aligned}$$

Therefore,

$$e^{2\alpha t}b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{u}^h) \le Ke^{2\alpha t} + \frac{\nu}{4}\|\hat{\mathbf{u}}^h\|_1^2. \quad (5.13)$$

Putting the estimates (5.12)-(5.13) in (5.11) gives us

$$\frac{d}{dt}\|\hat{\mathbf{u}}^h\|^2 - 2\alpha\|\hat{\mathbf{u}}^h\|^2 + \nu\|\hat{\mathbf{u}}^h\|_1^2 \le Ke^{2\alpha t} + \frac{1}{\nu\lambda_1}\|\hat{\mathbf{f}}\|^2.$$

Use Poincaré inequality to get

$$\frac{d}{dt}\|\hat{\mathbf{u}}^h\|^2 + \left(\nu - \frac{2\alpha}{\lambda_1}\right)\|\hat{\mathbf{u}}^h\|_1^2 \le Ke^{2\alpha t} + \frac{1}{\nu\lambda_1}\|\hat{\mathbf{f}}\|^2. \quad (5.14)$$

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Since $0 < \alpha < \nu\lambda_1/2$, we have that $\nu - 2\alpha/\lambda_1 > 0$. We integrate (5.14) with respect to time.

$$\|\hat{\mathbf{u}}^h(t)\|^2 + \left(\nu - \frac{2\alpha}{\lambda_1}\right)\int_0^t \|\hat{\mathbf{u}}^h(s)\|_1^2 ds \leq Ke^{2\alpha t}.$$

Multiply by $e^{-2\alpha t}$ to conclude (5.9).
For the next estimate, we choose $\boldsymbol{\phi}_h = e^{2\alpha t}\tilde{\Delta}_h \mathbf{u}^h(t)$ in (5.4) and proceed as above. For the non-linear term, we again follow similar proof technique of (4.14) to obtain

$$b(\mathbf{u}_H, \mathbf{u}_H, \tilde{\Delta}_h \mathbf{u}^h) \leq C\|\mathbf{u}_H\|^{1/2}\|\mathbf{u}_H\|_2^{1/2}\|\mathbf{u}_H\|_1\|\mathbf{u}^h\|_2. \tag{5.15}$$

These estimates let us have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\hat{\mathbf{u}}^h\|_1^2 - \alpha\|\hat{\mathbf{u}}^h\|_1^2 + \nu\|\hat{\mathbf{u}}^h\|_2^2 \\ &\leq \|\hat{\mathbf{f}}\|\|\hat{\mathbf{u}}^h\|_2 + Ce^{\alpha t}\|\mathbf{u}_H\|^{1/2}\|\mathbf{u}_H\|_2^{1/2}\|\mathbf{u}_H\|_1\|\hat{\mathbf{u}}^h\|_2. \end{aligned} \tag{5.16}$$

Using Cauchy-Schwarz inequality and Poincaré inequality as earlier, we find that

$$\frac{d}{dt}\|\hat{\mathbf{u}}^h\|_1^2 + \big(\nu - \frac{\alpha}{2\lambda_1}\big)\|\hat{\mathbf{u}}^h\|_2^2 \leq C\|\hat{\mathbf{f}}\|^2 + Ce^{2\alpha t}\Big(\|\mathbf{u}_H\|^2\|\mathbf{u}_H\|_1^4 + \|\mathbf{u}_H\|_2^2\Big). \tag{5.17}$$

Integrate (5.17) with respect to time, use Lemma 5.1 and finally multiply by $e^{-2\alpha t}$ to obtain (5.10). □

6. Error estimate

Here, we carry out the error analysis for the two-grid finite element approximation. We present below, the main results of our paper.

Theorem 6.1. *Let* $(\mathbf{u}_h, p_h)$ *and* $(\mathbf{u}^h, p^h)$ *be the solutions of* (3.2) *and* (5.1)-(5.2)*, respectively. Then, under the assumptions of Theorem* 4.1 *and with the additional assumption that* $\mathbf{u}^h(0) = \mathbf{u}_h(0)$*, the following estimate holds:*

$$\|(\mathbf{u}_h - \mathbf{u}^h)(t)\|_1 + \|(p_h - p^h)(t)\| \leq Kt^{-1}H^2, \quad t > 0. \tag{6.1}$$

(Note that here and in the rest of the paper, we write Ke^{Kt} simply as K.)

Remark 6.1. We have not presented the L^2 error bound in the velocity, since it is sub-optimal, i.e., $\|(\mathbf{u}_h - \mathbf{u}^h)(t)\| \sim O(H^2)$, see Lemma 6.2 below. Such a sub-optimal result, for two-grid schemes with linear elements is, in fact, the best we could hope for, see for example, [8, Remark 2].

We achieve the desired results through a series of Lemmas. Denote the error, due to two-grid method, as $\mathbf{e} = \mathbf{u}_h - \mathbf{u}^h$. From the Eqs. (3.3) and (5.4), we have the following error equation:

$$(\mathbf{e}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}, \boldsymbol{\phi}_h) = \Lambda_h(\boldsymbol{\phi}_h) \ \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{6.2}$$

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where

$$\begin{aligned}\Lambda_h(\boldsymbol{\phi}_h) &= b(\mathbf{u}_H, \mathbf{u}_H, \boldsymbol{\phi}_h) - b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) \\ &= b(\mathbf{u}_H, \mathbf{u}_H - \mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h). \end{aligned} \tag{6.3}$$

Lemma 6.1. *Under the assumptions of Theorem* 4.1 *and with the additional assumption that* $\mathbf{u}^h(0) = \mathbf{u}_h(0)$ *and for* $0 < \alpha < \nu\lambda_1/2$*, the following estimate*

$$\|\mathbf{e}(t)\|_{-1}^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{e}(s)\|^2 ds \le KH^4 \tag{6.4}$$

holds, for $t > 0$.

Proof. Choose $\boldsymbol{\phi}_h = e^{2\alpha t}(-\tilde{\Delta}_h)^{-1}\mathbf{e}(t) = e^{\alpha t}(-\tilde{\Delta}_h)^{-1}\hat{\mathbf{e}}(t)$ in (6.2) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\hat{\mathbf{e}}\|_{-1}^2 - \alpha\|\hat{\mathbf{e}}\|_{-1}^2 + \nu\|\hat{\mathbf{e}}\|^2 = e^{2\alpha t}\Lambda_h((-\tilde{\Delta}_h)^{-1}\mathbf{e}). \tag{6.5}$$

Following (4.14) and using Lemmas 3.2 and 5.1, we find

$$\begin{aligned}\Lambda((-\tilde{\Delta}_h)^{-1}\mathbf{e}) &= b(\mathbf{u}_H, \mathbf{u}_H - \mathbf{u}_h, (-\tilde{\Delta}_h)^{-1}\mathbf{e}) + b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_h, (-\tilde{\Delta}_h)^{-1}\mathbf{e}) \\ &\le \|\mathbf{u}_H - \mathbf{u}_h\|\|\mathbf{e}\|\big\{\|\mathbf{u}_H\|_1 + \|\mathbf{u}_h\|_1\big\} \le K\|\mathbf{u}_H - \mathbf{u}_h\|\|\mathbf{e}\|.\end{aligned}$$

Incorporate this in (6.5).

$$\frac{d}{dt}\|\hat{\mathbf{e}}\|_{-1}^2 + \nu\|\hat{\mathbf{e}}\|^2 \le 2\alpha\|\hat{\mathbf{e}}\|_{-1}^2 + K\|\hat{\mathbf{u}}_H - \hat{\mathbf{u}}_h\|^2. \tag{6.6}$$

Integrate (6.6) with respect to time. Using (4.29), we first observe that

$$\begin{aligned}&\int_0^t \|(\hat{\mathbf{u}}_H - \hat{\mathbf{u}}_h)(s)\|^2 ds \\ \le\ &\int_0^t \|(\hat{\mathbf{u}} - \hat{\mathbf{u}}_H)(s)\|^2 ds + \int_0^t \|(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)(s)\|^2 ds \le Ke^{2\alpha t}H^4.\end{aligned} \tag{6.7}$$

And hence

$$\|\hat{\mathbf{e}}(t)\|_{-1}^2 + \nu\int_0^t \|\hat{\mathbf{e}}(s)\|^2 ds \le 2\alpha\int_0^t \|\hat{\mathbf{e}}(s)\|_{-1}^2 ds + Ke^{2\alpha t}H^4. \tag{6.8}$$

Multiply by $e^{-2\alpha t}$ and use Gronwall's Lemma to complete the rest of the proof. □

Lemma 6.2. *Under the assumptions of Lemma* 6.1*, the following estimate*

$$\tau^*(t)\|\mathbf{e}(t)\|^2 + e^{-2\alpha t}\int_0^t \sigma(s)\|\mathbf{e}(s)\|_1^2 ds \le K(t)H^4 \tag{6.9}$$

holds, for $t > 0$*, where* $\sigma(t) = \tau^*(t)e^{2\alpha t}$, $\tau^*(t) = \min\{1, t\}$.

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Proof. Choose $\phi_h = \sigma(t)\mathbf{e}(t)$ in (6.2) to obtain

$$\frac{d}{dt}\{\sigma(t)\|\mathbf{e}\|^2\} + 2\nu\,\sigma(t)\|\mathbf{e}\|_1^2 = \sigma_t(t)\|\mathbf{e}\|^2 + 2\sigma(t)\Lambda_h(\mathbf{e}). \tag{6.10}$$

To estimate the non-linear term, we follow (4.19) and use Lemmas 5.1 and 3.2.

$$\begin{aligned}\Lambda_h(\mathbf{e}) &\le C\|\mathbf{u}_H - \mathbf{u}_h\|\|\mathbf{e}\|_1\{\|\mathbf{u}_H\|_1^{1/2}\|\mathbf{u}_H\|_2^{1/2} + \|\mathbf{u}_h\|_1^{1/2}\|\mathbf{u}_h\|_2^{1/2}\}\\ &\le K\|\mathbf{u}_H - \mathbf{u}_h\|\|\mathbf{e}\|_1\{\|\mathbf{u}_H\|_2^{1/2} + \|\mathbf{u}_h\|_2^{1/2}\}.\end{aligned} \tag{6.11}$$

And hence, we find from (6.10)

$$\begin{aligned}&\frac{d}{dt}\{\sigma(t)\|\mathbf{e}\|^2\} + 2\nu\,\sigma(t)\|\mathbf{e}\|_1^2\\ &\le C\|\hat{\mathbf{e}}\|^2 + K\sigma(t)\|\mathbf{e}\|_1\|\mathbf{u}_H - \mathbf{u}_h\|\Big(\|\mathbf{u}_H\|_2^{1/2} + \|\mathbf{u}_h\|_2^{1/2}\Big).\end{aligned}$$

Use Cauchy-Schwarz inequality and then kickback argument to yield

$$\begin{aligned}&\frac{d}{dt}\{\sigma(t)\|\mathbf{e}\|^2\} + \nu\,\sigma(t)\|\mathbf{e}\|_1^2\\ &\le C\|\hat{\mathbf{e}}\|^2 + K\sigma(t)\|\mathbf{u}_H - \mathbf{u}_h\|^2\Big(\|\mathbf{u}_H\|_2 + \|\mathbf{u}_h\|_2\Big).\end{aligned} \tag{6.12}$$

Use Lemmas 5.1 and 3.2 to estimate the last term of (6.12) as follows:

$$K\sigma(t)\|\mathbf{u}_H - \mathbf{u}_h\|^2\{\|\mathbf{u}_H\|_2 + \|\mathbf{u}_h\|_2\} \le K\|\hat{\mathbf{u}}_H - \hat{\mathbf{u}}_h\|^2.$$

We have used the fact that $\tau^*(t) \le 1$. Incorporate this in (6.12) and integrate in time. Use (6.7) to estimate the last term.

$$\sigma(t)\|\mathbf{e}(t)\|^2 + \nu\int_0^t \sigma(s)\|\mathbf{e}(s)\|_1^2 ds \le C\int_0^t \|\hat{\mathbf{e}}(s)\|^2 ds + Ke^{2\alpha t}H^4. \tag{6.13}$$

Use Lemma 6.1 and multiply the resulting inequality by $e^{-2\alpha t}$ to complete the rest of the proof. □

Lemma 6.3. *Under the assumptions of Lemma* 6.1*, the following estimate*

$$(\tau^*(t))^2\|\mathbf{e}(t)\|_1^2 + e^{-2\alpha t}\int_0^t \sigma_1(s)\|\mathbf{e}_s(s)\|^2 ds \le K(t)H^4 \tag{6.14}$$

holds, for $t > 0$, where $\sigma_1(t) = (\tau^*(t))^2 e^{2\alpha t}$.

Proof. Choose $\phi_h = \sigma_1(t)\mathbf{e}_t(t)$ in (6.2) to obtain

$$2\sigma_1(t)\|\mathbf{e}_t\|^2 + \nu\frac{d}{dt}\{\sigma_1(t)\|\mathbf{e}\|_1^2\} = \nu\sigma_{1,t}(t)\|\mathbf{e}\|_1^2 + 2\sigma_1(t)\Lambda_h(\mathbf{e}_t). \tag{6.15}$$

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Integrate (6.15) with respect to time.

$$2\int_0^t \sigma_1(s)\|\mathbf{e}_s(s)\|^2 ds + \nu\sigma_1(t)\|\mathbf{e}(t)\|_1^2$$
$$\leq C\int_0^t \sigma(s)\|\mathbf{e}(s)\|_1^2 ds + 2\int_0^t \sigma_1(s)\Lambda_h(\mathbf{e}_s)\, ds. \tag{6.16}$$

We rewrite the non-linear terms as follows:

$$\begin{aligned}\Lambda_h(\mathbf{e}_t) &= b(\mathbf{u}_H, \mathbf{u}_H - \mathbf{u}_h, \mathbf{e}_t) + b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_h, \mathbf{e}_t)\\ =&\frac{d}{dt}\big\{b(\mathbf{u}_H, \mathbf{u}_H - \mathbf{u}_h, \mathbf{e}) + b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_h, \mathbf{e})\big\} - b(\mathbf{u}_{H,t}, \mathbf{u}_H - \mathbf{u}_h, \mathbf{e})\\ &- b(\mathbf{u}_H, \mathbf{u}_{H,t} - \mathbf{u}_{h,t}, \mathbf{e}) - b(\mathbf{u}_{H,t} - \mathbf{u}_{h,t}, \mathbf{u}_h, \mathbf{e}) - b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_{h,t}, \mathbf{e}).\end{aligned}$$

And hence

$$\begin{aligned}\sigma_1(t)\Lambda_h(\mathbf{e}_t) =&\ \sigma_1(t)\big(b(\mathbf{u}_H, \mathbf{u}_H - \mathbf{u}_h, \mathbf{e}_t) + b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_h, \mathbf{e}_t)\big)\\ =&\frac{d}{dt}\big\{\sigma_1(t)\big(b(\mathbf{u}_H, \mathbf{u}_H - \mathbf{u}_h, \mathbf{e}) + b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_h, \mathbf{e})\big)\big\} - \sigma_{1,t}(t)\big(b(\mathbf{u}_H, \mathbf{u}_H - \mathbf{u}_h, \mathbf{e})\\ &+ b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_h, \mathbf{e})\big) - \sigma_1(t)b(\mathbf{u}_{H,t}, \mathbf{u}_H - \mathbf{u}_h, \mathbf{e}) - \sigma_1(t)\big(b(\mathbf{u}_H, \mathbf{u}_{H,t} - \mathbf{u}_{h,t}, \mathbf{e})\\ &+ b(\mathbf{u}_{H,t} - \mathbf{u}_{h,t}, \mathbf{u}_h, \mathbf{e}) + b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_{h,t}, \mathbf{e})\big).\end{aligned} \tag{6.17}$$

As seen earlier, we have

$$-b(\mathbf{u}_{H,t}, \mathbf{u}_H - \mathbf{u}_h, \mathbf{e}) - b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_{h,t}, \mathbf{e})$$
$$\leq C\|\mathbf{u}_H - \mathbf{u}_h\|\|\mathbf{e}\|_1\big\{\|\mathbf{u}_{H,t}\|_1^{1/2}\|\mathbf{u}_{H,t}\|_2^{1/2} + \|\mathbf{u}_{h,t}\|_1^{1/2}\|\mathbf{u}_{h,t}\|_2^{1/2}\big\}$$

Use Lemmas 3.3 and 5.2 to conclude that

$$-b(\mathbf{u}_{H,t}, \mathbf{u}_H - \mathbf{u}_h, \mathbf{e}) - b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_{h,t}, \mathbf{e}) \leq K(\tau^*(t))^{-3/2}\|\mathbf{u}_H - \mathbf{u}_h\|\|\mathbf{e}\|_1. \tag{6.18}$$

Similarly

$$-b(\mathbf{u}_H, \mathbf{u}_H - \mathbf{u}_h, \mathbf{e}) - b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_h, \mathbf{e}) \leq K(\tau^*(t))^{-1/2}\|\mathbf{u}_H - \mathbf{u}_h\|\|\mathbf{e}\|_1 \tag{6.19}$$
$$-b(\mathbf{u}_H, \mathbf{u}_{H,t} - \mathbf{u}_{h,t}, \mathbf{e}) - b(\mathbf{u}_{H,t} - \mathbf{u}_{h,t}, \mathbf{u}_h, \mathbf{e}) \leq K(\tau^*(t))^{-1/2}\|\mathbf{u}_{H,t} - \mathbf{u}_{h,t}\|\|\mathbf{e}\|_1. \tag{6.20}$$

Incorporate (6.18)-(6.20) in (6.17) and integrate with respect to time. Re-use (6.19)

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to find

$$\begin{aligned}
&\int_0^t \sigma_1(s)\Lambda_h(\mathbf{e}_s)\,ds\\
\le&\sigma_1(t)\big(b(\mathbf{u}_H,\mathbf{u}_H-\mathbf{u}_h,\mathbf{e})+b(\mathbf{u}_H-\mathbf{u}_h,\mathbf{u}_h,\mathbf{e})\big)\\
&+K\int_0^t e^{2\alpha s}\big\{(\tau^*(s))^{1/2}\|(\mathbf{u}_H-\mathbf{u}_h)(s)\|\\
&+(\tau^*(s))^{3/2}\|(\mathbf{u}_{H,s}-\mathbf{u}_{h,s})(s)\|\big\}\|\mathbf{e}(s)\|_1 ds\\
\le& Ke^{2\alpha t}(\tau^*(t))^{3/2}\|\mathbf{u}_H-\mathbf{u}_h\|\|\mathbf{e}\|_1+C\int_0^t\sigma(s)\|\mathbf{e}(s)\|_1^2 ds\\
&+K\int_0^t e^{2\alpha s}\big\{\|(\mathbf{u}_H-\mathbf{u}_h)(s)\|^2+(\tau^*(s))^2\|(\mathbf{u}_{H,s}-\mathbf{u}_{h,s})(s)\|^2\big\}\,ds\\
\le&\frac{\nu}{2}\sigma_1(t)\|\mathbf{e}\|_1^2+K\sigma(t)\|\mathbf{u}_H-\mathbf{u}_h\|^2+C\int_0^t\sigma(s)\|\mathbf{e}(s)\|_1^2 ds\\
&+K\int_0^t e^{2\alpha s}\big\{\|(\mathbf{u}_H-\mathbf{u}_h)(s)\|^2+(\tau^*(s))^2\|(\mathbf{u}_{H,s}-\mathbf{u}_{h,s})(s)\|^2\big\}\,ds. \qquad (6.21)
\end{aligned}$$

Put (6.21) in (6.16) to obtain

$$\begin{aligned}
2\int_0^t\sigma_1(s)\|\mathbf{e}_s(s)\|^2 ds+\frac{\nu}{2}\sigma_1(t)\|\mathbf{e}\|_1^2\le C\int_0^t\sigma(s)\|\mathbf{e}(s)\|_1^2 ds+K\sigma(t)\|\mathbf{u}_H-\mathbf{u}_h\|^2\\
+K\int_0^t e^{2\alpha s}\big\{\|(\mathbf{u}_H-\mathbf{u}_h)(s)\|^2+(\tau^*(s))^2\|(\mathbf{u}_{H,s}-\mathbf{u}_{h,s})(s)\|^2\big\}\,ds. \qquad (6.22)
\end{aligned}$$

Now, use Lemma 6.2. For the remaining part, rewrite $\mathbf{u}_H-\mathbf{u}_h$ as $\mathbf{u}-\mathbf{u}_h-(\mathbf{u}-\mathbf{u}_H)$. Then use (4.29) and Theorems 4.1 and 4.3 to complete the rest of the proof. □

Lemma 6.4. *Under the assumptions of Lemma* 6.1 *and additionally that the assumption* ($\mathbf{B2}'$) *holds, we have*

$$\|p_h-p^h\|\le K(t)(\tau^*(t))^{-1}H^2. \qquad (6.23)$$

Proof. The LBB condition ($\mathbf{B2}'$) tells us that, for $t>0$

$$\|p_h-p^h\|_{L^2/\mathbb{R}}\le K_0\sup_{0\neq\boldsymbol{\phi}_h\in\mathbf{H}_h}\frac{(p_h-p^h,\nabla\cdot\boldsymbol{\phi}_h)}{\|\boldsymbol{\phi}_h\|_1}. \qquad (6.24)$$

From (3.2) and (5.2), we have, for $\boldsymbol{\phi}_h\in\mathbf{H}_h$

$$\begin{aligned}
(p_h-p^h,\nabla\cdot\boldsymbol{\phi}_h)&=(\mathbf{e}_t,\boldsymbol{\phi}_h)+\nu a(\mathbf{e},\boldsymbol{\phi}_h)-\Lambda_h(\boldsymbol{\phi}_h)\\
&\le C\|\boldsymbol{\phi}_h\|_1\big\{\|\mathbf{e}_t\|_{-1,h}+\|\mathbf{e}\|_1\big\}+K(\tau^*(t))^{-1/4}\|\mathbf{u}_H-\mathbf{u}_h\|\|\boldsymbol{\phi}_h\|_1.
\end{aligned}$$

We have estimated Λ_h as in (6.11) and have used Lemmas 5.1 and 5.3. Using (6.24), we obtain, thanks to Lemma 4.1 and 6.3

$$\|p_h-p^h\|_{L^2/\mathbb{R}}\le C\|\mathbf{e}_t\|_{-1,h}+K(\tau^*(t))^{-1}H^2. \qquad (6.25)$$

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Following (4.24), we observe

$$\|\mathbf{e}_t\|_{-1,h} \leq \|\mathbf{e}_t\|_{-1}.$$

Keeping in mind that, for $\boldsymbol{\phi} \in \mathbf{H}_0^1, P_h\boldsymbol{\phi} \in \mathbf{J}_h$, we use (6.2) to write $(\mathbf{e}_t, \boldsymbol{\phi})$ as in (4.26) and similar to estimates in (4.27), we find

$$\begin{aligned}(\mathbf{e}_t, \boldsymbol{\phi}) \leq & C\|\mathbf{e}\|_1\|\boldsymbol{\phi}\|_1 + C\|\mathbf{u}_H - \mathbf{u}_h\|\{\|\mathbf{u}_H\|_1^{1/2}\|\mathbf{u}_H\|_2^{1/2} \\ & + \|\mathbf{u}_h\|_1^{1/2}\|\mathbf{u}_h\|_2^{1/2}\}\|\boldsymbol{\phi}\|_1 + Ch\|\mathbf{e}_t\|\|\boldsymbol{\phi}\|_1. \end{aligned} \tag{6.26}$$

Use Lemmas 5.1, 5.3 and 6.3. We again rewrite $\mathbf{u}_H - \mathbf{u}_h$ as $\mathbf{u} - \mathbf{u}_h - (\mathbf{u} - \mathbf{u}_H)$ and use Theorem 4.1. Now, for $\boldsymbol{\phi} \neq 0$, we divide (6.26) by $\boldsymbol{\phi}$ to finally obtain

$$\|\mathbf{e}_t\|_{-1} \leq K(\tau^*(t))^{-1}H^2 + Ch\|\mathbf{e}_t\|.$$

And hence, from (6.25), we find

$$\|p_h - p^h\|_{L^2/\mathbb{R}} \leq CH\|\mathbf{e}_t\| + K(\tau^*(t))^{-1}H^2. \tag{6.27}$$

Now the following lemma completes the proof. □

Lemma 6.5. *Under the assumptions of Lemma* 6.1*, the following estimate holds, for* $t > 0$.

$$\|\mathbf{e}_t\| \leq K(\tau^*(t))^{-1}H.$$

Proof. Differentiate the error Eq. (6.2) with respect to time.

$$(\mathbf{e}_{tt}, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}_t, \boldsymbol{\phi}_h) = \Lambda_{h,t}(\boldsymbol{\phi}_h) \ \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{6.28}$$

where

$$\begin{aligned}\Lambda_{h,t}(\boldsymbol{\phi}_h) = & b(\mathbf{u}_{H,t}, \mathbf{u}_H - \mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_H, \mathbf{u}_{H,t} - \mathbf{u}_{h,t}, \boldsymbol{\phi}_h) \\ & + b(\mathbf{u}_{H,t} - \mathbf{u}_{h,t}, \mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_H - \mathbf{u}_h, \mathbf{u}_{h,t}, \boldsymbol{\phi}_h). \end{aligned} \tag{6.29}$$

Choose $\boldsymbol{\phi}_h = \sigma_1(t)\mathbf{e}_t$ in (6.28) to find

$$\frac{d}{dt}\{\sigma_1(t)\|\mathbf{e}_t\|^2\} + 2\nu\sigma_1(t)\|\mathbf{e}_t\|_1^2 \leq \sigma_{1,t}(t)\|\mathbf{e}_t\|^2 + 2\sigma_1(t)\Lambda_{h,t}(\mathbf{e}_t). \tag{6.30}$$

Similar to (6.18)-(6.20), we obtain

$$\begin{aligned}2\sigma_1(t)\Lambda_{h,t}(\mathbf{e}_t) \leq & C\sigma_1(t)\|\mathbf{e}_t\|_1\{\|\mathbf{u}_{H,t}\|_1 + \|\mathbf{u}_{h,t}\|_1\}\|\mathbf{u}_H - \mathbf{u}_h\|_1 \\ & + C\sigma_1(t)\|\mathbf{e}_t\|_1\{\|\mathbf{u}_H\|_1 + \|\mathbf{u}_h\|_1\}\|\mathbf{u}_{H,t} - \mathbf{u}_{h,t}\|_1. \end{aligned} \tag{6.31}$$

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Incorporate (6.31) in (6.30). Use kickback argument and then integrate with respect to time.

$$\begin{aligned}
&\sigma_1(t)\|\mathbf{e}_t(t)\|^2 + \nu\int_0^t \sigma_1(s)\|\mathbf{e}_t\|_1^2 \\
&\leq C\int_0^t \sigma(s)\|\mathbf{e}_t(s)\|^2 ds + K\int_0^t \sigma_1(s)\|(\mathbf{u}_{H,s}-\mathbf{u}_{h,s})(s)\|_1^2 ds \\
&\quad + KH^2\int_0^t \sigma(s)\big(\|\mathbf{u}_{H,s}(s)\|_1^2 + \|\mathbf{u}_{h,s}(s)\|_1^2\big)\, ds
\end{aligned} \tag{6.32}$$

Use Lemmas 2.1 and 5.3 to bound the last term of (6.32). For the second last term, we rewrite $\mathbf{u}_{H,s}-\mathbf{u}_{h,s}$ as $\mathbf{u}_s-\mathbf{u}_{h,s}-(\mathbf{u}_s-\mathbf{u}_{H,s})$ and use Lemma 4.2.

$$\sigma_1(t)\|\mathbf{e}_t(t)\|^2 + \nu\int_0^t \sigma_1(s)\|\mathbf{e}_t\|_1^2 \leq C\int_0^t \sigma(s)\|\mathbf{e}_t(s)\|^2 ds + KH^2. \tag{6.33}$$

To estimate the first term on the right hand-side of (6.33), we put $\phi_h = \sigma(t)\mathbf{e}_t$ in (6.2).

$$\sigma(t)\|\mathbf{e}_t\|^2 + \frac{\nu}{2}\frac{d}{dt}\{\sigma(t)\|\mathbf{e}\|_1^2\} = \sigma_t(t)\|\mathbf{e}\|_1^2 + \sigma(t)\Lambda_h(\mathbf{e}_t). \tag{6.34}$$

We take care of the non-linear term as earlier.

$$\begin{aligned}
\Lambda_h(\mathbf{e}_t) &= b(\mathbf{u}_H, \mathbf{u}_H-\mathbf{u}_h, \mathbf{e}_t) + b(\mathbf{u}_H-\mathbf{u}_h, \mathbf{u}_h, \mathbf{e}_t) \\
&\leq C\|\mathbf{e}_t\|\|\mathbf{u}_H-\mathbf{u}_h\|_1\Big(\|\mathbf{u}_H\|^{1/2}\|\mathbf{u}_H\|_2^{1/2} + \|\mathbf{u}_h\|^{1/2}\|\mathbf{u}_h\|_2^{1/2}\Big) \\
&\quad + C\|\mathbf{e}_t\|\|\mathbf{u}_H-\mathbf{u}_h\|^{1/2}\|\mathbf{u}_H-\mathbf{u}_h\|_1^{1/2}\Big(\|\mathbf{u}_H\|_1^{1/2}\|\mathbf{u}_H\|_2^{1/2} + \|\mathbf{u}_h\|_1^{1/2}\|\mathbf{u}_h\|_2^{1/2}\Big) \\
&\leq K\|\mathbf{e}_t\|\|\mathbf{u}_H-\mathbf{u}_h\|_1\Big(\|\mathbf{u}_H\|_2^{1/2} + \|\mathbf{u}_h\|_2^{1/2}\Big).
\end{aligned}$$

We have used Lemmas 5.1 and 3.2. Incorporate in (6.34) and after kickback argument, integrate with respect to time.

$$\int_0^t \sigma(s)\|\mathbf{e}_s(s)\|^2 ds + \sigma(t)\|\mathbf{e}(t)\|_1^2 \leq C\int_0^t e^{2\alpha s}\|\mathbf{e}(s)\|_1^2 ds + K\int_0^t e^{2\alpha s}\|\mathbf{E}(s)\|_1^2 ds.$$

Use (4.52) to obtain

$$\int_0^t \sigma(s)\|\mathbf{e}_s(s)\|^2 ds + \sigma(t)\|\mathbf{e}(t)\|_1^2 \leq C\int_0^t e^{2\alpha s}\|\mathbf{e}(s)\|_1^2 ds + KH^2. \tag{6.35}$$

To estimate the first term on the right hand-side of (6.35), we choose $\phi_h = e^{2\alpha t}\mathbf{e}$ in (6.2) to find

$$\frac{1}{2}\frac{d}{dt}\|\hat{\mathbf{e}}\|^2 + \Big(\nu - \frac{\alpha}{\lambda_1}\Big)\|\hat{\mathbf{e}}\|_1^2 \leq e^{2\alpha t}\Lambda_h(\mathbf{e}) \leq K\|\hat{\mathbf{e}}\|_1\Big(e^{\alpha t}\|\mathbf{u}_H-\mathbf{u}_h\|_1\Big).$$

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Use kickback argument and then integrate with respect to time.

$$e^{2\alpha t}\|\mathbf{e}(t)\|^2 + \int_0^t e^{2\alpha s}\|\mathbf{e}(s)\|_1^2 ds \leq K\int_0^t e^{2\alpha s}\|\mathbf{E}(s)\|_1^2 ds.$$

We recall that $\mathbf{e}(0) = 0$. Apply (4.52) to get

$$e^{2\alpha t}\|\mathbf{e}(t)\|^2 + \int_0^t e^{2\alpha s}\|\mathbf{e}(s)\|_1^2 ds \leq KH^2. \tag{6.36}$$

Now, from (6.35), we have

$$\int_0^t \sigma(s)\|\mathbf{e}_s(s)\|^2 ds + \sigma(t)\|\mathbf{e}(t)\|_1^2 \leq KH^2. \tag{6.37}$$

And from (6.33)

$$\sigma_1(t)\|\mathbf{e}_t(t)\|^2 + \nu\int_0^t \sigma_1(s)\|\mathbf{e}_t\|_1^2 \leq KH^2. \tag{6.38}$$

And this completes the rest of the proof. □

Remark 6.2. Note that the estimate of the Lemma 6.5 is sub-optimal. An optimal result would read $\|e_t\| \sim O(t^{-3/2}H^2)$. And if we plug in this optimal estimate in (6.27), then we find

$$\|p_h - p^h\|_{L^2/\mathbb{R}} \leq K(\tau^*(t))^{-3/2}H^3 + K(\tau^*(t))^{-1}H^2 \leq K(\tau^*(t))^{-3/2}H^2.$$

But it is sub-optimal in comparison to (6.23). Hence a sub-optimal estimate of $\|e_t\|$ is obtained.

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