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Rokhlin extensions and lifting disjointness

MARIUSZ LEMAŃCZYK† and FRANÇOIS PARREAU‡

† *Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87–100 Toruń, Poland*
(e-mail: mlem@mat.uni.torun.pl)

‡ *Laboratoire d'Analyse, Géométrie et Applications, UMR 7539, Université Paris 13 et CNRS, 99, av. J.-B. Clément, 93430 Villetaneuse, France*
(e-mail: parreau@math.univ-paris13.fr)

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Abstract. Assuming that two ergodic automorphisms T and R are disjoint, we prove that an ergodic extension $\tilde{T}$ of T remains disjoint with R when, in a representation of $\tilde{T}$ as a skew product over T, the fiber automorphisms belong to a mildly mixing action of a second countable locally countable abelian (LCA) group and the corresponding cocycle with values in this group is ergodic. This allows us to show that the class of ergodic automorphisms which are disjoint from all weakly mixing automorphisms is not closed under taking ergodic joinings, thereby solving a problem stated by E. Glasner and B. Weiss.

0. *Introduction*

The notion of disjointness for dynamical systems was introduced by Furstenberg in 1967 [**6**]. An important task followed to determine the disjointness between some classes of automorphisms (see [**6, 17, 18, 23, 30, 37, 38**]). One motivation of our work was a better understanding of Furstenberg's result that the class of weakly mixing automorphisms is disjoint from the class of distal automorphisms (see also [**11**]). A related problem consists of finding conditions under which disjointness lifts to extensions (see [**6, 11, 21, 28**]) and this problem is taken up in the present paper.

Let T and R be two ergodic automorphisms of standard probability Borel spaces $(X, \mathcal{B}, \mu)$ and $(Z, \mathcal{D}, \rho)$, and let $\tilde{T}$ be an ergodic extension of T. The question is, if T and R are disjoint, under which assumptions can we assert that $\tilde{T}$ and R are still disjoint. From a theorem of Abramov and Rokhlin [**4**], $\tilde{T}$ can be considered as a skew product over T; that is, $\tilde{T}$ acts on a product probability Borel space $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ by

$$\tilde{T}(x, y) = (Tx, S_x(y)),$$

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where $(S_x)_{x\in X}$ is a family of automorphisms of $(Y, \mathcal{C}, \nu)$ such that the map $(x, y) \mapsto S_x y$ is measurable. Furstenberg in [**6**] shows that, if $\tilde{T}$ is a compact group extension of T (that is, if there exists a compact subgroup $\mathcal{G}$ of automorphisms of $(Y, \mathcal{C}, \nu)$ such that $(S_x)_{x\in X} \subset \mathcal{G}$), then $\tilde{T}$ and R remain disjoint under the necessary condition that $\tilde{T}$ and R have no common eigenvalues.

In the present paper, we study extensions $\tilde{T} = T_{\varphi,\mathcal{S}}$ of the form

$$T_{\varphi,\mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y)),$$

where φ is a 'cocycle' from X to a second countable locally compact abelian (LCA) group G, which we throughout assume to be *non-compact*, and $\mathcal{S} = (S_g)_{g\in G}$ is an ergodic measure-preserving action of G on a standard probability Borel space $(Y, \mathcal{C}, \nu)$. We assume that the map $(g, y) \mapsto S_g y$ is measurable when G is endowed with its σ-algebra of Borel sets and we denote a Haar measure of G by λ_G.

In [**21**] it was proved that if T and R are disjoint then $T_{\varphi,\mathcal{S}}$ is disjoint from R under the following assumptions: the skew product $T_\varphi : (x, g) \mapsto (Tx, \varphi(x) + g)$ on $(X \times G, \mathcal{B} \otimes \mathcal{B}(G), \mu \otimes \lambda_G)$ is ergodic and its group of L^∞-eigenvalues is a null set for the maximal spectral type of the associated unitary action of R on the subspace of the zero mean functions of $L^2(Z, \mathcal{D}, \rho)$.

One of our main results is that this latter spectral assumption can be omitted if the action $\mathcal{S}$ is mildly mixing (Theorem 3.2). We show that there exists a correspondence from the set of joinings between $T_{\varphi,\mathcal{S}}$ and R to the set of joinings between the G-action $(S_g^{-1})_{g\in G}$ and the Mackey G-action of the cocycle $\varphi^R(x, z) = \varphi(x)$ on $X \times Z$. Then we show that this latter action preserves a probability measure and is rigid.

Glasner and Weiss [**11**] proved that the class $\mathrm{WM}^\perp$ of ergodic automorphisms which are disjoint from the class WM of all weakly-mixing automorphisms is strictly larger than the class DS of distal automorphisms and since DS is closed under ergodic joinings, they ask whether $\mathrm{WM}^\perp$ is also closed under ergodic joinings. In [**10**], Glasner proved that the class $\mathcal{M}(\mathrm{WM}^\perp)$ of *multipliers* of $\mathrm{WM}^\perp$, that is the class of automorphisms whose ergodic joinings with an arbitrary member of $\mathrm{WM}^\perp$ remain in $\mathrm{WM}^\perp$, is strictly larger than the class of distal automorphisms. As a consequence of our disjointness results and an earlier theorem of Smorodinsky and Thouvenot [**35**] on the joinings of Bernoulli shifts, we show that the class $\mathrm{WM}^\perp$ is not closed under taking ergodic self-joinings and so we have

$$\mathrm{DS} \subsetneq \mathcal{M}(\mathrm{WM}^\perp) \subsetneq \mathrm{WM}^\perp.$$

Other applications, e.g. on lifting multiple mixing, are given in the last two sections of this paper.

1. *Preliminaries*

Throughout we assume that $(X, \mathcal{B}, \mu, T)$, G, $(Y, \mathcal{C}, \nu, \mathcal{S})$ and $\varphi : X \to G$ are given as under the assumptions stated in the introduction. For the spectral theory of unitary G-actions we refer to [**26**].

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1.1. *Cocycles.* By a *cocycle* we mean any measurable function $\varphi : X \to G$, although in fact the cocycle is the corresponding function $\varphi^{(\cdot)}(\cdot) : \mathbb{Z} \times X \to G$ defined by

$$\varphi^{(N)}(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{N-1}x) & \text{if } N > 0, \\ 0 & \text{if } N = 0, \\ -(\varphi(T^N x) + \cdots + \varphi(T^{-1}x)) & \text{if } N < 0. \end{cases}$$

The cocycle φ is a *coboundary* if $\varphi = f - f \circ T$ for some measurable $f : X \to G$ and it is a *quasi-coboundary* if it is *cohomologous* to some constant; that is, if there exists $g_0 \in G$ such that $\varphi - g_0$ is a coboundary.

Let

$$T_\varphi : (x, g) \mapsto (Tx, \varphi(x) + g)$$

denote the associated skew product automorphism on $(X \times G, \mathcal{B} \otimes \mathcal{B}(G), \mu \otimes \lambda_G)$. We say that φ is *ergodic* when T_φ is ergodic.

Following [**33**], an element $g \in G$ is called an *essential value* of φ if for each $\varepsilon > 0$, for each $A \in \mathcal{B}$ of positive measure and for each neighbourhood V of g in G there exists an integer N such that

$$\mu(A \cap T^{-N}A \cap [\varphi^{(N)} \in V]) > 0. \tag{1}$$

The set $E(\varphi)$ of essential values of φ is a closed subgroup of G and φ is ergodic iff $E(\varphi) = G$ [**33**]. We say that φ is *regular* if it is cohomologous to a cocycle with values in $E(\varphi)$. It is *recurrent* [**1, 33**] if zero is a limit point of $\{\varphi^{(n)}(x) : n \in \mathbb{Z}\}$ for μ-a.e. $x \in X$, equivalently if T_φ is conservative, or if whenever $A \in \mathcal{B}$, $\mu(A) > 0$ and V is a neighbourhood of 0 in G, there exists a *positive* N such that (1) is satisfied.

Given a cocycle $\varphi : X \to G$, one associates with it the action of G by translations on the space of ergodic components of T_φ, called the Mackey action (see e.g. [**1, 13, 20**]). Since the Mackey action is central in our study, we recall the precise definition.

Fix λ, a *probability* measure on $\mathcal{B}(G)$ which is equivalent to λ_G. We emphasize that for all of our subsequent study the measure λ will be fixed. Denote by $J_\varphi \subset \mathcal{B} \otimes \mathcal{B}(G)$ the σ-algebra of T_φ-invariant sets and let

$$C_\varphi = (X \times G)/J_\varphi, \quad \mu_\varphi = (\mu \otimes \lambda)|_{J_\varphi}.$$

The standard probability space $(C_\varphi, J_\varphi, \mu_\varphi)$ will be called *the space of ergodic components of* T_φ. Given $g \in G$, the infinite measure-preserving automorphism $(x, h) \mapsto (x, h + g)$ of $(X \times G, \mathcal{B} \otimes \mathcal{B}(G), \mu \otimes \lambda_G)$ commutes with T_φ. It follows that it acts on the measurable space (C_φ, J_φ) and this defines a non-singular action $\tau := (\tau_g)_{g \in G}$ of G on $(C_\varphi, J_\varphi, \mu_\varphi)$. This action is called the *Mackey action* (or the *associated action*) of φ. The Mackey action is always ergodic. Indeed, each measurable function $F = F(x, g)$ which is T_φ-invariant and which at the same time is τ-invariant, depends only on the first coordinate and must be constant since T is ergodic.

Following [**13, 33**] we say that an ergodic non-singular action τ of G on a standard finite or σ-finite measure space $(C, \mathcal{J}, \omega)$ is of type I if μ_φ is concentrated on a single orbit. Otherwise τ is said to be *properly* ergodic and of type II_1 if there exists a τ-invariant probability measure equivalent to ω on $\mathcal{J}$, of type II_∞ if there exists an infinite τ-invariant

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measure equivalent to ω on $\mathcal{J}$, or of type III otherwise. The Mackey action of φ is of type I iff φ is regular. If φ is recurrent then T_φ is properly ergodic on almost all of its ergodic components.

By $e(T_\varphi)$ we denote the subgroup of the torus group $\mathbb{T}$ (taken as $\mathbb{R}/\mathbb{Z}$) of all L^∞-eigenvalues of T_φ corresponding to eigenfunctions of modulus 1. The group $e(T_\varphi)$ has a unique and natural Polish topology whose Borel σ-algebra is the restriction of $\mathcal{B}(\mathbb{T})$. More precisely, let $U(X,\mu)$ denote the Polish group of equivalence classes of modulus 1 measurable functions on X endowed with the $L^2(X,\mu)$-topology. Then, the group $\mathcal{E}(T_\varphi)$ of modulus 1 eigenfunctions of T_φ is a closed subgroup of $U(X\times G,\mu\otimes\lambda)$, and the Polish topology on $e(T_\varphi)$ is obtained by identifying $e(T_\varphi)$ to the quotient group of $\mathcal{E}(T_\varphi)$ by the subgroup of invariant functions. For these facts and further information on groups of eigenvalues we refer the reader to [**3, 16, 26**].

We also denote the subgroup of $\widehat{G}$ of all characters χ for which $\chi\circ\varphi$ is a quasi-coboundary by Σ_φ, i.e. for which there exist $t\in\mathbb{T}$ and a measurable function $j: X\to\mathbf{C}$ of modulus 1 such that

$$\chi\circ\varphi = e^{2\pi it}\, j/j\circ T \quad \mu\text{-a.e.}$$

Then the function $f:(x,g)\mapsto j(x)\chi(g)$ is an eigenfunction of T_φ corresponding to $e^{2\pi it}$ and, for each $h\in G$,

$$\chi(h) = f(x,g+h)/f(x,g) \quad \mu\otimes\lambda\text{-a.e.} \tag{2}$$

It follows that the group $\mathcal{E}_0(T_\varphi)$ of eigenfunctions of this form is a closed subgroup of $\mathcal{E}(T_\varphi)$ and, on $\mathcal{E}_0(T_\varphi)$, the map which to $f=j\otimes\chi$ associates χ is continuous. Considering the quotient topology, we also get a Polish topology on Σ_φ. It also follows that Σ_φ is a Borel subset of $\widehat{G}$ and its Borel σ-algebra is the restriction of $\mathcal{B}(\widehat{G})$.

Conversely, it is shown in [**2**] that, when φ is ergodic, $\mathcal{E}_0(T_\varphi)=\mathcal{E}(T_\varphi)$ (it is enough to notice that, if $f\in\mathcal{E}(T_\varphi)$, then for every $h\in G$ the function $(x,g)\mapsto f(x,g+h)$ is still an eigenfunction with the same eigenvalue, so in this case $f(x,g+h)/f(x,g)$ must be almost everywhere constant and (2) yields a character of G). Then the quotient map $\mathcal{E}(T_\varphi)\to\Sigma_\varphi$ factorizes through $e(T_\varphi)$ and more precisely we have the following.

PROPOSITION 1.1. *Assume that φ is ergodic. Then there exists a continuous group homomorphism $t\mapsto\chi_t$ from $e(T_\varphi)$ to Σ_φ such that every eigenfunction of T_φ corresponding to $t\in e(T_\varphi)$ is of the form $(x,g)\mapsto j(x)\chi_t(g)$, where j satisfies $\chi_t(\varphi(x))\, j(Tx) = e^{2\pi it} j(x)$ μ-a.e.*

1.2. *Joinings and disjointness.* Let us now recall basic notions concerning probability-preserving automorphisms. A *factor* of $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is the quotient action of T on the space $(X/\mathcal{A},\mathcal{A},\mu|_{\mathcal{A}})$, where $\mathcal{A}$ is a T-invariant sub-σ-algebra of $\mathcal{B}$. The σ-algebra $\mathcal{A}$ itself will also be called a factor. The *centralizer* $C(T)$ of T is the set of all automorphisms of $(X,\mathcal{B},\mu)$ commuting with T.

Given two automorphisms $T_i:(X_i,\mathcal{B}_i,\mu_i)\to(X_i,\mathcal{B}_i,\mu_i)$, $i=1,2$, we denote by $J(T_1,T_2)$ the set of *joinings* of T_1 and T_2, taken as the set of $T_1\times T_2$-invariant probability measures η on $\mathcal{B}_1\otimes\mathcal{B}_2$ whose projections on $\mathcal{B}_1$ and $\mathcal{B}_2$ are μ_1 and μ_2, respectively (the corresponding automorphism $T_1\times T_2$ on $(X_1\times X_2,\mathcal{B}_1\otimes\mathcal{B}_2,\eta)$ will be denoted

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$(T_1 \times T_2, \eta)$). By $J^e(T_1, T_2)$ we denote the subset of ergodic joinings of T_1 and T_2. If we consider a finite or infinite sequence $T_1, T_2, \dots$ of automorphisms, we define the sets $J(T_1, T_2, \dots)$ and $J^e(T_1, T_2, \dots)$ similarly, and in the case of $T_1 = T_2 = T$ we will use the notation $J_2(T)$, $J_2^e(T)$, or $J_n(T)$, $J_n^e(T)$ with $n \in \mathbb{N}$ or $n = \infty$ when we consider more copies of T. We refer the reader to [**37**] for results about joinings.

Following [**6**], we say that two automorphisms T_1 and T_2 are *disjoint* and we write $T_1 \perp T_2$ if $J(T_1, T_2) = \{\mu_1 \otimes \mu_2\}$. Recall that if T_1 and T_2 are *spectrally disjoint*, that is when the maximal spectral types of the associated unitary actions on the subspaces of the zero mean functions of L^2 are mutually singular, then $T_1 \perp T_2$.

Given a class Θ of ergodic automorphisms, we denote the class of all ergodic automorphisms which are disjoint from all members of Θ by $\Theta^\perp$. The class $\mathcal{M}(\Theta)$ of *multipliers* of Θ is defined as the class of those automorphisms whose ergodic joinings with an arbitrary member of Θ all give rise to automorphisms that are still in Θ (see [**10**]). If the one-point system is in Θ, then $\mathcal{M}(\Theta)$ is contained in Θ.

Assume that $\mathcal{A}$ is a factor of T and let $\mu = \int_{X/\mathcal{A}} \delta_{\overline{x}} \otimes \mu_{\overline{x}}\, d\mu|_{\mathcal{A}}(\overline{x})$ be the disintegration of μ over its projection on $(X, \mathcal{A})$. Following [**7**], we say that T is *relatively weakly mixing over* $\mathcal{A}$ if the relative product over $\mathcal{A}$, that is the self-joining of T given by

$$\int_{X/\mathcal{A}} \delta_{\overline{x}} \otimes (\mu_{\overline{x}} \otimes \mu_{\overline{x}})\, d\mu|_{\mathcal{A}}(\overline{x}),$$

remains ergodic. The class of *distal* automorphisms can be defined as the smallest class of automorphisms containing the one-point dynamical system, closed under ergodic compact group extensions, factors and passage to inverse limits (see [**6, 7**]). It follows from the structure theorem of Furstenberg [**7**] that if T is relatively weakly mixing over some proper factor then it cannot be distal.

We need the following results from [**21**] (recall that we assume that the action $\mathcal{S}$ is ergodic).

PROPOSITION 1.2. *If φ is ergodic then $T_{\varphi,\mathcal{S}}$ is ergodic.*

PROPOSITION 1.3. *If φ is ergodic and $\mathcal{S}$ is weakly mixing then $T_{\varphi,\mathcal{S}}$ is relatively weakly mixing over T; in particular, $T_{\varphi,\mathcal{S}}$ is not distal.*

The definitions of factors, centralizer and joinings given above for $\mathbb{Z}$-actions and their notation have their natural extensions for G-actions.

In [**31**], Rudolph and Silva introduced the concept of joinings between non-singular automorphisms. In fact, they show that we have to restrict ourselves to so called *rational joinings*. Such joinings will appear here, but only in the particular situation when we join a probability measure-preserving G-action $\mathcal{S} = (S_g)_{g\in G}$ on $(Y, \mathcal{C}, \nu)$ with a non-singular G-action $\tau = (\tau_g)_{g\in G}$ on a standard measure space $(C, \mathcal{J}, \omega)$. We denote by $\tau \times \mathcal{S}$ the product action $(\tau_g \times S_g)_{g\in G}$ and we define $J(\tau, \mathcal{S})$ as the set of $\tau \times \mathcal{S}$-quasi-invariant measures η on $(C \times Y, \mathcal{J} \otimes \mathcal{C})$ whose projections are ω and ν, respectively, and such that the corresponding Radon–Nikodym cocycle of $\tau \times \mathcal{S}$ satisfies, for all $g \in G$,

$$\frac{d(\tau_g \times S_g)\eta}{d\eta}(c, y) = \frac{d\tau_g\omega}{d\omega}(c) \quad \text{for } \eta\text{-a.a. } (c, y).$$

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Then $J(\tau, \mathcal{S})$ is precisely the set of rational joinings, in the sense of Rudolph and Silva, between the two G-actions.

1.3. *Mildly mixing actions and Dirichlet measures.* In what follows, *rigid* and *mildly mixing* actions play an important role. The notion of mild mixing for $\mathbb{Z}$-actions was introduced by Furstenberg and Weiss in [**9**]. For the definition in the case of general G-actions, we follow [**34**]. Recall that we constantly assume that G is non-compact and that its action $\mathcal{S} = (S_g)_{g\in G}$ is finite measure-preserving and ergodic.

The G-action $\mathcal{S}$ is said to be *rigid* if there is a sequence (g_n) going to infinity in G such that $\nu(S_{g_n} B \triangle B) \to 0$ for every set $B \in \mathcal{C}$. It is said to be *mildly mixing* if, for each properly ergodic non-singular G-action τ, the product G-action $\tau \times \mathcal{S}$ remains ergodic. It is then shown in [**34**] that $\mathcal{S}$ is mildly mixing iff it has no (non-trivial) rigid factor.

By a standard argument, $\mathcal{S}$ is rigid iff its spectral type σ (which is a positive Borel measure on $\widehat{G}$) satisfies

$$\limsup_{g\to\infty} \operatorname{Re} \hat{\sigma}(g) = \sigma(\widehat{G}). \tag{3}$$

According to [**15, 16**], we say that a positive Borel measure σ on $\widehat{G}$ is a *Dirichlet* measure if it satisfies (3) and that a Borel subgroup of $\widehat{G}$ is a *Dirichlet subgroup* if every positive Borel measure concentrated on it is Dirichlet.

Remark 1.1. There is some difference between these definitions and others given in the literature. For example, in [**21**], we still find that a G-action is mildly mixing iff it has no rigid factor, but this corresponds to a slightly different definition of rigidity: $\mathcal{S}$ is rigid if there is a sequence (g_n) in G *which does not converge to zero* such that $\nu(S_{g_n} B \triangle B) \to 0$ for every set $B \in \mathcal{C}$. In this case, either $\mathcal{S}$ is rigid in our sense, or there exists a non-zero $g \in G$ such that S_g is the identity automorphism and the subgroup of all such g is compact.

It follows that the definitions are equivalent if G has no non-trivial compact subgroup, that is if $\widehat{G}$ is connected. In the case when G has some non-trivial compact subgroup, the definition of [**21**] forbids any G-action to be mildly mixing, which brings an unnecessary restriction in the present paper.

We also recall that a probability measure σ on $\widehat{G}$ is *full* [**25**] if it is not Dirichlet ($\limsup_{g\to\infty} \operatorname{Re} \hat{\sigma}(g) < 1$) and moreover it is not supported by any proper closed subgroup, and the latter condition can be omitted when G has no non-trivial compact subgroup [**25**, Remark 3.3]. As noted in [**21**, Proposition 2], the action $\mathcal{S}$ is mildly mixing (in the sense of [**21**]) iff the spectral measure $\sigma_{f,\mathcal{S}}$ of every zero-mean $f \in L^2(Y, \mathcal{C}, \nu)$ is full, and the same proof shows that $\mathcal{S}$ is mildly mixing in our sense iff no spectral measure $\sigma_{f,\mathcal{S}}$ for any zero-mean $f \in L^2(Y, \mathcal{C}, \nu)$ is Dirichlet. In particular, the class $\mathrm{MM}(G)$ of all mildly mixing G-actions is spectrally disjoint from the class of rigid G-actions.

The role of full measures and Dirichlet measures in the theory of cocycles is well known (see [**16, 25**]). We need the following theorem, which has been proved by Moore and Schmidt [**25**] in the case when G has no non-trivial compact subgroups. In Appendix A, we give a short and simple proof in the general case.

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THEOREM 1.1. *Assume that Σ_φ is not a Dirichlet subgroup of $\widehat{G}$. Then there exists a compact subgroup H of G and a second countable LCA group $\tilde{G}$ containing G/H as a closed subgroup such that the quotient cocycle $\tilde{\varphi} : X \to G/H$ is a quasi-coboundary as a cocycle with values in $\tilde{G}$. If moreover φ is recurrent, then φ is cohomologous to a cocycle with values in a compact subgroup of G.*

Since an ergodic cocycle is recurrent and not cohomologous to any cocycle with values in a proper subgroup, in particular we have the following.

COROLLARY 1.1. *If φ is ergodic, then Σ_φ is a Dirichlet subgroup of $\widehat{G}$.*

2. *Invariant measures of $T_{\varphi,\mathcal{S}}$ and non-singular joinings*

In order to formulate the main result of this section, we need some more notation. Let $\mathcal{P}(T_{\varphi,\mathcal{S}}; \mu)$ denote the set of $T_{\varphi,\mathcal{S}}$-invariant probability measures η on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ whose marginal $\eta|_X$ on $(X, \mathcal{B})$ is equal to μ (we do not make any assumption on $\eta|_Y$). We establish a correspondence between $\mathcal{P}(T_{\varphi,\mathcal{S}}; \mu)$ and the set of measures defined as follows.

We denote by $\mathcal{S}^{-1}$ the inverse action $(S_g^{-1})_{g\in G}$ and by $\mathcal{P}(\tau, \mathcal{S}^{-1}; \mu_\varphi)$ the set of all $\tau \times \mathcal{S}^{-1}$-quasi-invariant probability measures η on $(C_\varphi \times Y, J_\varphi \otimes \mathcal{C})$ such that $\eta|_{C_\varphi} = \mu_\varphi$ and, for all $g \in G$,

$$\frac{d(\tau_g \times S_g^{-1})\eta}{d\eta}(c, y) = \frac{d\tau_g \mu_\varphi}{d\mu_\varphi}(c) \quad \text{for } \eta\text{-a.a. } (c, y) \in C_\varphi \times Y. \tag{4}$$

If η is a probability measure on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ such that $\eta|_X = \mu$ and if we let

$$\eta = \int_X \delta_x \otimes \eta_x \, d\mu(x) \tag{5}$$

be its disintegration over μ, by the unicity of this disintegration, we obtain that

$$\eta \in \mathcal{P}(T_{\varphi,\mathcal{S}}; \mu) \quad \text{iff} \quad \eta_{Tx} = S_{\varphi(x)}\eta_x \text{ for } \mu\text{-a.a. } x \in X. \tag{6}$$

A straightforward calculation similarly yields the following.

LEMMA 2.1. *Let η be a probability measure on $(C_\varphi \times Y, J_\varphi \otimes \mathcal{C})$ whose projection on (C_φ, J_φ) equals μ_φ, and let $\eta = \int_{C_\varphi} \delta_c \otimes \eta_c \, d\mu_\varphi(c)$ be its disintegration over μ_φ. Then $\eta \in \mathcal{P}(\tau, \mathcal{S}^{-1}; \mu_\varphi)$ if and only if, for all $g \in G$,*

$$\eta_{\tau_g c} = S_g^{-1}\eta_c \quad \textit{for } \mu_\varphi\textit{-a.a. } c \in C_\varphi. \tag{7}$$

PROPOSITION 2.1. *There exists an affine isomorphism Λ between $\mathcal{P}(T_{\varphi,\mathcal{S}}; \mu)$ and $\mathcal{P}(\tau, \mathcal{S}^{-1}; \mu_\varphi)$, which moreover satisfies:*

(i) *if ν' is $\mathcal{S}$-invariant, then $\Lambda(\mu \otimes \nu') = \mu_\varphi \otimes \nu'$;*

(ii) *if $\eta \in \mathcal{P}(T_{\varphi,\mathcal{S}}; \mu)$ and $\eta|_Y$ is $\mathcal{S}$-invariant, then $\Lambda(\eta)|_Y = \eta|_Y$.*

Proof. Take $\eta \in \mathcal{P}(T_{\varphi,\mathcal{S}}; \mu)$ and let $\eta = \int_X \delta_x \otimes \eta_x \, d\mu(x)$ be its disintegration. From (6), we have

$$S_{\varphi(x)}\eta_x = \eta_{Tx} \quad \text{for } \mu\text{-a.a. } x \in X. \tag{8}$$

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Let $\overline{\eta}$ be the probability measure on $(X \times G \times Y, \mathcal{B} \otimes \mathcal{B}(G) \otimes \mathcal{C})$ given by

$$\int_{X\times G\times Y} f \, d\overline{\eta} = \int_{X\times G\times Y} f(x, g, S_g^{-1} y) \, d\eta(x, y) \, d\lambda(g)$$

for each measurable bounded function f on $X \times G \times Y$. In other words, $\overline{\eta} = \int_{X\times G} \delta_{x,g} \otimes \overline{\eta}_{x,g} \, d\mu(x) \, d\lambda(g)$, where

$$\overline{\eta}_{x,g} = S_g^{-1} \eta_x \quad \text{for } \mu \otimes \lambda\text{-a.a. } (x, g) \in X \times G. \tag{9}$$

We define $\tilde{\eta} = \Lambda(\eta)$ as the projection of $\overline{\eta}$ on $(C_\varphi \times Y, J_\varphi \otimes \mathcal{C})$. Clearly, $\tilde{\eta}|_{C_\varphi} = \mu_\varphi$. Since T_φ is non-singular with respect to $\mu \otimes \lambda$, it follows from (8) and (9) that

$$\overline{\eta}_{T_\varphi(x,g)} = \overline{\eta}_{x,g} \quad \text{for } \mu \otimes \lambda\text{-a.a. } (x, g) \in X \times G. \tag{10}$$

Hence, in the disintegration $\tilde{\eta} = \int_{C_\varphi} \delta_c \otimes \tilde{\eta}_c \, d\mu_\varphi(c)$ we have

$$\tilde{\eta}_{c(x,g)} = \overline{\eta}_{x,g} \quad \text{for } \mu \otimes \lambda\text{-a.a. } (x, g) \in X \times G, \tag{11}$$

where $c(x, g)$ denotes the element of C_φ corresponding to (x, g). Moreover, for each $h \in G$ we have $S_h^{-1} \overline{\eta}_{x,g} = \overline{\eta}_{x,g+h}$ for $\mu \otimes \lambda$-a.a. $(x, g) \in X \times G$, so

$$\tilde{\eta}_{\tau_h c} = S_h^{-1} \tilde{\eta}_c \quad \text{for } \mu_\varphi\text{-a.a. } c \in C_\varphi \tag{12}$$

and, by Lemma 2.1, $\tilde{\eta} \in \mathcal{P}(\tau, S^{-1}; \mu_\varphi)$.

Suppose conversely that $\tilde{\eta} \in \mathcal{P}(\tau, S^{-1}; \mu_\varphi)$ and $\tilde{\eta} = \int_{C_\varphi} \delta_c \otimes \tilde{\eta}_c \, d\mu_\varphi(c)$. The formula (11) gives rise to a lift of $\tilde{\eta}$ to a measure $\overline{\eta}$ on $(X \times G \times Y, \mathcal{B} \otimes \mathcal{B}(G) \otimes \mathcal{C})$ whose projection on $(X \times G, \mathcal{B} \otimes \mathcal{B}(G))$ equals $\mu \otimes \lambda$ and whose corresponding disintegration satisfies (10). Then we define η on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ by its disintegration (5) with $\eta_x = \int_G S_g \overline{\eta}_{x,g} \, d\lambda(g)$ μ-a.e. or equivalently by putting for each bounded measurable f on $X \times Y$

$$\int_{X\times Y} f \, d\eta = \int_{X\times G\times Y} f(x, S_g y) \, d\overline{\eta}(x, g, y).$$

In view of Lemma 2.1, (12) is satisfied, so for each $h \in G$ we have $S_h \overline{\eta}_{x,g+h} = \overline{\eta}_{x,g}$ for $\mu \otimes \lambda$-a.a. $(x, g) \in X \times G$. It follows that the function $(x, g) \mapsto S_g \overline{\eta}_{x,g}$ is invariant by the translations on the second coordinate and, therefore, it is $\mu \otimes \lambda$-a.e. equal to a function of x alone, whence

$$\eta_x = S_g \overline{\eta}_{x,g} \quad \text{for } \mu \otimes \lambda\text{-a.a. } (x, g) \in X \times G.$$

Since T_φ is non-singular, we have for $\mu \otimes \lambda$-a.a. $(x, g) \in X \times G$

$$\eta_{Tx} = S_{\varphi(x)+g} \overline{\eta}_{Tx,\varphi(x)+g} = S_{\varphi(x)+g} \overline{\eta}_{x,g} = S_{\varphi(x)} \eta_x$$

and, therefore, $\eta \in \mathcal{P}(T_{\varphi,\mathcal{S}}; \mu)$.

It is clear that the two constructions are mutually inverse and that the map Λ is affine. Also, (i) easily follows from the construction.

In order to prove (ii), note that if $\eta \in \mathcal{P}(T_{\varphi,\mathcal{S}}; \mu)$ and $\nu' = \eta|_Y$ is $\mathcal{S}$-invariant then, by (9)–(11),

$$\int_{C_\varphi} \tilde{\eta}_c \, d\mu_\varphi = \int_{X\times G} S_g^{-1} \eta_x \, d\mu(x) \, d\lambda(g) = \int_G S_g^{-1} \nu' \, d\lambda(g) = \nu'. \qquad \square$$

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Remark 2.1. Note that given $\tilde{\eta} \in \mathcal{P}(\tau, \mathcal{S}^{-1}; \mu_\varphi)$ the above proof does not tell us whether the projections on $(Y, \mathcal{C})$ of $\tilde{\eta}$ and $\Lambda^{-1}(\tilde{\eta})$ are equal.

A particular case appears when φ is ergodic. Then τ is the one-point G-action and every measure η in $\mathcal{P}(\tau, \mathcal{S}^{-1}; \mu_\varphi)$ is of the form $\mu_\varphi \otimes \nu'$ where ν' is $\mathcal{S}$-invariant, whence, by Proposition 2.1(i) and the fact that Λ is one-to-one, $\eta = \mu \otimes \nu'$. We have thus reproved a result from [**21**].

COROLLARY 2.1. [**21**] If φ is ergodic then each measure in $\mathcal{P}(T_{\varphi,\mathcal{S}}; \mu)$ is of the form $\mu \otimes \nu'$, where ν' is an $\mathcal{S}$-invariant measure on $(Y, \mathcal{C})$.

3. *On lifting disjointness*

The aim of this section is to give a sufficient (and in a sense necessary) condition for lifting the disjointness of T from an automorphism R to $T_{\varphi,\mathcal{S}}$.

Throughout we assume that the cocycle $\varphi : X \to G$ is ergodic. Let R be an ergodic automorphism of a standard probability space $(Z, \mathcal{D}, \rho)$, such that the direct product transformation $T \times R$ is still ergodic; that is, we assume that the only common eigenvalue of T and R is zero (in the additive notation of $\mathbb{T}$).

We denote by $\varphi^R : X \times Z \to G$ the cocycle $(x, z) \mapsto \varphi^R(x, z) = \varphi(x)$, by $(C_{\varphi^R}, J_{\varphi^R}, (\mu \otimes \rho)_{\varphi^R})$ the space of ergodic components of $(T \times R)_{\varphi^R} = T_\varphi \times R$ and by $\tau^R = (\tau_g^R)_{g \in G}$ its Mackey action.

3.1. *A case where the Mackey action preserves a finite measure.*

PROPOSITION 3.1. *Under the above assumptions the Mackey action τ^R preserves the probability measure $(\mu \otimes \rho)_{\varphi^R}$.*

Proof. Let $\tilde{R} := Id_{X \times G} \times R$. Then $\tilde{R}$ preserves $\mu \otimes \lambda \otimes \rho$ and it commutes with $T_\varphi \times R$, so it preserves the σ-algebra J_{φ^R} of $T_\varphi \times R$-invariant sets. The quotient action $\tilde{R}_\varphi$ of $\tilde{R}$ on $(C_{\varphi^R}, J_{\varphi^R}, (\mu \otimes \rho)_{\varphi^R})$ is ergodic. Indeed, if a $\mathcal{B} \otimes \mathcal{B}(G) \otimes \mathcal{D}$-measurable function F satisfies $F(x, g, z) = F(x, g, Rz)$ for $\mu \otimes \lambda \otimes \rho$-a.a. $(x, g, z) \in X \times G \times Z$, then it depends only on the first two coordinates since R is ergodic. If moreover F is $T_\varphi \times R$-invariant, then it is almost everywhere constant since T_φ is ergodic. Now, for each $g \in G$, we have that $\tilde{R}_\varphi$ commutes with τ_g^R. Hence the measure $\tau_g^R(\mu \otimes \rho)_{\varphi^R}$, which is equivalent to $(\mu \otimes \rho)_{\varphi^R}$, is still $\tilde{R}_\varphi$-invariant. As $\tilde{R}_\varphi$ is ergodic, it must be equal to $(\mu \otimes \rho)_{\varphi^R}$ and the result follows. □

It follows that either φ^R is regular and cohomologous to a cocycle taking values in a cocompact subgroup H of G or τ^R is of type II_1.

Remark 3.1. We can identify τ^R with an ergodic (although, not necessarily faithful) representation of G in the centralizer of a factor of R.

Indeed, the ergodic decomposition of $\tilde{R}$ on the space $(X \times G \times Z, \mu \otimes \lambda \otimes \rho)$ is given by

$$\mu \otimes \lambda \otimes \rho = \int_{X \times G} \delta_{(x,g)} \otimes \rho \, d(\mu \otimes \lambda)(x, g).$$

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Since the quotient action $\tilde{R}_\varphi$ is ergodic, for $\mu\otimes\lambda$-a.a. $(x,g)\in X\times G$ the quotient action of $\tilde{R}$ on $(C_{\varphi^R}, J_{\varphi^R}, \delta_{(x,g)}\otimes\rho)$ is isomorphic to $\tilde{R}_\varphi$. Since the action of $\tilde{R}$ on each ergodic component is isomorphic with R through the map $z\mapsto(x,g,z)$, our claim follows (see also §3.3 below).

Now, suppose moreover that T and R are disjoint. Then we may identify $J(T_{\varphi,\mathcal{S}}, R)$ to the set of measures in $\mathcal{P}((T\times R)_{\varphi^R,\mathcal{S}};\mu\otimes\rho)$ whose marginal on $(Y,\mathcal{C})$ is ν. Also, $J(\tau^R,\mathcal{S}^{-1})$ is the subset of all measures in $\mathcal{P}(\tau^R,\mathcal{S}^{-1};(\mu\otimes\rho)_{\varphi^R})$ whose marginal on $(Y,\mathcal{C})$ is ν. Denote by Λ the affine isomorphism from $\mathcal{P}((T\times R)_{\varphi^R,\mathcal{S}};\mu\otimes\rho)$ onto $\mathcal{P}(\tau^R,\mathcal{S}^{-1};(\mu\otimes\rho)_{\varphi^R})$ arising from Proposition 2.1.

THEOREM 3.1. *If T and R are disjoint and φ is ergodic then:*
(i) $\Lambda(J(T_{\varphi,\mathcal{S}}, R)) = J(\tau^R,\mathcal{S}^{-1})$;
(ii) $T_{\varphi,\mathcal{S}}\perp R$ *if and only if* $\tau^R\perp\mathcal{S}^{-1}$.

Proof. Notice first that (ii) is a consequence of (i) and also that $\Lambda(J(T_{\varphi,\mathcal{S}},R))\subset J(\tau^R,\mathcal{S}^{-1})$ follows from Proposition 2.1(ii).

So, we only have to prove $\Lambda^{-1}(J(\tau^R,\mathcal{S}^{-1}))\subset J(T_{\varphi,\mathcal{S}},R)$. Take $\tilde{\eta}\in J(\tau^R,\mathcal{S}^{-1})$ and let $\eta=\Lambda^{-1}(\tilde{\eta})$. We have $\eta|_{X\times Y}\in\mathcal{P}(T_{\varphi,\mathcal{S}};\mu)$ and by Corollary 2.1, since φ is ergodic, $\eta|_{X\times Y}$ is a product measure $\mu\otimes\nu'$ where ν' is $\mathcal{S}$-invariant. Then ν' is also the marginal $\eta|_Y$ and, by Proposition 2.1(ii), we get that $\Lambda(\eta)|_Y=\nu'$, so $\eta|_Y=\tilde{\eta}|_Y=\nu$ and $\eta\in J(T_{\varphi,\mathcal{S}},R)$. □

3.2. *Spectral analysis of τ^R.* We now investigate the spectral properties of the unitary action of τ^R. Denote by σ_R the maximal spectral type of the unitary action of R and by $\sigma_{\mathcal{S}}$ the maximal spectral type of the unitary G-action $\mathcal{S}$ on their spaces of zero-mean functions and, given $f\in L^2(C_{\varphi^R}, J_{\varphi^R}, (\mu\otimes\rho)_{\varphi^R})$, denote by σ_{f,τ^R} the spectral measure of f with respect to the unitary G-action of τ^R.

Let us recall (see [**16**, Remark, p. 53] or [**1**, Theorem 2.7.1.]) that $T_\varphi\times R$ is ergodic iff $\sigma_R(e(T_\varphi))=0$ and then its Mackey action is trivial. In this case, by Theorem 3.1, $T_{\varphi,\mathcal{S}}\perp R$ whenever $T\perp R$, a result which was already proved in [**21**]. We will now consider the case when $T_\varphi\times R$ is not ergodic and develop some arguments from [**16**] in order to derive a rigidity property of the Mackey action τ^R.

We identify $L^2(C_{\varphi^R},(\mu\otimes\rho)_{\varphi^R})$ with the closed subspace of J_{φ^R}-measurable functions in $L^2(X\times G\times Z,\mu\otimes\lambda\otimes\rho)$; then we pass to a spectral representation with respect to the third coordinate. If we denote the unitary action of R on $L^2(Z,\rho)$ by U_R and identify $f\in L^2(X\times G\times Z,\mu\otimes\lambda\otimes\rho)$ with a measurable function

$$(x,g)\mapsto f((x,g),\cdot),\quad X\times G\to L^2(Z,\rho)$$

then f is J_{φ^R}-measurable iff

$$f(T_\varphi(x,g),\cdot)=U_R^{-1}f((x,g),\cdot)\quad\mu\otimes\lambda\text{-a.e.}\tag{13}$$

Consider a decomposition $L^2(Z,\rho)=\bigoplus_{n=0}^\infty H_n$ of $L^2(Z,\rho)$ into U_R cyclic subspaces, then we may write

$$f((x,g),\cdot)=\sum_{n=0}^\infty f_n((x,g),\cdot),$$

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where $f_n((x,g),\cdot) \in H_n$ $\mu\otimes\lambda$-a.e. and the series converges in $L^2(Z,\rho)$. Since T_φ is non-singular and U_R^{-1} is unitary on $L^2(Z,\rho)$, $\mu\otimes\lambda$-a.e.,

$$f(T_\varphi(x,g),\cdot) = \sum_{n=0}^{\infty} f_n(T_\varphi(x,g),\cdot),$$

$$U_R^{-1} f((x,g),\cdot) = \sum_{n=0}^{\infty} U_R^{-1} f_n((x,g),\cdot).$$

Thus, by the unicity of the decomposition and (13), if f is J_{φ^R}-measurable, then so is each f_n. Moreover, for each $h \in G$,

$$f((x,g+h),\cdot) = \sum_{n=0}^{\infty} f_n((x,g+h),\cdot),$$

$\mu\otimes\lambda$-a.e. and $f_n((x,g+h),\cdot) \in H_n$.

It follows that $\sigma_{f,\tau^R} = \sum_{n=0}^{\infty} \sigma_{f_n,\tau^R}$ and thus in order to compute the spectral type of τ^R it is enough to consider the functions $f : X\times G \to L^2(Z,\rho)$ taking values in some cyclic subspace H of $L^2(Z,\rho)$. Then, we have a spectral isomorphism $H \approx L^2(\mathbb{T},\sigma)$, where $\sigma \ll \sigma_R$ and U_R corresponds to the multiplication by $e^{2\pi i t}$. We denote by $f \mapsto \hat{f}$ the corresponding isomorphism

$$L^2(X\times G,\mu\otimes\lambda;H) \to L^2(X\times G,\mu\otimes\lambda;L^2(\mathbb{T},\sigma)) = L^2(X\times G\times\mathbb{T},\mu\otimes\lambda\otimes\sigma),$$

so that $(U_R f)\hat{}\,(x,g,t) = e^{2\pi i t}\hat{f}(x,g,t)$, $\mu\otimes\lambda\otimes\sigma$-a.e.

By (13), f is J_{φ^R}-measurable iff

$$\hat{f}(T_\varphi(x,g),t) = e^{-2\pi i t}\hat{f}(x,g,t) \quad \mu\otimes\lambda\otimes\sigma\text{-a.e.} \tag{14}$$

Since T_φ is ergodic, for σ-a.a. $t\in\mathbb{T}$ either $(x,g)\mapsto \hat{f}(x,g,t)$ vanishes $\mu\otimes\lambda$-a.e. or it is an eigenfunction of T_φ (of constant non-zero modulus). Hence, if we let

$$d\sigma'(t) = \left(\int_{X\times G} |\hat{f}(x,g,t)|^2\, d\mu(x)\, d\lambda(g)\right) d\sigma(t),$$

σ' is concentrated on $e(T_\varphi)$. Recall (Proposition 1.1) that we have a continuous map $t\mapsto \chi_t$ from $e(T_\varphi)$ to Σ_φ such that any eigenfunction of T_φ corresponding to the eigenvalue t is of the form $j(x)\chi_t(g)$. Thus we may write

$$\hat{f}(x,g,t) = j(x,t)\overline{\chi}_t(g) \quad \mu\otimes\lambda\otimes\sigma'\text{-a.e.} \tag{15}$$

Let us consider now the unitary action of τ^R. We still have $f\circ\tau_g^R \in L^2(X\times G,\ \mu\otimes\lambda;H)$ and in the spectral representation

$$(f\circ\tau_g^R)\hat{}\,(x,h,t) = \hat{f}(x,g+h,t) \quad \mu\otimes\lambda\otimes\sigma\text{-a.e.}$$

If f is J_{φ^R}-measurable then in view of (15), the unitary action of τ_g^R is the multiplication by $\overline{\chi_t(g)}$ and thus

$$\begin{aligned}\hat{\sigma}_{f,\tau^R}(g) = \langle f\circ\tau_g^R, f\rangle &= \int_{X\times G\times e(T_\varphi)} \overline{\chi_t(g)}\,|\hat{f}(x,h,t)|^2\, d\mu(x)\, d\lambda(h)\, d\sigma(t)\\ &= \int_{e(T_\varphi)} \overline{\chi_t(g)}\, d\sigma'(t).\end{aligned}$$

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This shows that σ_{f,τ^R} is the image of σ' via the map $t \mapsto \overline{\chi}_t$ from $e(T_\varphi)$ to Σ_φ. We have proved the following.

PROPOSITION 3.2. *If φ is ergodic then the maximal spectral type of τ^R is concentrated on Σ_φ.*

Now, as a consequence of Theorem 3.1 we get the following.

COROLLARY 3.1. *If T and R are disjoint, φ is ergodic and $\sigma_{\mathcal{S}}(\Sigma_\varphi) = 0$, then $T_{\varphi,\mathcal{S}}$ is still disjoint from R.*

By Corollary 1.1, if φ is ergodic, then Σ_φ is a Dirichlet subgroup of $\widehat{G}$ and so, by Proposition 3.2, τ^R is rigid. In particular,

$$\tau^R \perp \mathrm{MM}(G)$$

and, therefore, we obtain the following.

THEOREM 3.2. *If T and R are disjoint, φ is ergodic and $\mathcal{S}$ is mildly mixing, then $T_{\varphi,\mathcal{S}}$ is still disjoint from R.*

In [**11**] it was proved that the class $\mathrm{WM}^\perp$ is larger than the class of distal automorphisms by constructing non-distal ergodic automorphisms lying in $\mathrm{WM}^\perp$. A similar result was obtained in [**21**]. By Proposition 1.3, Corollary 3.2 below gives rise to a class of non-distal elements in $\mathrm{WM}^\perp$ containing both classes obtained in [**11**] and in [**21**].

COROLLARY 3.2. *If $T \perp \mathrm{WM}$, φ is ergodic and $\mathcal{S}$ is mildly mixing, then $T_{\varphi,\mathcal{S}} \perp \mathrm{WM}$.*

3.3. *Further properties.* We investigate a little further the properties of τ^R in the case when it is not trivial, under our general assumptions ($T \times R$ is still ergodic, φ is ergodic).

Let $H_{e(T_\varphi)}$ be the spectral subspace of U_R of all functions in $L^2(Z, \rho)$ whose spectral measure are concentrated on $e(T_\varphi)$. From the above discussion, for every J_{φ^R}-measurable $f \in L^2(X \times G \times Z, \mu \otimes \lambda \otimes \rho)$, for almost all (x, g), the function $z \mapsto f(x, g, z)$ belongs to $H_{e(T_\varphi)}$.

Let $\sigma_\varphi = \sigma_R|_{e(T_\varphi)}$. We recall that, since σ_φ is concentrated on $e(T_\varphi)$, there exists a measurable function $\hat{f}_0$ of modulus 1 on $X \times G \times \mathbb{T}$ satisfying

$$\hat{f}_0(T_\varphi(x, g), t) = e^{-2\pi i t}\, \hat{f}_0(x, g, t) \quad \mu \otimes \lambda \otimes \sigma_\varphi\text{-a.e.}, \tag{16}$$

i.e. which yields an eigenfunction selector for T_φ (see e.g. [**1, 16, 26**]). Moreover we can choose $\hat{f}_0$ so that for $\mu \otimes \lambda$-a.a. (x, g), the function $t \mapsto \hat{f}_0(x, g, t)$ belongs to the closure of the exponentials $t \mapsto e^{2\pi i n t}$, $n \in \mathbb{Z}$, in $L^2(\mathbb{T}, \sigma_\varphi)$ (see e.g. [**16**] or [**26**, Theorem 11.19]).

If $H \subset H_{e(T_\varphi)}$ is any cyclic subspace with spectral measure $\sigma \ll \sigma_\varphi$, then $\hat{f}_0$ satisfies (14) and the function $f_0 \in L^2(X \times G, \mu \otimes \lambda; H)$ corresponding spectrally to $\hat{f}_0$ is J_{φ^R}-measurable. More precisely, it follows from (14) and the ergodicity of T_φ that the J_{φ^R}-measurable functions in $L^2(X \times G, \mu \otimes \lambda; H)$ correspond exactly to the functions $\hat{f} \in L^2(X \times G \times \mathbb{T}, \mu \otimes \lambda \otimes \sigma)$ of the form $\hat{f}(x, g, t) = \hat{f}_0(x, g, t)\psi(t)$, where $\psi \in L^2(\sigma)$. Therefore, for $\mu \otimes \lambda$-a.a. (x, g), $f \mapsto f(x, g, \cdot)$ maps the subspace of J_{φ^R}-measurable functions in $L^2(X \times G, \mu \otimes \lambda; H)$ onto H.

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If we consider a decomposition $H_{e(T_\varphi)} = \oplus_n H_n$ into an orthogonal sum of cyclic subspaces, by the proof of Proposition 3.2, we obtain that $L^2(C_{\varphi^R}, (\mu \otimes \rho)_{\varphi^R})$ is naturally identified with the sum of the subspaces of J_{φ^R}-measurable functions in $L^2(X \times G, \mu \otimes \lambda; H_n)$. It follows that, for $\mu \otimes \lambda$-a.a. (x, g), $f \mapsto f(x, g, \cdot)$ maps $L^2(C_{\varphi^R}, (\mu \otimes \rho)_{\varphi^R})$ onto $H_{e(T_\varphi)}$.

This means that, for $\mu \otimes \lambda$-a.a. (x, g), the L^2-space of the factor of R defined by the map $z \mapsto (x, g, z)$ from $(Z, \mathcal{D}, \rho)$ to $(C_{\varphi^R}, J_{\varphi^R}, (\mu \otimes \rho)_{\varphi^R})$ (Remark 3.1) is equal to $H_{e(T_\varphi)}$. In particular, this factor does not depend on (x, g). Let us denote by $R_\varphi : (Z_\varphi, \mathcal{D}_\varphi, \rho) \to (Z_\varphi, \mathcal{D}_\varphi, \rho)$ the quotient action of R.

Given $h \in G$, the automorphism of $(Z_\varphi, \mathcal{D}_\varphi, \rho)$ corresponding to τ_h^R is given spectrally by the multiplication by $\overline{\chi}_{(\cdot)}(h)$ and does not depend on the choice of (x, g) either. We still denote it by τ_h^R.

Since $\overline{\chi}_t(h) = \hat{f}_0(x, g + h, t)/\hat{f}_0(x, g, t)$ and for almost all (x, g), the function $t \mapsto \hat{f}_0(x, g, t)$ is in the closure of the exponentials, so is the function $t \mapsto \overline{\chi}_t(h)$. If we write $\hat{f}_0(x, g, t) = j(x, t)\overline{\chi}_t(g)$, it follows that for almost every x, the unit modulus function $t \mapsto j(x, t)$ is also the limit in $L^2(\sigma_\varphi)$ of some sequence $e^{2\pi i n_k \cdot}$ (depending on x). Then, the sequence $R_\varphi^{n_k}$ converges weakly to an automorphism J_x of $(Z_\varphi, \mathcal{D}_\varphi, \rho)$, which corresponds spectrally to the multiplication by $j(x, \cdot)$. Replacing $\hat{f}_0(x, g, t)$ by $j(x, t)\overline{\chi}_t(g)$ in (16) yields

$$\overline{\chi}_{\varphi(x)} j(Tx, t) = e^{-2\pi i t} j(x, t) \quad \mu \otimes \sigma_\varphi\text{-a.e.},$$

and thus

$$\tau^R_{\varphi(x)} J_{Tx} = R_\varphi^{-1} J_x \quad \mu\text{-a.e.}$$

In other words, the Rokhlin cocycle $x \mapsto \tau^R_{\varphi(x)}$ is cohomologous to the constant cocycle $x \mapsto R_\varphi^{-1}$ and it follows that the extension $T_{\varphi,(\tau^R)^{-1}}$ is isomorphic to $T \times R_\varphi$ (the isomorphism moreover being equal to the identity on X).

PROPOSITION 3.3. *Assume that φ is ergodic and that R is weakly mixing. If $\sigma_R(e(T_\varphi)) > 0$ then there exists a weakly mixing G-action $\mathcal{S}$ such that $T_{\varphi,\mathcal{S}}$ and R have a common factor.*

Proof. We can take $\mathcal{S} = (\tau^R)^{-1}$. Since the kernel of the map $t \mapsto \overline{\chi}_t$ is the eigenvalue group $e(T)$, it is countable and thus the spectral type of τ^R, which is the image of $\sigma_\varphi = \sigma_R|_{e(T_\varphi)}$ under this map, is a continuous measure. So τ^R is weakly mixing. □

4. *The class* $\mathrm{WM}^\perp$ *is not closed under ergodic joinings*

Recall that we denote the class of multipliers of $\mathrm{WM}^\perp$ by $\mathcal{M}(\mathrm{WM}^\perp)$. The aim of this section is to prove the following theorem which brings an answer to the problem raised by Glasner and Weiss [**11**]. When $G = \mathbb{Z}$ and $\mathcal{S} = (S^n)_{n\in\mathbb{Z}}$ we shall use the notation $T_{\varphi,\mathcal{S}} = T_{\varphi,S}$.

THEOREM 4.1. *The class* $\mathrm{WM}^\perp$ *is not closed under ergodic joinings. More precisely, assume that $T \perp \mathrm{WM}$, $\varphi : X \mapsto \mathbb{Z}$ is ergodic, $e(T_\varphi)$ is uncountable and let S be a Bernoulli automorphism. Then $T_{\varphi,S} \in \mathrm{WM}^\perp \setminus \mathcal{M}(\mathrm{WM}^\perp)$.*

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Proof. First, we get $T_{\varphi,S} \in \mathrm{WM}^{\perp}$ directly from Corollary 3.2.

Since $e(T_\varphi)$ is an uncountable Borel set, by Kuratowski's theorem there exists a continuous probability Borel measure concentrated on $e(T_\varphi)$. It follows that there exists a weakly mixing $R : (Z, \mathcal{D}, \rho) \to (Z, \mathcal{D}, \rho)$ whose maximal spectral type is concentrated on $e(T_\varphi)$; we can take here any ergodic Gaussian automorphism generated by a Gaussian process whose spectral measure is concentrated on $e(T_\varphi)$.

Denote by τ^R the Mackey automorphism associated to $T_\varphi \times R$. Since $\tau^R \perp \mathrm{MM}$, its entropy vanishes. Therefore, by a result of Smorodinsky and Thouvenot [**35**], τ^R appears as a factor of some ergodic self-joining η of the Bernoulli automorphism S^{-1}. By Theorem 3.1, it follows that $T_{\varphi,(S\times S,\eta)}$ is not disjoint from R. In view of Proposition 1.2, $T_{\varphi,(S\times S,\eta)}$ is ergodic and it is isomorphic to $(T_{\varphi,S} \times T_{\varphi,S}, \mu_{Id_X} \otimes \eta)$, where μ_{Id_X} is the diagonal measure on $X \times X$ (the graph self-joining of T given by the identity automorphism of X). So we have an ergodic self-joining of $T_{\varphi,S}$ which is not disjoint from R.

We still have to show that the construction is not vacuous, i.e. that we can find an ergodic automorphism $T \perp \mathrm{WM}$ and an ergodic $\mathbb{Z}$-valued cocycle φ with $e(T_\varphi)$ uncountable. In fact, such examples are known for real-valued cocycles. Indeed, let $\overline{T}$ be an irrational rotation $x \mapsto x + \alpha$ on the torus group $\mathbb{T}$ and ψ be a step function of the form $\mathbf{1}_{[0,\beta)} - \beta : \mathbb{T} \to \mathbb{R}$. Oren [**27**] proved that ψ is ergodic whenever $\beta \notin \mathbb{Z}\alpha$. On the other hand, Stewart [**36**] gives a sufficient condition on $s \in \mathbb{R}$ in order that $\exp(2\pi i s\psi)$ be a quasi-coboundary and this condition shows that, whenever α has unbounded partial quotients, we can find values of $\beta \notin \mathbb{Z}\alpha$ such that $e(\overline{T}_\psi)$ is uncountable (see also [**24**], or [**12**] for more precise conditions).

Now, let T be the $\mathbb{T}$-extension of $\overline{T}$ defined on $X = \mathbb{T} \times \mathbb{T}$ by

$$T(x, s) = (\overline{T}x, \psi(x) \ (\mathrm{mod}\ 1) + s)$$

and let $\varphi : X \to \mathbb{Z}$ be given by

$$\varphi(x, s) = [\psi(x) + s],$$

where s is taken in $[0, 1)$ and $[t]$ denotes the integer part of the real t.

Since ψ is ergodic, T is ergodic and, moreover, as it is a compact group extension of $\overline{T}$, we still have $T \perp \mathrm{WM}$. It is easy to check that the map $(x, t) \mapsto (x, t \ (\mathrm{mod}\ 1), [t])$ settles an isomorphism between $\overline{T}_\psi$ and T_φ, so φ is ergodic and $e(T_\varphi)$ is uncountable. □

Remark 4.1. Since, in the construction, τ^R is a factor of $(S^{-1} \times S^{-1}, \eta)$, it follows from the proof of Proposition 3.3 that, actually, $T_{\varphi,(S\times S,\eta)}$ has a weakly-mixing factor.

In addition, Danilenko and Park [**5**] have recently extended the result of Smorodinsky and Thouvenot to countable amenable group actions. So, a similar construction can be carried out with a Bernoulli action of a countable abelian group G, provided that we are able to find an ergodic G-valued cocycle with uncountable group $e(T_\varphi)$.

5. *A lemma on multipliers and applications*

In this section we will formulate a sufficient condition for an automorphism to be a multiplier of a certain class and then obtain some applications.

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5.1. *A lemma on multipliers.* The aim of this subsection is to give a sufficient condition for an automorphism T to belong to $\mathcal{M}(\{R\}^{\perp})$, where R is a weakly-mixing automorphism. When instead of a single automorphism R, we consider a subfamily Θ of weakly-mixing automorphisms closed under taking Cartesian squares, the proposition below shows that an ergodic automorphism T is a multiplier of $\Theta^{\perp}$ as soon as every ergodic self-joining of T is disjoint from Θ.

First, we will briefly recall that joinings are in one-to-one correspondence with certain Markov operators. Assume that T_i is an automorphism of $(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$. If a probability measure η is defined on $\mathcal{B}_1 \otimes \mathcal{B}_2$ and its marginal on X_i equals μ_i $(i = 1, 2)$ then the operator $\Phi_\eta : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$ given by

$$\langle \Phi_\eta f, g\rangle = \int_{X_1 \times X_2} f(x_1)\overline{g(x_2)}\, d\eta(x_1, x_2)$$

is Markov (that is, it is positive and $\Phi_\eta 1 = \Phi_\eta^* 1 = 1$). On the other hand, each Markov operator $\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$ is necessarily of the form $\Phi = \Phi_\eta$, for a unique measure η which is a joining iff

$$\Phi \circ U_{T_1} = U_{T_2} \circ \Phi,$$

where $U_{T_i} f = f \circ T_i$ (see e.g. [**23**] for more details). Then, moreover, $\Phi^*\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_1, \mathcal{B}_1, \mu_1)$ is a Markov operator and it commutes with U_{T_1}, so it corresponds to a self-joining of T_1. Finally denote by $\Pi_{X_1} : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_1, \mathcal{B}_1, \mu_1)$ the projection onto the space of constant functions, which is also the Markov operator corresponding to $\mu_1 \otimes \mu_1$.

Assume that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, $R : (Z, \mathcal{D}, \rho) \to (Z, \mathcal{D}, \rho)$ are two automorphisms.

PROPOSITION 5.1. *Let T be ergodic and R be weakly mixing. Assume moreover that for every $\zeta \in J_2^e(T)$, the automorphism $(T \times T, \zeta)$ is disjoint from the direct product $R \times R$.*

Then $T \in \mathcal{M}(\{R\}^{\perp})$.

Proof. All we need to show is that given $S : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ disjoint with R, any $\eta \in J(T, S, R)$ is equal to the direct product $\eta|_{X\times Y} \otimes \rho$. It is enough to prove that for each bounded measurable f on X, g on Y and h on Z

$$\int_{X\times Y\times Z} f \otimes g \otimes h\, d\eta = \int_{X\times Y} f \otimes g\, d\eta|_{X\times Y} \int_Y h\, d\rho.$$

Hence, it is enough to prove that

$$\text{if } \int_Y h\, d\rho = 0 \quad \text{then} \quad \int_{X\times Y\times Z} f \otimes g \otimes h\, d\eta = 0. \tag{17}$$

We regard η as a joining of $(T \times R, \mu \otimes \rho)$ and S. Denote the corresponding Markov operator by Φ, where $\Phi : L^2(X \times Z, \mu \otimes \rho) \to L^2(Y, \nu)$. Since $S \perp R$,

$$\int_Y \Phi(1 \otimes h) g\, d\nu = \int_{Z\times Y} h \otimes g\, d\rho\, d\nu = 0$$

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and therefore $\Phi(1\otimes h)=0$. It follows that $\Phi^*\Phi(1\otimes h)=0$. Let $\tilde{\eta}\in J_2((T\times R,\mu\otimes\rho))$ be given by $\Phi_{\tilde{\eta}}:=\Phi^*\Phi$. Then for each measurable bounded functions f_i on X and h_i on Z, $i=1,2$,

$$\int_{(X\times Z)\times(X\times Z)} f_1\otimes h_1\otimes f_2\otimes h_2\,d\tilde{\eta}=\int_{X\times Z}\Phi^*\Phi(f_1\otimes h_1)f_2\otimes h_2\,d\mu\otimes\rho.$$

If we take $f_1=f_2=1$, we obtain

$$\int_{Z\times Z} h_1\otimes h_2\,d\tilde{\eta}|_{Z\times Z}=\int_Z h_1\,d\rho\int_Z h_2\,d\rho.$$

Indeed, it suffices to notice that the above integral vanishes whenever $\int_Z h_1\,d\rho=0$.

This proves $\tilde{\eta}|_{Z\times Z}=\rho\otimes\rho$. If we let $\zeta=\tilde{\eta}|_{X\times X}$, then $\tilde{\eta}\in J((T\times T,\zeta),(R\times R,\rho\otimes\rho))$. By considering the ergodic decomposition of ζ and applying our assumption we get $\tilde{\eta}=\zeta\otimes(\rho\otimes\rho)$. Since $\tilde{\eta}$ is a tensor product, $\Phi^*\Phi$ must also be a tensor product and we obtain

$$\Phi^*\Phi=\Phi_\zeta\otimes\Pi_Z.$$

Therefore, if $\int_Y h\,d\rho=0$,

$$\Phi^*\Phi(f\otimes h)=\Phi_\zeta(f)\int_Z h\,d\rho=0,$$

whence $\Phi(f\otimes h)=0$ and

$$\int_{X\times Y\times Z} f\otimes g\otimes h\,d\eta=\int_Y\Phi(f\otimes h)g\,d\nu=0$$

and (17) follows. □

5.2. *Existence of non-distal multipliers of* $\mathrm{WM}^\perp$. Glasner [**10**] proved that non-distal multipliers of $\mathrm{WM}^\perp$ do exist. We will now show how, using our results and a result from [**22**], we can obtain some new non-distal multipliers of $\mathrm{WM}^\perp$.

We still assume that T is an ergodic automorphism of $(X,\mathcal{B},\mu)$ and $\mathcal{S}$ is an ergodic action of G on $(Y,\mathcal{C},\nu)$ and we assume moreover here that G has no non-trivial compact subgroup.

Let $\varphi: X\to G$ be an ergodic cocycle and let $\tilde{\eta}\in J_2^e(T_{\varphi,\mathcal{S}})$. We denote its projection on $X\times X$ by η. Consider the cocycle $(\varphi\times\varphi,\eta)$ for the action $(T\times T,\eta)$ of $T\times T$ on $(X\times X,\mathcal{B}\otimes\mathcal{B},\eta)$ given by $(x_1,x_2)\mapsto(\varphi(x_1),\varphi(x_2))$.

Assume that this cocycle is regular and cohomologous to a cocycle ψ with values in its essential value group $H=E(\varphi\times\varphi,\eta)\subset G\times G$. Then ψ, as a H-valued cocycle, is ergodic.

Given the closed subgroup $H\subset G\times G$, we denote by $\mathcal{P}^e(Y\times Y,H;\nu)$ the set of $(S_{g_1}\times S_{g_2})_{(g_1,g_2)\in H}$-invariant ergodic probability measures on $(Y\times Y,\mathcal{C}\otimes\mathcal{C})$ whose projections on $(Y,\mathcal{C})$ are both equal to ν and, given $\rho\in\mathcal{P}^e(Y\times Y,H;\nu)$, we denote by $(\mathcal{S}\times\mathcal{S})_{H,\rho}$ the action of $(S_{g_1}\times S_{g_2})_{(g_1,g_2)\in H}$ on $(Y\times Y,\mathcal{C}\otimes\mathcal{C},\rho)$.

Under these assumptions the following result is proved in [**22**].

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PROPOSITION 5.2. *There exists* $\rho \in \mathcal{P}^e(Y \times Y, H; \nu)$ *such that the automorphisms* $(T_{\varphi,\mathcal{S}} \times T_{\varphi,\mathcal{S}}, \tilde{\eta})$ *and* $(T \times T, \eta)_{\psi,(\mathcal{S}\times\mathcal{S})_{H,\rho}}$ *are isomorphic.*

The following proposition gives rise to a class of non-distal multipliers of $\mathrm{WM}^{\perp}$. Let Δ_G denote the diagonal subgroup of $G \times G$.

PROPOSITION 5.3. *Assume that* $T \in \mathcal{M}(WM^{\perp})$, *G has no non-trivial compact subgroup,* $\varphi : X \to G$ *is ergodic and for each* $\eta \in J_2^e(T)$:

(i) *the cocycle* $(\varphi \times \varphi, \eta)$ *is regular;*

(ii) *if* $H = E(\varphi \times \varphi, \eta)$, *then either* $H = \Delta_G$ *or* $\mathcal{P}^e(Y \times Y, H; \nu) = \{\nu \otimes \nu\}$. *Assume moreover that* $\mathcal{S} \in MM(G)$ *and:*

(iii) *for each ergodic self-joining of* $\mathcal{S}$ *the resulting G-action remains mildly mixing.*

Then $T_{\varphi,\mathcal{S}} \in \mathcal{M}(WM^{\perp})$.

Proof. Due to Propositions 5.1 and 5.2, it is enough to show that $(T \times T, \eta)_{\psi,(\mathcal{S}\times\mathcal{S})_{H,\rho}}$ belongs to $\mathrm{WM}^{\perp}$, whenever $\eta \in \mathcal{J}_2^e(T)$, $H = E(\varphi \times \varphi, \eta)$, $\psi : X \times X \to H$ is ergodic for the action of $(T \times T, \eta)$ and $\rho \in \mathcal{P}^e(Y \times Y, H; \nu)$.

Suppose first that $\rho = \nu \otimes \nu$. The product $G \times G$-action is mildly mixing and so is each non-trivial subaction of it, by the assumption that G has no non-trivial compact subgroup, so the result follows from Corollary 3.2.

Otherwise $H = \Delta_G$ and $(\mathcal{S} \times \mathcal{S})_{H,\rho} = ((S_g \times S_g)_{g\in G}, \rho)$ with, in fact, $\rho \in J_2^e(\mathcal{S})$. Then the result follows from (iii) and Corollary 3.2 again. □

Note that because of Proposition 1.3, the automorphism $T_{\varphi,\mathcal{S}}$ above is not distal. In [**22**] for $G = \mathbb{R}$ and some irrational rotations, ergodic cocycles $\varphi : X \to G$ satisfying (i) and (ii) are shown to exist. Moreover, mildly-mixing $\mathbb{R}$-actions satisfying (iii) can be found either in the class of Gaussian systems [**23**] or in the class of two-fold simple systems [**18**].

5.3. *A sufficient condition for multiple disjointness.* In connection with the problem of decoding some multiplexed signals, Furstenberg *et al.* [**8**] introduced the notion of double disjointness: an automorphism T is *doubly disjoint* from R if all self-joinings of T are disjoint from R. They ask whether systems in $\mathrm{WM}^{\perp}$ are doubly disjoint from WM. In view of Theorem 4.1 the answer to this question is negative. Another question raised in [**8**] is that whether double disjointness implies triple disjointness. In order to give a partial answer to this question first observe that if the assumption of Proposition 5.1 is satisfied then

$$\text{every ergodic infinite self-joining of } T \text{ belongs to } \mathcal{M}(\{R\}^{\perp}), \tag{18}$$

in other words T is disjoint of all orders from R. Indeed, this is done by induction for finite-fold self-joinings and then by a passage to inverse limits.

In [**14**] (see also [**26**, Chapter 10]), Host proved that if $R_1, \ldots, R_m$ are automorphisms and at least $m - 2$ of them are weakly mixing with singular maximal spectral types, then each joining of $R_1, \ldots, R_m$ which is pairwise independent is globally independent. Therefore, if T is doubly disjoint from R and R has continuous singular spectrum, then the assumptions of Proposition 5.1 are satisfied and, from (18), we obtain that all ergodic infinite self-joinings of T are disjoint from R.

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5.4. *Multipliers of some known classes.* We will show that using Proposition 5.1 we can strengthen some known disjointness results.

The main result from [**17**] states the following. The class SP of simple automorphisms is disjoint from the class GS of Gaussian automorphisms. In fact, we have more:

$$\mathrm{SP} \subset \mathcal{M}(\mathrm{GS}^{\perp}).$$

Indeed, if T is simple and R is Gaussian, then as shown in [**17**], the infinite Cartesian product T^{∞} and R have no common factors. If for an ergodic self-joining η of T $(T \times T, \eta)$ and the Cartesian product $R \times R$ are not disjoint, then by a disjointness result from [**23**], a certain ergodic infinite self-joining of $(T \times T, \eta)$ and $R \times R$ has a common factor. By the simplicity of T this infinite self-joining automorphism is a factor of T^{∞} while $R \times R$ is still Gaussian and we obtain a contradiction.

We have already noted that the class RG of rigid automorphisms is disjoint from the class MM. Clearly RG is closed under self-joining, while MM is closed under direct product. Therefore, by Proposition 5.1, we obtain

$$\mathrm{RG} \subset \mathcal{M}(\mathrm{MM}^{\perp}).$$

It should however, be emphasized that in general an ergodic joining of two different rigid automorphisms is not rigid (in fact, $\mathcal{M}(\mathrm{RG})$ is precisely the class of ergodic rotations).

6. *On lifting multiple mixing*

Rokhlin's problem of whether mixing implies three-fold mixing remains one of the major open problems in ergodic theory and positive answers are only known in certain classes of automorphisms (see [**14, 19**]). In connection, Rudolph showed that multiple mixing property lifts to weakly-mixing compact group extensions [**29**] and asked the question of whether mixing and multiple mixing lift to extensions of the form $T_{\varphi,\mathcal{S}}$ when $\mathcal{S} \in \mathrm{MM}(G)$ and $G = \mathbb{Z}^d$ or $G = \mathbb{R}^d$ [**30**]. The question on lifting mixing is solved in [**21**], but in Appendix A we give a more precise sufficient condition for the general case. Ryzhikov [**32**] gives a condition for lifting multiple mixing in the case where $\mathcal{S}$ and $T_{\varphi,\mathcal{S}}$ are mixing and $G = \mathbb{Z}$. In [**21**], another partial answer is given under strong assumptions on φ. In this section we show that the answer is positive whenever $T_{\varphi,\mathcal{S}}$ is mildly mixing and φ is recurrent.

The following result is a consequence of the construction in the proof of Proposition 2.1. We again consider another automorphism R of a standard probability space $(Z, \mathcal{D}, \rho)$.

PROPOSITION 6.1. *Assume that φ is recurrent and R is mildly mixing. Then $\mu \otimes \nu \otimes \rho$ is the only joining of $T_{\varphi,\mathcal{S}}$ and R whose marginal on $X \times Z$ is equal to $\mu \otimes \rho$.*

Proof. We prove a little more, namely, with the notation of §2 and §3, that for every $\eta \in \mathcal{P}((T \times R)_{\varphi^R}, \mu \otimes \rho)$ we have $\eta = \eta|_{X\times Y} \otimes \rho$.

Following the first steps of the proof of Proposition 2.1 (for $T \times R$ and φ^R instead of T and φ), we consider a disintegration

$$\eta = \int_{X\times Z} \delta_{(x,z)} \otimes \eta_{x,z}\, d\mu(x)\, d\rho(z),$$

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and the family of measures $\overline{\eta}_{x,g,z} = S_g^{-1}\eta_{x,z}$ which define the measure $\overline{\eta}$ on $X \times G \times Z \times Y$. Then $(x, g, z) \mapsto \overline{\eta}_{x,g,z}$ is $(T \times R)_{\varphi^R}$-invariant (formula (10)), i.e. it is J_{φ^R}-measurable.

Since φ is recurrent, the action of T_φ on μ_φ-a.a. ergodic component $c \in C_\varphi$ is properly ergodic and, since R is mildly mixing, then the action of $T_\varphi \times R$ on $c \times Z$ is still ergodic. Therefore, $J_{\varphi^R} = J_\varphi \times Z \pmod{\mu \otimes \lambda \otimes \rho}$.

In particular, $(x, g, z) \mapsto \overline{\eta}_{x,g,z}$ is almost everywhere equal to a function of x and g only and it follows that $\eta_{x,z} = \int_G S_g \overline{\eta}_{x,g,z}\, d\lambda(g)$ depends only on x. This clearly implies the result. □

Before coming to the problem of lifting multiple mixing, let us state an immediate consequence of Proposition 6.1.

COROLLARY 6.1. *If $T \perp R$, φ is recurrent and R is mildly mixing, then $T_{\varphi,\mathcal{S}} \perp R$.*

Now, assume that φ is recurrent and $T_{\varphi,\mathcal{S}}$ is mildly mixing and let $\eta \in J_r(T_{\varphi,\mathcal{S}})$, where r is an integer greater than or equal to two.

For $0 \le k \le r$, let $\eta_k = \eta|_{(X\times Y)^k \times X^{r-k}}$. Given $k \ge 1$, suppose that η_{k-1} is the product measure $(\mu \otimes \nu)^{\otimes k-1} \otimes \mu^{\otimes r-k+1}$. Then let R be the direct product of $k-1$ factors $T_{\varphi,\mathcal{S}}$ and of $r-k$ factors T on $Z = (X \times Y)^{k-1} \times X^{r-k}$ equipped with the product measure $\rho = (\mu \otimes \nu)^{\otimes k-1} \otimes \mu^{\otimes r-k}$. Up to a rearrangement of the coordinates, η_k appears as a joining of $T_{\varphi,\mathcal{S}}$ and R whose marginal η_{k-1} on $X \times Z$ is equal to $\mu \otimes \rho$. Moreover R is still mildly mixing, so we can apply Proposition 6.1 and thus $\eta_k = \mu \otimes \nu \otimes \rho = (\mu \otimes \nu)^{\otimes k} \otimes \mu^{\otimes r-k}$.

It follows by induction that

$$\text{if } \eta \in J_r(T_{\varphi,\mathcal{S}}) \text{ and } \eta|_{X^r} = \mu^{\otimes r}, \text{ then } \eta = (\mu \otimes \nu)^{\otimes r}. \tag{19}$$

THEOREM 6.1. *Assume that T is r-fold mixing and that φ is recurrent. If $T_{\varphi,\mathcal{S}}$ is mildly mixing then it is mixing of order r.*

Proof. We give the proof when $r = 3$; the argument is the same in the general case. If $T_{\varphi,\mathcal{S}}$ were not three-fold mixing, we could find sequences of integers (m_k) and (n_k) with m_k, n_k and $m_k - n_k$ going to infinity and a joining $\eta \in J_3(T_{\varphi,\mathcal{S}})$ obtained as a weak limit of graph joinings by

$$\eta(A \times B \times C) = \lim_{k\to+\infty} \mu \otimes \nu(A \cap T_{\varphi,\mathcal{S}}^{-m_k} B \cap T_{\varphi,\mathcal{S}}^{-n_k} C),$$

for A, B, $C \in \mathcal{B} \times \mathcal{C}$, which would not be the product measure $(\mu \otimes \nu)^{\otimes 3}$ (see e.g. [**26**, Chapter 10]). However, since T is three-fold mixing, we then have $\eta|_{X^3} = \mu^{\otimes 3}$ and this contradicts (19). □

In Appendix A, we show that, if T is mixing, $T_{\varphi,\mathcal{S}}$ is mixing as soon as $\sigma_{\mathcal{S}}(\Sigma_\varphi) = 0$ (Theorem A.1). When φ is recurrent, by Theorem 1.1, Σ_φ is a Dirichlet subgroup of $\widehat{G}$ unless φ is cohomologous to a cocycle with values in some compact subgroup. We hence obtain the following.

COROLLARY 6.2. *Let T be r-fold mixing, $\mathcal{S}$ be mildly mixing and assume that φ is recurrent and not cohomologous to a cocycle with values in a compact subgroup of G. Then $T_{\varphi,\mathcal{S}}$ is r-fold mixing.*

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A. *Appendix*

A.1. *Proof of Theorem 1.1.* Our proof will rely on a similar result for the subgroup $\Lambda_\varphi \subset \Sigma_\varphi$ of all $\chi \in \widehat{G}$ such that $\chi \circ \varphi$ is a coboundary; that is,

$$\Lambda_\varphi = \{\chi \in \widehat{G} : \chi \circ \varphi = j/j \circ T \text{ for some } j \in U(X, \mu)\}.$$

Moore and Schmidt [**25**] proved that if Λ_φ carries a full measure then φ is a coboundary. We make use of both a version of this result in the case when we only suppose that Λ_φ is not Dirichlet and of the main arguments of the proof, which we recall briefly. The assumption that T is ergodic is not needed here and we remove it. We need some general properties of Polish groups and standard spaces, for which references are given in [**25**] and [**16**, §3] (however, the result on Λ_φ given in [**16**] is ambiguous since no definition of a Dirichlet subgroup for a general LCA group is given there).

Recall that Σ_φ may be considered as a quotient of a closed subgroup of $U(X \times G, \mu \otimes \lambda)$, namely of the subgroup $\mathcal{E}_0(T_\varphi)$ of all L^∞-eigenfunctions of T_φ which can be written as $f = j \otimes \chi$ (§1.1). Similarly, let $\mathcal{I}_0(T_\varphi)$ be the closed subgroup of $U(X \times G, \mu \otimes \lambda)$ of all T_φ-invariant functions of this form. Clearly $j \otimes \chi$ is T_φ-invariant iff $(\chi \circ \varphi)(j \circ T) = j$, so Λ_φ is the image of $\mathcal{I}_0(T_\varphi)$ under the continuous homomorphism $j \otimes \chi \mapsto \chi$. It follows that Λ_φ is a Borel subgroup of $\widehat{G}$ and that there exists a Borel cross section $\Lambda_\varphi \to \mathcal{I}_0(T_\varphi)$, hence we can measurably choose a transfer function j_χ for $\chi \in \Lambda_\varphi$. Furthermore, given a positive measure σ concentrated on Λ_φ, as $\chi \mapsto j_\chi$ is a measurable mapping from Λ_φ to $U(X, \mu)$, there exists a function $J \in U(X \times \widehat{G}, \mu \otimes \sigma)$ such that $J(x, \chi) = j_\chi(x)$ $\mu \otimes \sigma$-a.e., whence

$$\chi(\varphi(x)) = J(x, \chi)/J(Tx, \chi) \quad \mu \otimes \sigma\text{-a.e.}$$

If we denote by $g \mapsto \tilde{g}$ the natural homomorphism from G to $U(\Lambda_\varphi, \sigma)$ given by the duality, that is by $\tilde{g}(\chi) = \chi(g)$, and $\tilde{\varphi}(x) = \widetilde{\varphi(x)}$, this equation becomes

$$\tilde{\varphi}(x) = J(x, \cdot)/J(Tx, \cdot) \quad \mu \otimes \sigma\text{-a.e.}, \tag{A.1}$$

so that the $U(\widehat{G}, \sigma)$-valued cocycle $\tilde{\varphi}$ is a coboundary.

On the other hand, $\tilde{\varphi}$ takes its values in the subgroup $G(\sigma) = \{\tilde{g} : g \in G\}$ of $U(\widehat{G}, \sigma)$. A main part of the argument is then the classical fact that $\tilde{\varphi}$ will still be a coboundary as a $G(\sigma)$-valued cocycle if this subgroup is closed. If σ is full then $G(\sigma)$ is actually isomorphic to G [**15, 25**]. When we only suppose that Λ_φ is not Dirichlet, we obtain the following.

PROPOSITION A.1. *Let T be any automorphism of a standard probability space and let $\varphi : X \to G$ be a cocycle. If Λ_φ is not a Dirichlet subgroup of $\widehat{G}$, then φ is cohomologous to a cocycle taking values in a compact subgroup of G and Λ_φ is open.*

Proof. Choose for σ a probability measure concentrated on Λ_φ which is not Dirichlet, i.e. $\limsup_{g \to \infty} \operatorname{Re} \hat{\sigma}(g) < 1$. The kernel of the homomorphism $g \mapsto \tilde{g}$ is the closed

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subgroup $H = \{g \in G : \tilde{g}(\chi) = \chi(g) = 1, \sigma\text{-a.e.}\} = \{g \in G : \hat{\sigma}(g) = 1\}$. Since σ is not Dirichlet, H is compact and, moreover, $\tilde{g}_k \to 1$ in $U(\widehat{G}, \sigma)$ iff the sequence (g_k) stays in some compact set and all its limit points are in H; that is, iff it converges to 1 in G/H. Thus $G(\sigma)$ is isomorphic to G/H and it follows, in particular, that $G(\sigma)$ is closed in $U(\widehat{G}, \sigma)$. Hence $\tilde{\varphi}$ is still a coboundary as a $G(\sigma)$-valued cocycle and composing the transfer equation with a measurable cross section $G(\sigma) \to G$ yields that φ is cohomologous to a cocycle with values in H.

As for the last assertion, Λ_φ contains the annihilator $H^\perp = \{\chi \in \widehat{G} : \chi(g) = 1$ for all $g \in H\}$ of H. Since H is compact, $H^\perp$ is open and so is Λ_φ. □

We are not able to directly apply the same technique with Σ_φ instead of Λ_φ, but the next lemma shows that these groups belong to the same class.

By $D(\varphi) : X \times X \to G$ we denote the cocycle $X \times X \to G$ defined by $D(\varphi)(x_1, x_2) = \varphi(x_1) - \varphi(x_2)$ for the action of the direct product $T \times T$. Also, to a cocycle $\psi : X \to \mathbb{T}$, we associate the operator $V_\psi : f \mapsto e^{2\pi i\psi} f \circ T$ on $L^2(X, \mu)$. Notice that ψ is a quasi-coboundary (respectively a coboundary) iff V_ψ has an eigenvalue (respectively has the eigenvalue 1) with a corresponding eigenfunction of modulus 1.

LEMMA A.1. *Assume that T is ergodic and let $\varphi : X \to G$ be any cocycle. Then $\Sigma_\varphi = \Lambda_{D(\varphi)}$.*

Proof. We have to prove, for $\chi \in \widehat{G}$, that $\chi \circ \varphi$ is a quasi-coboundary iff $\chi \circ D(\varphi)$ is a coboundary or, equivalently, that a cocycle $\psi : X \to \mathbb{T}$ is a quasi-coboundary iff $D(\psi)$ is a coboundary.

The 'only if' part is obvious. Suppose that $D(\psi)$ is a coboundary. Then 1 is an eigenvalue of $V_{D(\psi)}$. However, if we identify $L^2(X \times X, \mu \otimes \mu)$ with $L^2(X, \mu) \otimes L^2(X, \mu)$, then the operator $V_{D(\psi)}$ is simply $V_\psi \otimes V_{-\psi}$. So 1 is an eigenvalue of $V_\psi \otimes V_{-\psi}$. From elementary spectral theory it follows that V_ψ has an eigenvalue and, since T is ergodic, any corresponding eigenfunction must have constant modulus. This proves that ψ is a quasi-coboundary. □

Proof of Theorem 1.1. Assume that T is ergodic and Σ_φ is not Dirichlet. We first have to find a compact subgroup H of G and a second countable LCA group $\tilde{G}$ such that the quotient cocycle is a quasi-coboundary as a $\tilde{G}$-valued cocycle. Now, it follows from Lemma A.1 and Proposition A.1 that $\Sigma_\varphi = \Lambda_{D(\varphi)}$ is open (we apply Proposition A.1 with $T \times T$ instead of T but recall that the ergodicity is not needed there). In particular, Σ_φ is locally compact in the topology inherited from $\widehat{G}$. Also, the annihilator ${\Sigma_\varphi}^\perp$ of Σ_φ in G is compact and this will be our subgroup H.

We define $\tilde{G}$ by duality. Consider the intermediate group $\tilde{\Sigma}_\varphi$ of all pairs (χ, t) in $\widehat{G} \times \mathbb{T}$ for which there exists $j \in U(X, \mu)$ satisfying

$$\chi(\varphi(x)) = e^{2\pi it} j(x)/j(Tx) \quad \mu\text{-a.e.}; \tag{A.2}$$

that is, the image of $\mathcal{E}_0(T_\varphi)$ by the continuous homomorphism which to each eigenfunction of the form $j \otimes \chi$ associates χ and the corresponding eigenvalue t. Again, $\tilde{\Sigma}_\varphi$ is a Borel subgroup of $\widehat{G} \times \mathbb{T}$ and the quotient topology yields a Polish topology on $\tilde{\Sigma}_\varphi$. In addition, the homomorphism $j \otimes \chi \mapsto \chi$ from $\mathcal{E}_0(T_\varphi)$ to $\widehat{G}$ factorizes through $\tilde{\Sigma}_\varphi$

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and the projection $\pi : (\chi, t) \mapsto \chi$, $\tilde{\Sigma}_\varphi \to \widehat{G}$ which is hence continuous. Observe that, up to obvious identification, $\ker \pi$ is the eigenvalue group $e(T)$ of T. The quotient map $\tilde{\Sigma}_\varphi / e(T) \to \Sigma_\varphi \subset \widehat{G}$ is a continuous one-to-one homomorphism of a Polish group onto a locally compact group. Thus, it must be an isomorphism and $\tilde{\Sigma}_\varphi / e(T)$ is locally compact. Moreover, since $e(T)$ is countable, hence discrete in $\tilde{\Sigma}_\varphi$, it follows that $\tilde{\Sigma}_\varphi$ itself is locally compact.

We let $\tilde{G}$ be the dual group $\hat{\tilde{\Sigma}}_\varphi$. By duality, $G/H = \hat{\Sigma}_\varphi$ is embedded in $\tilde{G}$ as a closed subgroup. More precisely, this embedding is given by the homomorphism $g \to \tilde{g}$, $G \to \tilde{G}$, where $\tilde{g}(\chi, t) = \chi(g)$. We also define $\tilde{\varphi} : x \to \widetilde{\varphi(x)}$ by composition.

Now, if we choose a probability measure $\tilde{\sigma}$ equivalent to the Haar measure of $\tilde{\Sigma}_\varphi$, then $\tilde{G}$ appears as a closed subgroup of $U(\tilde{\Sigma}_\varphi, \tilde{\sigma})$ and, as from (A.2), we can find a function $J \in U(X \times \tilde{\Sigma}_\varphi, \mu \otimes \tilde{\sigma})$ such that

$$\chi(\varphi(x)) = e^{2\pi i t} J(x, (\chi, t))/J(Tx, (\chi, t)) \quad \mu \otimes \tilde{\sigma}\text{-a.e.}$$

Let c be the character $(\chi, t) \mapsto e^{2\pi i t}$ of $\tilde{\Sigma}_\varphi$. Since $\chi(\varphi(x)) = \tilde{\varphi}(x)(\chi, t)$, the above equation becomes

$$\tilde{\varphi}(x) = c\, J(x, \cdot)/J(Tx, \cdot) \quad \mu \otimes \tilde{\sigma}\text{-a.e.}$$

Hence, as a $U(\tilde{\sigma})$-valued cocycle, $\tilde{\varphi}$ is cohomologous to the constant $c \in \tilde{G}$, but then it is still cohomologous to c as a $\tilde{G}$-valued cocycle, which proves the main assertion of the theorem.

Finally, assume moreover that φ is recurrent. Then $\tilde{\varphi}$ is recurrent and so is the cohomologous constant cocycle $x \mapsto c$. Therefore, the monothetic group K generated by c cannot be isomorphic to $\mathbb{Z}$ and so K is compact. It follows that $\Lambda_{\tilde{\varphi}}$ contains the open subgroup $K^\perp$ of $\tilde{\Sigma}_\varphi = \hat{\tilde{G}}$. However, from $\chi(\varphi(x)) = \tilde{\varphi}(\chi, t)$, Λ_φ is clearly the image of $\Lambda_{\tilde{\varphi}}$ under the projection $\pi : \tilde{\Sigma}_\varphi \to \Sigma_\varphi$ and thus Λ_φ is also open in $\widehat{G}$. We conclude by Proposition A.1 that φ is cohomologous to a cocycle with values in a compact subgroup of G. □

Remark A.1. If G is divisible, then G/H is a direct factor of $\tilde{G}$ and we can compose the transfer equation by a projection $\tilde{G} \to G/H$. Thus the quotient cocycle is already a quasi-coboundary as a G/H-valued cocycle. This was already noted by Moore and Schmidt [**25**] in the case $G = \mathbb{R}^n$. Also, if T is weakly mixing, we get in the construction $\tilde{\Sigma}_\varphi = \Sigma_\varphi$ and $\tilde{G} = G/H$.

Note that more generally, if T is weakly mixing but we allow Σ_φ to be Dirichlet, the eigenvalue t in (A.2) depends only on χ and we can write $e^{2\pi i t} = c(\chi)$ where c is a measurable function of modulus 1 on Σ_φ. Given a positive measure σ on Σ_φ, we can then find $J \in U(X \times \Sigma_\varphi, \mu \otimes \sigma)$ so that

$$\chi(\varphi(x)) = c(\chi) J(x, \chi)/J(Tx, \chi) \quad \mu \otimes \sigma\text{-a.e.} \tag{A.3}$$

Thus, $\tilde{\varphi} : X \to U(\Sigma_\varphi, \sigma)$ given by $\tilde{\varphi}(x)(\chi) = \chi(\varphi(x))$ is cohomologous to c. It follows in particular that, when φ is recurrent, so is the constant cocycle $x \mapsto c \in U(\Sigma_\varphi, \sigma)$ and there exists a sequence (n_k) going to infinity such that $c(\chi)^{n_k} \to 1$ σ-a.e.

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A.2. *When mixing lifts.* Assume that T is mixing. In [**21**], it is proved that $T_{\varphi,\mathcal{S}}$ is mixing when $\sigma_{\mathcal{S}}(\Sigma_{k\varphi}) = 0$ for all $k \neq 0$. Thanks to the above analysis, we are able to show that this is the case if we only suppose $\sigma_{\mathcal{S}}(\Sigma_{\varphi}) = 0$ and, moreover, that this condition is necessary when φ is recurrent. For the direct part, the proof is essentially the same as the proof in [**21**], except for the following lemma.

LEMMA A.2. *Assume that T is mixing and let $\psi : X \to \mathbb{T}$ be a cocycle. If ψ is not a quasi-coboundary then, for every $f \in L^2(X, \mu)$,*

$$\int_X e^{2\pi i \psi^{(n)}(x)} f(T^n x)\overline{f(x)}\, d\mu(x) \to 0$$

as $n \to \infty$.

Proof. Consider the skew product T_ψ on $X \times \mathbb{T}$ ($\mathbb{T}$ is equipped with its Lebesgue measure $\lambda_{\mathbb{T}}$) and let e denote the exponential function $y \mapsto e^{2\pi i y}$. For $f \in L^2(X, \mu)$, we have $f \otimes e(T_\psi(x, y)) = e^{2\pi i \psi(x)} f(Tx)e(y)$, so that the integral in the statement of this lemma can be written as

$$\int_{X\times\mathbb{T}} (f \otimes e) \circ T_\psi^n \, \overline{(f \otimes e)}\, d\mu \otimes \lambda_{\mathbb{T}}. \tag{A.4}$$

Suppose that this integral does not converge to zero for some $f \in L^2(X, \mu)$, which we can suppose bounded. Then we can find a subsequence (n_k) going to infinity such that the corresponding graph-joinings of T_ψ converge weakly towards some $\eta \in J_2(T_\psi)$ such that

$$\int_{X\times\mathbb{T}\times X\times\mathbb{T}} f \otimes e \otimes \overline{f} \otimes \overline{e}\, d\eta \neq 0. \tag{A.5}$$

However, since T is mixing, the projection of η on $X \times X$ is equal to $\mu \otimes \mu$. Let $h \in L^2(X \times X, \mu \otimes \mu)$ be the conditional expectation of $(x_1, y_1, x_2, y_2) \mapsto e(y_1)\overline{e(y_2)}$ with respect to the σ-algebra generated by the x-coordinates, for the measure η. So, for every $f_1, f_2 \in L^2(X, \mu)$,

$$\int_{X\times\mathbb{T}\times X\times\mathbb{T}} f_1 \otimes e \otimes f_2 \otimes \overline{e}\, d\eta = \int_{X\times X} (f_1 \otimes f_2)h\, d\mu \otimes \mu. \tag{A.6}$$

Now, we express the $T_\psi \times T_\psi$-invariance of η. In terms of the operators V_ψ (defined before Lemma A.1), we have $(f_1 \otimes e) \circ T_\psi = V_\psi f_1 \otimes e$ and $(f_2 \otimes \overline{e}) \circ T_\psi = V_{-\psi} f_2 \otimes \overline{e}$. So, from (A.6) we get

$$\int_{X\times X} (V_\psi f_1 \otimes V_{-\psi} f_2)h\, d\mu \otimes \mu = \int_{X\times X} (f_1 \otimes f_2)h\, d\mu \otimes \mu.$$

Because of (A.5), h is not almost everywhere equal to zero. This proves that $V_\psi \otimes V_{-\psi} = V_{D(\psi)}$ has a nonidentically zero invariant function and we conclude as in Lemma A.1 that ψ is a quasi-coboundary. □

THEOREM A.1. *Assume that T is mixing. If $\sigma_{\mathcal{S}}(\Sigma_\varphi) = 0$ then $T_{\varphi,\mathcal{S}}$ is mixing. Conversely, if $\sigma_{\mathcal{S}}(\Sigma_\varphi) > 0$ and φ is recurrent, then $T_{\varphi,\mathcal{S}}$ is not mixing.*

Proof. Since T is mixing, $T_{\varphi,\mathcal{S}}$ is mixing iff $\hat{\sigma}_{f\otimes h, T_{\varphi,\mathcal{S}}}(n) \to 0$ for every $f \in L^2(X, \mu)$ and for every zero-mean $h \in L^2(Y, \nu)$. For sake of simplicity we denote $\sigma_{f\otimes h} = \sigma_{f\otimes h, T_{\varphi,\mathcal{S}}}$

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and $\sigma_h = \sigma_{h,\mathcal{S}}$. Following [**21**], we write

$$\begin{aligned}\hat{\sigma}_{f\otimes h}(n) &= \int_X f(T^n x)\overline{f(x)}\left(\int_Y h(S_{\varphi^{(n)}(x)}y)\overline{h(y)}\,d\nu(y)\right)d\mu(x)\\ &= \int_X f(T^n x)\overline{f(x)}\hat{\sigma}_h(\varphi^{(n)}(x))\,d\mu(x)\\ &= \int_X f(T^n x)\overline{f(x)}\left(\int_{\widehat{G}}\chi(\varphi^{(n)}(x))\,d\sigma_h(\chi)\right)d\mu(x)\end{aligned}$$

and so

$$\hat{\sigma}_{f\otimes h}(n) = \int_{\widehat{G}}\left(\int_X \chi(\varphi^{(n)}(x))f(T^n x)\overline{f(x)}\,d\mu(x)\right)d\sigma_h(\chi). \tag{A.7}$$

According to Lemma A.2, the inner integral converges to zero whenever $\chi\circ\varphi$ is not a quasi-coboundary, i.e. whenever $\chi\notin\Sigma_\varphi$. So, if $\sigma_{\mathcal{S}}(\Sigma_\varphi)=0$, it converges σ_h-a.e. to zero and $\hat{\sigma}_{f\otimes h}(n)\to 0$ follows.

Conversely, assume that φ is recurrent and that $\sigma_{\mathcal{S}}(\Sigma_\varphi)>0$. We choose a zero-mean $h\in L^2(Y,\nu)$, not almost everywhere equal to zero and such that σ_h is concentrated on Σ_φ, so that we can restrict the outer integral in (A.7) to Σ_φ. The function f will be chosen later on.

Since T is weakly mixing, according to Remark A.1, there is a measurable function J of modulus 1 on $X\times\Sigma_\varphi$ satisfying (A.3) with $\sigma=\sigma_h$, where $c(\chi)$ is the unique constant which is cohomologous to $\chi\circ\varphi$. When we replace $\chi(\varphi^{(n)}(x))$ according to (A.3) in (A.7), we get

$$\hat{\sigma}_{f\otimes h}(n) = \int_{\Sigma_\varphi}\left(\int_X c(\chi)^n J(x,\chi)\overline{J(T^n x,\chi)}f(T^n x)\overline{f(x)}\,d\mu(x)\right)d\sigma_h(\chi).$$

Moreover, following Remark A.1, since φ is recurrent, there exists a sequence (n_k) going to infinity such that $c(\chi)^{n_k}\to 1$ σ_h-a.e. Then, since T is mixing, we find

$$\lim\hat{\sigma}_{f\otimes h}(n_k) = \int_{\Sigma_\varphi}\left|\int_X J(x,\chi)\overline{f(x)}\,d\mu(x)\right|^2 d\sigma_h(\chi).$$

It remains to note that, since $\chi\mapsto J(\cdot,\chi)$ is a measurable function from Σ_φ to $U(X,\mu)$, we can choose the function $f\in L^2(X,\mu)$ so that the above limit is non-zero. □

Remark A.2. By Theorem 6.1, assuming that T is mixing, if $T_{\varphi,\mathcal{S}}$ is not mixing, it is not even mildly mixing.

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