THE REVIEW OF SYMBOLIC LOGIC
Volume 7, Number 3, September 2014

HARMONIOUS RULES FOR IDENTITY

OWEN GRIFFITHS

Abstract. Logical inferentialists claim that the meanings of the logical constants are given by their inference rules. To rule out *tonk*-like expressions, it is often demanded that pairs of inference rules must be *harmonious*. The usual inference rules for the identity predicate are not harmonious, but most inferentialists want identity to be logical. Stephen Read has tried to formulate alternative, harmonious inference rules for identity. It will be proved, however, that his rules are precisely as strong as the old rules and that, because the old rules are not harmonious (as Read argues), nor are his. Further, it will be shown that no *sound* rules will be any improvement. Identity remains in need of satisfactory inferentialist treatment.

Logical inferentialists claim that the meanings of the logical constants are given by their introduction and elimination rules.[1] Not just any inference rules are acceptable, however. In order to rule out trivializing pairs of inference rules, such as those for *tonk*, a common inferentialist requirement is that the rules must be *harmonious*. But the usual rules governing identity are not harmonious. Read (2004, 2012) tries to formulate alternative, harmonious rules and, although his method is plausible at first glance, I shall argue that it fails to improve on the usual, unharmonious rules. Further, I show that no *sound* inference rules for identity can do any better, and explore the consequences for inferentialism. Identity remains in need of satisfactory inferentialist treatment.

§1. Inferentialism and harmony. Inferentialism, generally, is the view that the meaning of an expression is given by the rules for its use. Thus Dummett:

> Crudely expressed, there are always two aspects of the use of a given form of sentence: the conditions under which an utterance of that sentence is appropriate, which include, in the case of an assertoric sentence, what counts as an acceptable ground for asserting it; and the consequences of an utterance of it, which comprise both what the speaker commits himself to by the utterance and the appropriate response on the part of the hearer, including, in the case of assertion, what he is entitled to infer from it if he accepts it. (Dummett, 1973, 396)

Received: August 15, 2013.

[1] *Logical* inferentialism is generally contrasted with the *global* inferentialism of, e.g., Brandom (1994), who holds that the meanings of *all* expressions can be given by introduction and elimination rules. I will only be concerned with logical inferentialism in this paper but, if my argument is effective against an inferentialist treatment of this logical fragment of language, then it is of course effective against the stronger form of inferentialism too.

© Association for Symbolic Logic, 2014
doi:10.1017/S1755020314000161

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press

Dummett separates the principles that regulate linguistic practices into introduction (I-) principles and elimination (E-) principles:

I-principles: give the conditions under which an assertion of a sentence containing a certain subsentential expression is warranted;
E-principles: give the consequences of the assertion of a sentence containing the same subsentential expression.

Dummett ties I-principles to a broadly verificationist understanding of meaning: the content of a statement is determined by what it would take to verify it. E-principles, however, get tied to a pragmatist theory of meaning: the content of a statement is determined by the consequences for one who accepts it as true (see, e.g., Dummett, 1991, 212). These I- and E-principles are used to explain the contribution that subsentential expressions make to sentences in which they occur.

Dummett argues that, for a language to be in good working order, 'we may legitimately demand a certain consonance between the two aspects of the use of a given form of expression' (Dummett, 1973, 397). In particular, the E-principles should allow us to infer no more and no less than can be deduced directly from the premises of the corresponding I-principles. When this balance is in place, the I- and E-principles are in *harmony*.[2]

This is an outline of Dummett's use-theoretic approach to meaning in general. He applies this approach specifically to the logical constants in, most notably, his (1991, 215-7). In the case of the logical constants, the demand for harmony is motivated by what Steinberger (2011) calls the *principle of innocence*:

Principle of innocence: we should not be able, by deductive reasoning alone, to discover nonlogical truths that we would not have been able to discover independently of logic.

Logical inference should allow us to manipulate truths that we have already discovered, but should not allow us to discover any previously unknown atomic truths. Steinberger further motivates the principle by pointing out that 'logic ought not to tinker with the meanings of the nonlogical expressions' (2011, 620). Again, when logic is innocent, we can be confident that it does not interfere with the meanings of nonlogical expressions in this sense.

The I- and E-principles for the logical constants are obviously their introduction and elimination rules. For pairs of inference rules to be in harmony, the E-rules ought to exploit *all* and *only* the inferential powers given by the I-rules.

When this balance fails, the I- and E-principles are in disharmony, which Steinberger (2009, 656) notes can take one of two forms:

E-strong disharmony: the E-principles are unduly permissive relative to the corresponding I-principles;
E-weak disharmony the E-principles are unduly prohibitive relative to the corresponding I-principles.

Of course, E-strong disharmony could be called I-weak disharmony, and E-weak disharmony could be called I-strong disharmony. The decision to take the I-principles as the

[2] This is an intuitive understanding of harmony, which Dummett (1991) tries to explicate with a number of formal tests e.g., using the Gentzen-Prawitz *inversion principle*. See Steinberger (2011) for a discussion of Dummett's various tests for harmony.

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press

standard against which to measure the strength of the E-principles follows Gentzen's original privileging of I-principles:

> the introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions (Gentzen, 1969, 80)

We could instead privilege the E-principles, and contend that the meanings of logical expressions are given by their elimination rules. This is what Rumfitt (2000, 790) claims is true of the material conditional, and Dummett (1991, 275) claims is true of the universal quantifier. Or we could adopt a hybrid view for some expressions and claim that their meaning is shared between the I- and E-principles. Nothing in this paper requires me to take a stance on the question of whether I-rules, E-rules or both are the meaning-constituting rules: I argue that the rules governing identity are not harmonious, so they fail regardless of whether the I-rules, E-rules, or both, are privileged.

This is an informal characterization of harmony, which many writers have attempted to capture with a precise, formal notion. For example, Read's (2010, 563) preferred formal notion is general elimination harmony, which I will discuss in more detail in Section 4.[3]

I do not need to commit myself to any particular formal test for harmony in this paper. It will be sufficient for my purposes to argue that the rules for identity, in either the traditional form or Read's new form, fail to be harmonious by the informal understanding of harmony.

§2. Identity and harmony. The usual introduction (I-) and elimination (E-) rules for the identity predicate are:

$$\frac{}{a = a}\ {=}\mathrm{I} \qquad \frac{a = b \quad A}{A(a/b)}\ {=}\mathrm{E}$$

The =I rule captures the reflexivity of identity, while the =E rule follows from the indiscernibility of identicals (Leibniz's law). In =E, $A(a/b)$ is the result of replacing any or all of the occurrences of a in A with b.

Pairs of rules are in harmony 'when the E-rules do no more and no less than spell out what may be inferred from the assertion of the conclusion of the I-rules, given the grounds for its assertion' (Read, 2010, 562). Against this standard, =I and =E are clearly not harmonious. As Milne (2007, 37) notes, 'we are stuck as far as harmony is concerned because the formulas introduced by the =-I ($t = t$ and the like) are distinct from the formulas eliminated by =-E ($t = u$ and its ilk)'. In particular, =I and =E display *E-strong disharmony* because the E-rule is unduly permissive relative to the I-rule.

The challenge for inferentialists is to formulate harmonious first-order inference rules for identity.[4] As far as I am aware, the only attempt to give identity harmonious treatment is that of Read (2004, 2012), who adopts the most natural strategy: since =E captures the indiscernibility of identicals, the introduction rule should capture the *identity of indiscernibles*.[5] He is limited to the resources of first-order logic in this project.

---

[3] See Steinberger (2011) for a taxonomy of precise definitions of harmony.

[4] Of course, identity can be *defined* in second-order logic.

[5] The identity of indiscernibles is a controversial principle. If you reject the principle, then you will reject Read's =I′ from the start. Here I will assume that the identity of indiscernibles is correct, and argue that Read's proposal fails elsewhere.

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press

His introduction rule states that we may conclude $a = b$ if *every* predicate true of a is true of b:[6]

$$\begin{array}{c} [Fa] \\ \vdots \\ \dfrac{Fb}{a = b}\ =\mathrm{I}' \end{array}$$

In $=\mathrm{I}'$, F is a schematic variable ranging over predicates of any adicity and degree of complexity. F is used schematically because, for a and b to be indiscernible, *any* predicate true of a must be true of b. Further, Read (2004, 116) demands that F must not appear in any assumption other than Fa. This restriction is essential because, if F were allowed to appear in other assumptions, the following derivation would be possible, using $\vee$I and disjunctive syllogism (DS):

$$\dfrac{\neg Fa \quad \dfrac{[Fa]}{Fa \vee Fb}\ \vee\mathrm{I}}{\dfrac{Fb}{a = b}\ =\mathrm{I}'}\ \mathrm{DS}$$

But this is plainly invalid: assign 2 to a, 1 to b and the set of odd numbers to F.[7]

§3. Mixed identity. Read's identity rules have met with general approval by harmony-minded proof theorists (e.g., Milne, 2007; Francez & Dyckhoff, 2010; and Restall, 2012). To my knowledge, Read's only critic is Kremer (2007), who accepts Read's rule but corrects his soundness proof. Kremer's correction is needed, as Read (2012) admits, but he is wrong to accept that Read's $=\mathrm{I}'$ rule is an improvement, as I will now argue.

According to Read's $=\mathrm{I}'$ rule, when will the grounds for asserting $a = b$ hold? If we assume Fa, we can trivially deduce Fa and conclude $a = a$, so $=\mathrm{I}'$ does at least specify when we may conclude $a = a$. But when does it allow the assertion of a mixed identity statement, such as $a = b$, for distinct a and b?

Consider the circumstances under which a derivation of Fb from Fa and some other premises not involving F will be possible. One possibility is that (i) the identity predicate already features amongst the collateral premises in such a way as to allow the derivation of $a = b$. For example:

$$\dfrac{\dfrac{[Fa] \quad a = c}{Fc}\ =\mathrm{E} \qquad c = b}{\dfrac{Fb}{a = b}\ =\mathrm{I}'}\ =\mathrm{E}$$

Another possibility is that (ii) amongst the collateral premises, there is a contradiction not involving F, such as $Ga \wedge \neg Ga$, so Fb can be derived by *ex falso quodlibet* (EFQ).

In fact, (i) and (ii) are the *only* conditions under which Fb can be derived. Take the set of premises $\{Fa\} \cup X$, where X is a (possibly empty) set of premises not containing predicate

[6] Read's $=\mathrm{I}'$ rule is clearly not acceptable to many free logicians without the additional premise $\exists x\ x = b$. Milne (2007, 39) formulates inference rules for the identity predicate in the spirit of Read's that are acceptable to free logicians. I only argue here against Read, but my arguments also apply against Milne.

[7] Read, of course, is a relevant logician who could resist this derivation by blocking the appropriate form of DS (see e.g., Read, 1981). There is no hint in Read (2004), however, that he only intends $=$I to be restricted to relevant logic.

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press

variable F. Assume for simplicity that F ranges over atomic predicates only. Since Read's rule must apply to all predicates of any complexity, *including the atomic predicates*, it is sufficient to show that $\{Fa\} \cup X \nvdash Fb$ for atomic F.

I will show that (i) and (ii) are in fact the *only* circumstances under which the premises of Read's =I′ can obtain by showing that there is no derivation of Fb from $\{Fa\} \cup X$ when (i) and (ii) are excluded. The derivation of Fb by (i) is ruled out because the language does not contain identity. To rule out the derivation of Fb by (ii), assume $\{Fa\} \cup X \nvdash \bot$. With this setup, I will prove:

THEOREM. *There is no derivation of Fb from $\{Fa\} \cup X$, where a and b are distinct and F does not appear in X, unless X contains the identity predicate or $\{Fa\} \cup X \vdash \bot$.*

I will proceed semantically, because the proof is significantly quicker. Read cannot object to my use of soundness and completeness, as he justifies his =I′ rule in this way (2004, 117-9). Of course, other inferentialists could reject as illegitimate any attempt to measure their formalism against model theory. For this reason, I offer a longer, purely syntactic, proof of the theorem in the appendix.

We are assuming that $\{Fa\} \cup X$ is consistent so, by completeness, there is a model $\mathcal{M}$ of $\{Fa\} \cup X$ (on an interpretation). Let a denote the object o in the domain of $\mathcal{M}$. Define a new model $\mathcal{M}'$ such that the domain of $\mathcal{M}'$ is the domain of $\mathcal{M}$ with one additional object o'. For all atomic predicates other than F, expand their extensions in $\mathcal{M}$ to $\mathcal{M}'$ by treating o' the same as o. Let all names have the same extension in $\mathcal{M}$ as $\mathcal{M}'$ except b, and let b denote o' in $\mathcal{M}'$. The extension of F in $\mathcal{M}'$ is $\{o\}$. Now $\mathcal{M}'$ is a model of $\{Fa\} \cup X$, but not of Fb. Therefore $\{Fa\} \cup X \not\models Fb$. By soundness, $\{Fa\} \cup X \nvdash Fb$.

Why should this result trouble the inferentialist? It shows that, by Read's rule, the assertion of $a = a$ is always warranted, but that assertion of mixed identity statements is only warranted when we already have identity among the collateral premises or a contradiction. But, of course, these are *precisely* the circumstances under which we can make assertions of identity with the old identity rules. Read's rule is, therefore, equivalent to the old identity rule, in the sense that both rules allow the warranted assertion of identity statements, mixed or otherwise, under exactly the same circumstances.

But now Read's claim that the old identity introduction rule is 'too weak' (2004, 115) to justify Leibniz's Law is incorrect: his suggested rule is *precisely* as strong as the old rule, so if the old rule is too weak, then so is Read's rule. When he justifies the harmony of his proposed pair of rules, Read writes that 'Something follows from $a = b$ if and only if it follows from the grounds for the assertion of $a = b$, that is, from a derivation of Fb from Fa for some predicate variable F' (2004, 116). But a derivation of Fb from Fa will only occur when we either have identity amongst the collateral premises, or a contradiction. And in these circumstances, we would similarly have a derivation of Fb from Fa with the old introduction rule. If Read is correct, therefore, that his =I′ is harmonious with =E because something follows from $a = b$ iff it follows from the grounds for asserting $a = b$, then the old identity rules =I and =E must also be harmonious. But, as we have seen, and as Read accepts, the old rules are not harmonious, so nor are Read's.

§4. Inferentialist responses. One response that the inferentialist could make is that I have actually shown the familiar identity rules, =I and =E, to be harmonious: they are precisely as strong as Read's =I′ and =E, and Read argues that his are harmonious, so the old rules must be harmonious too.

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press

The veracity of this response depends on how we understand 'harmonious'. If we intend, as I have throughout this paper, the intuitive concept of two rules' being exactly commensurate in strength, then the response has no force. It is uncontroversial that the original rules are unharmonious by this intuitive notion: all of the inferentialists I have found who comment on identity, including Read, admit this. Indeed, that =I concerns only a subclass of the formulas with which the E-rule is concerned is a paradigmatic case of E-strong disharmony. It follows that Read's rules must be similarly disharmonious, by this notion, as I have shown that they are *precisely* as strong as the familiar rules.

If we instead understand by 'harmony' not the intuitive notion but some formal model of this notion, such as Read's general elimination (GE) harmony, then the response is also unconvincing. Read demonstrates that his =I′ rule is in general elimination harmony with =E, and *states* that the old rules are unharmonious by the same standard, though offers no proof of this. In fact, it is unclear quite how the old identity rules would fail to be in GE harmony. To see why, it is necessary to say a little more about this test for harmony.

Let a local peak with respect to some candidate logical constant \$ in a proof of a conclusion, C, from a set of premises, Γ, occur where \$ is introduced and *immediately* eliminated. The inference rules for \$ are in GE harmony if there is a reduction procedure by which all local peaks with respect to \$ can be eliminated.

Consider, for example, the following proof in which & is introduced and then eliminated:

$$\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A \& B}\ \&\text{I}}{A}\ \&\text{E}$$

This local peak with respect to & can straightforwardly be eliminated:

$$\begin{array}{c}\vdots \\ A\end{array}$$

But, of course, the only local peak that can be set up involving the old identity rules involves self-identity statements, $a = a$, and these *can* be eliminated, because

$$\dfrac{\dfrac{}{a = a}\ =\text{I} \quad \begin{array}{c}\vdots \\ Fa\end{array}}{Fa}\ =\text{E}$$

reduces to

$$\begin{array}{c}\vdots \\ Fa\end{array}$$

If this is correct, then the old identity rules *are* in GE harmony. But this should not reassure the inferentialist, as it merely demonstrates that their formal test for harmony has misfired: it has *overgenerated* by finding harmony where there is none. If this demonstration is incorrect, then the old identity rules may not be in GE harmony. But, again, this shows a deficiency in the test for GE harmony: it was intended to model the intuitive notion of harmony but judges =I and =E differently to =I′ and =E, which are precisely the same in strength. Again the test has misfired. Indeed, we may worry that Read's particular test is problematic for independent reasons, as Steinberger (2011, 628–632) shows (Steinberger's arguments are aimed at Dummett's notion of *intrinsic* harmony, but they apply equally

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press

to Read's). In short, the inferentialist cannot avoid my argument by moving from talk of the intuitive notion to some particular formal model.

Instead, they could claim that my talk of 'the intuitive notion' is misguided. The intuitive notion, they may claim, does not merely concern the relative strength of logical rules, but also the way in which those rules are *presented*. It is true, they may claim, that $=$I and $=$I′ are precisely equal in strength, but this does not entail that they are equal in terms of harmony, for they are presented differently, and it is in Read's presentation that the harmony becomes apparent.

My concern with this form of response is that a *presentational* notion of harmony is not one that any inferentialists I have found discuss. Indeed, it is not obvious how such a notion could be motivated. The motivation for harmony, we have seen, is the principle of innocence: we do not want to be able, by logical reasoning alone, to discover novel truths we would have been unable to discover independently of logic. To ensure innocence, a demand for equivalence of logical power is both motivated and natural. But no presentational criterion follows straightforwardly from this philosophical motivation. And the motivation to correctly adjudicate on the case of identity is obviously *ad hoc* and unsatisfactory. I leave it as a challenge to inferentialists therefore, to formulate an intuitive notion of harmony that includes a presentational criterion that delivers the correct results and is philosophically motivated.

Another possible response is that the theorem can be avoided by weakening the restriction that F must not appear in X. For example, we could allow F to appear in X but only in the antecedents of conditionals. This is not possible, however, as the result will be true of any *sound* introduction rule for identity.

Assume that X is satisfiable and does not contain identity. Then there is a model $\mathcal{M}$ of X. Let a denote the object o in the domain of $\mathcal{M}$. Define a new model $\mathcal{M}'$ with the same domain as $\mathcal{M}$ plus one object o'. All names have the same extension in $\mathcal{M}'$ as $\mathcal{M}$, except b, which has the extension $\{o'\}$ in $\mathcal{M}'$. Expand the extension of all atomic predicates in X to treat o' the same as o. Now $\mathcal{M}'$ is a model of X but not of $a = b$, so $X \not\models a = b$.

In this proof, we placed no restrictions on instances of F in X. Therefore, if X is satisfiable and does not contain the identity predicate, then it will never semantically entail a mixed identity sentence. If a rule for introducing identity could be put forward that avoided the theorem, it would therefore be unsound. Once again, this may not bother an inferentialist who refused to measure their proof theory against model-theoretic standards, but it prevents Read from trying to revise his rule as he explicitly wants his rule to be sound.

§5. Harmony and logical constanthood. What is the upshot of this discussion for inferentialists? One problem in answering that question is that inferentialists differ about just how formulability in terms of harmonious inference rules relates to logical constanthood.

Tennant, for example, takes the former to be both necessary and sufficient for the latter (see Tennant, 1997, Section 9.1). Dummett, however, takes harmonious formulability to be necessary but not sufficient for logical constanthood (see, e.g., Dummett, 1991: 215-7).

I have argued that identity has not been given harmonious inference rules, by our intuitive understanding of harmony. This is enough to show that it is not a logical constant for either Tennant or Dummett, both of whom accept that formulability in terms of harmonious inference rules is necessary for logical constanthood.

Read differs on the relation between harmony and logical constanthood from both Tennant and Dummett. In his most recent writings, Read holds that formulability in terms of harmonious inference rules is neither necessary nor sufficient for logical constanthood

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press

(see e.g., Read, 2010, 575). Rather, he believes that harmony is desirable because harmonious rules are perspicuous and allow the meaning to be contained in, and read from, the introduction rule (2010, 562), but it is not necessary for logical constanthood. For example, Read (2010, 561) notes that the usual inference rules governing '$\Diamond$' are not harmonious, though he takes it to be logical constant. That identity cannot be given harmonious treatment is insufficient, therefore, to show that it is not a logical constant in Read's sense.

Read avoids, therefore, the conclusion that identity is not a logical constant, because he denies that harmonious formulability is necessary for logical constanthood. This is only possible, however, because his criteria for logical constanthood are left completely underspecified. He offers no necessary features for constanthood, nor any sufficient features. He leaves us in need of an argument that the identity predicate could be a logical constant, despite the absence of harmonious rules. Further, if Read does claim that harmonious formulability is neither necessary nor sufficient for logical constanthood, he must give another reason for ruling out *tonk* as a legitimate logical constant. The usual inferentialist reason for denying *tonk* the status of logical constant is that harmony is necessary for logical constanthood. If Read is denied this reason, he must offer another.

The inferentialist about the logical constants is left with two options: either deny that identity is a logical constant, or formulate some new introduction rule that is harmonious.

§6. Identity and logical constanthood. The inferentialist may, at this point, accept that identity is not harmonious and so not a logical constant, but argue that this is unproblematic because we should not want identity to qualify as a logical constant. I do not think that this is a plausible, reply, however, because we have good reasons to accept identity as a logical constant.

First, there are good reasons for everyone to accept identity as a logical constant. This is because it allows us to capture the validity of a large number of intuitively valid arguments. There are, famously, many intuitively valid arguments (in the sense of, say, necessarily truth-preserving), whose formalizations are not valid by the canonical means of formalization. For example, the argument 'the ball is red; *so* the ball is colored' is plausibly valid, but would not normally get formalized as a valid argument.

If we wanted to capture the validity of this argument, one means of doing so would be to take the predicates 'x is red' and 'x is colored' as logical constants. But this seems to be an implausible way to go, since we are holding fixed items of vocabulary that do not intuitively seem logical merely to capture a small number of valid arguments.

Accepting identity as a logical constant, however, allows us to capture a large range of valid arguments, e.g., those involving locutions such as 'there are at most . . . ', 'there are at least . . . ', definite descriptions, and numerically definite quantification. Further, in a footnote, Dummett (1973, 22) argues that identity should be accepted as a logical constant because it allows the expression of 'quantifier conditions', which are the properties of predicates that are invariant under permutations of the elements of the domain. Again, we would not be able to express such conditions if we did not have identity as a logical constant.

There are good reasons, then, for everyone to accept identity as a logical constant. There are special reasons, however, for the *inferentialist* to accept identity as a logical constant. This is because inferentialists frequently appeal to identity in order to distinguish terms from predicates. It is obvious why this should be the case: inferentialists cannot distinguish terms as e.g., those subsentential expressions that refer to objects, since this is to invoke reference in just the way the inferentialist must avoid. Rather, they often claim that singular

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press

terms are those expressions that are able to flank the identity sign, and predicates are then defined as subsentential expressions formed by the deletion of singular terms (and perhaps replacing them with variables). Thus Brandom:

> The paradigm of subsentential explicating vocabulary is provided by identity locutions, which make substitutional commitments explicit. . . . these are semantically explicating locutions, which make explicit the inferential involvements of the nonlogical expressions they operate on. (Brandom, 1994, 498)

The thought here is that identity allows us to state the conditions under which singular terms can be substituted, and so distinguish them from predicates. And, Brandom contends, if the distinction between terms and predicates is to be of logical significance, then we will want the identity sign to be part of logical vocabulary in order to play this role.

Aside from the reasons for everyone, therefore, the inferentialist has special reasons to want identity as part of the logical vocabulary. If they are to accept my arguments and rule identity out as a logical constant, then as well as limiting the number of logically valid arguments that can be captured, they must also find another means of distinguishing terms from predicates.

§7. Conclusion. I have proved that Read's $=\mathrm{I}'$ rule is precisely as strong as the old $=\mathrm{I}$ rule. It follows, on the informal notion of harmony with which Read and I have both been working, that the disharmony of the old $=\mathrm{I}$ rule and $=\mathrm{E}$ carries over to the disharmony of Read's $=\mathrm{I}'$ rule with $=\mathrm{E}$. And, indeed, any *sound* introduction rule will be similarly disharmonious. This should trouble the inferentialist as harmony is usually taken as a necessary condition for logical constanthood, in order to exclude recalcitrant tonk-like phenomena. But inferentialists generally want identity to be a logical constant, in order to capture many seemingly valid inferences and to be able to syntactically distinguish terms from predicates.

§8. Appendix. Consider a language with $\wedge$, $\neg$ and $\forall$ as the only logical constants. This language is sufficient for my purposes, given the expressive adequacy of the rules. The logical constants are governed by the familiar inference rules[8]:

$$\frac{A \quad B}{A \wedge B}\ \wedge\mathrm{I} \qquad \frac{A \wedge B}{A}\ \wedge\mathrm{E} \qquad \frac{A \wedge B}{B}\ \wedge\mathrm{E}$$

$$\frac{\begin{array}{c}[A]\\ \vdots\\ \bot\end{array}}{\neg A}\ \neg\mathrm{I} \qquad \frac{\neg\neg A}{A}\ \neg\mathrm{E}$$

$$\frac{A(y/x)}{\forall x\, A}\ \forall\mathrm{I} \qquad \frac{\forall x\, A \qquad \begin{array}{c}[A(t/x)]\\ \vdots\\ C\end{array}}{C}\ \forall\mathrm{E}$$

[8] This proof takes place in a natural deduction setting. See Negri and von Plato (2001, Ch. 6.5) for a presentation of the inference rules for identity (they call it 'equality') in a sequent calculus setting. They prove the cut elimination for first-order logic with identity and the result proved here is a corollary.

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press

$\forall$I is subject to the usual variable restriction that y must not occur free in $\forall x A$, nor in any assumption on which $A(y/x)$ depends. I am using the rules for classical negation, which are not harmonious, but this is unproblematic for two reasons: first, the focus of Read (2004) is classical logic; second, since intuitionistic logic is a sublogic of classical logic, if Fb cannot be derived in a classical setting, it cannot in an intuitionistic setting, so my argument carries across.

THEOREM. *There is no derivation of Fb from $\{Fa\} \cup X$, where a and b are distinct and F does not appear in X, unless X contains the identity predicate or $\{Fa\} \cup X \vdash \bot$.*

Proof. Proceed by complete induction on the height of proofs.

Basis clause There is no derivation of Fb from $\{Fa\} \cup X$ of height 1.

Proof. Since F is atomic, Fb cannot be derived using $\wedge$I, $\neg$I or $\forall$I. I take the E-rules in turn:

($\wedge$E) Suppose there is a derivation of Fb from $\{Fa\} \cup X$ of height 1, using $\wedge$E. Then X must contain a sentence of the form $\phi \wedge Fb$ (or $Fb \wedge \phi$). But X does not contain such a sentence, because it does not contain F.

($\neg$E) Similarly.

($\forall$E) Suppose there is a derivation of Fb from $\{Fa\} \cup X$ of height 1, using $\forall$E. Then X must contain a sentence of the form $\forall x A$ such that $A(t/x) \vdash Fb$. But the subderivation of Fb from $A(t/x)$ must be of height ≥ 1. So the derivation of Fb using $\forall$E must be of height ≥ 2, contrary to the basis clause.

Inductive clause If there is no derivation of Fb from $\{Fa\} \cup X$ of height $m < n + 1$, then there is no derivation of height $n + 1$.

Proof. Again, since F is atomic, only the elimination rules could derive Fb at the final line of the proof. I take them in turn:

($\wedge$E) Suppose there is a derivation of Fb from $\{Fa\} \cup X$ of height $n + 1$, using $\wedge$E at the last line. Then at some earlier line, a sentence of the form $\phi \wedge Fb$ must have been derived. But there is no sentence of this form in X, since it contains F, so $\phi \wedge Fb$ must have been derived at some earlier line using $\wedge$I or $\neg$E.
If $\phi \wedge Fb$ has been derived by $\wedge$I, there is a derivation of Fb from $\{Fa\} \cup X$ of height $m < n + 1$. But the existence of such a derivation contradicts the inductive clause.
If $\phi \wedge Fb$ has been derived by $\neg$E, then there is an earlier derivation of $\neg\neg(\phi \wedge Fb)$. *That* could only have been derived from $\{Fa\} \cup X$ by $\neg$I, so there is a derivation of $\bot$ from $\{Fa\} \cup X \cup \{\neg(\phi \wedge Fb)\}$ of height $m < n$. But we have that $\{Fa\} \cup X \nvdash \bot$, so there is a derivation of $\phi \wedge Fb$ to contradict $\neg(\phi \wedge Fb)$ from $\{Fa\} \cup X$ without the detour through $\neg$I and $\neg$E. By $\wedge$E, there is a derivation of Fb of height $m < n + 1$, contradicting the inductive clause.

($\neg$E) Suppose there is a derivation of Fb from $\{Fa\} \cup X$ of height $n + 1$, using $\neg$E at the last line. Then at some earlier line, $\neg\neg Fb$ has been derived using $\neg$I. So $\{Fa\} \cup X \cup \{\neg Fb\} \vdash \bot$. But we are assuming that $\{Fa\} \cup X \nvdash \bot$ so, to derive a contradiction, there must be a derivation of Fb of height $m < n + 1$ to contradict the assumption of $\neg Fb$. But the existence of such a derivation contradicts the inductive clause.

($\forall$E) Suppose there is a derivation of Fb from $\{Fa\} \cup X$ of height $n + 1$, using $\forall$E at the last line. Then there is a derivation of $\forall x A$ from $\{Fa\} \cup X$ and a subderivation of Fb from the assumption of $A(t/x)$ for any term t. For there to be a subderivation of

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press

Fb from $A(t/x)$, A must contain F and so $\forall x A$ cannot appear in X. It must therefore have been derived by ∀I or ¬E. The proof that it cannot have been derived by ¬E mirrors that in (∧E) above.

But if there is a derivation of $\forall x A$ using ∀I, then there must be a derivation of $A(y/x)$ from $\{Fa\} \cup X$. If there is such a derivation, then in particular there is a derivation of $A(a/x)$ from $\{Fa\} \cup X$. And, if there is a subderivation of Fb on the assumption of $A(t/x)$ then in particular there is a subderivation of Fb on the assumption of $A(a/x)$. So there is a derivation of Fb from $\{Fa\} \cup X$ without the detour through ∀I and ∀E. This derivation will be of height $m < n + 1$ so its existence contradicts the inductive clause.

□

§9. Acknowledgments. I am grateful to Arif Ahmed, Tim Button, Rosalie Iemhoff, Alex Oliver, Michael Potter, Stephen Read, Florian Steinberger, Rob Trueman and two anonymous referees for comments on previous drafts. I am especially grateful to the referee who pointed out that my general result applies to any *sound* pair of rules. I am also grateful to audiences in Madrid and at the Cambridge Moral Sciences Club, and to the Analysis Trust for their studentship.

BIBLIOGRAPHY

Brandom, R. (1994). *Making it Explicit*. Cambridge, MA: Harvard University Press.

Dummett, M. (1973). *Frege: Philosophy of Language*. London, UK: Duckworth.

Dummett, M. (1991). *The Logical Basis of Metaphysics*. Cambridge, MA: Harvard University Press.

Francez, N., & Dyckhoff, R. (2010). Proof-theoretic semantics for a natural language fragment. *Linguistics and Philosophy*, *33*, 447–477.

Gentzen, G. (1969). Unterschungen über das logische schliessen. In *The Collected Papers of Gerhard Gentzen* (Translated by M. Szabo), pp. 68–131. Amsterdam: North Holland.

Kremer, M. (2007). Read on identity and harmony – a friendly correction and simplification. *Analysis*, *67*, 157–159.

Milne, P. (2007). Existence, freedom, identity, and the logic of abstractionist realism. *Mind*, *116*, 23–53.

Negri, S., & von Plato, J. (2001). *Structural Proof Theory*. Cambridge: Cambridge University Press.

Read, S. (1981). What is wrong with disjunctive syllogism? *Analysis*, *41*, 66–70.

Read, S. (2004). Identity and harmony. *Analysis*, *64*, 113–119.

Read, S. (2010). General elimination harmony and the meaning of the logical constants. *Journal of Philosophical Logic*, *39*, 557–576.

Read, S. (2012). Identity and harmony revisited. Unpublished. Available at: http://www.st-andrews.ac.uk/∼slr/identity_revisited.pdf.

Restall, G. (2012). Anti-realist classical logic and realist mathematics. In Rahman, S., editor. *The Realism-Antirealism Debate in the Age of Alternative Logics*. vol. 23, Logic, epistemology, and the unity of science, pp. 269–284.

Rumfitt, I. (2000). 'Yes' and 'No'. *Mind*, *109*, 781–823.

Steinberger, F. (2009). Not so stable. *Analysis*, *69*, 655–661.

Steinberger, F. (2011). What harmony could and could not be. *Australasian Journal of Philosophy*, *89*, 617–639.

Tennant, N. (1997). *The Taming of the True*. Oxford: Oxford University Press.

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press

ST JOHN'S COLLEGE
CAMBRIDGE
CB2 1TP
E-mail: oeg21@cam.ac.uk

https://doi.org/10.1017/S1755020314000161 Published online by Cambridge University Press