6

Scattering from Linear Structures

When the target is endowed with two or more scales, the image, or cross section distribution, can get rather involved. Before exploring scattering from various two- and three-dimensional structures, we shall study scattering from one, two, and three parallel one-dimensional slits here. These will however be endowed with some structure, allowing us to exhibit some features of scattering from multiscale objects.

6.1 One Slit

Scattering from an infinite slit is given by the Fourier transform

$$G(q) = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{iqx} A(x), \tag{6.1}$$

obtained by integrating across the slit, and where $A(x)$ is a normalized "transparency distribution",

$$\int \mathrm{d}x\, A(x) = 1. \tag{6.2}$$

Below, we consider a few simple examples of such transparency profiles, restricting ourselves to cases for which the integral in Eq. (6.1) can be evaluated analytically. The functions $A(x)$ that we consider are shown in Fig. 6.1, both in linear and logarithmic representations.

Let us start by first considering a constant profile,

$$A_1(x) = \begin{cases} \dfrac{1}{2\beta}, & |x| \le \beta \\ 0 \quad \text{outside} \end{cases} \tag{6.3}$$

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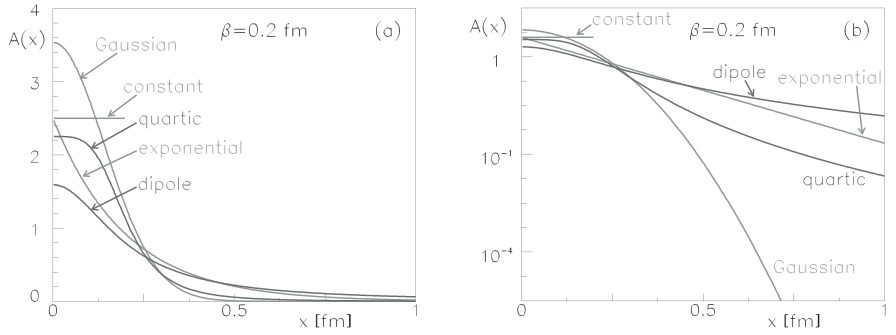


Figure 6.1 Some simple slit profiles, given by Eqs. (6.3), (6.5), (6.7), (6.9), and (6.11). The same functions are shown both in linear (a) and logarithmic (b) representations.

with β characterizing the width. Then

$$G_1(q) = \frac{\sin(q\beta)}{q\beta}. \tag{6.4}$$

The scattering cross section will have a modulation, given by $\sin^2(q\beta)$, and it will be damped at increasing momentum transfer by $1/(q\beta)^2$.

Next, we consider a "dipole" distribution,

$$A_2(x) = \frac{1}{\pi}\frac{\beta}{x^2+\beta^2}, \tag{6.5}$$

for which

$$G_2(q) = \exp(-q\beta). \tag{6.6}$$

This is recognized as the distribution discussed in Chapter 1, see Eq. (1.2). For $q\beta \gg 1$ the cross section will be exponentially damped, as already observed.

Also a quartic distribution,

$$A_3(x) = \frac{\sqrt{2}}{\pi}\frac{\beta^3}{x^4+\beta^4} \tag{6.7}$$

leads to an analytic amplitude

$$G_3(q) = \sqrt{2}\exp(-q\beta\sqrt{2})\sin(q\beta/\sqrt{2}+\pi/4). \tag{6.8}$$

For $q\beta \gg 1$ the cross section will be exponentially damped, but there is a superimposed oscillation, reminiscent of the Fourier transform in Eq. (6.4) of the constant distribution.

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Next, we consider a profile that is exponential in shape, but symmetric around its center, $x = 0$:

$$A_4(x) = \frac{1}{2\beta} \exp[-|x|/\beta]. \tag{6.9}$$

One finds

$$G_4(q) = \frac{1}{1 + (q\beta)^2}. \tag{6.10}$$

For $q\beta \gg 1$ the cross section will be damped as a power law.

As a final example, we consider a Gaussian profile,

$$A_5(x) = \frac{1}{\sqrt{\pi}\beta} \exp[-x^2/\beta^2], \tag{6.11}$$

for which the amplitude is also a Gaussian:

$$G_5(q) = \exp[-q^2\beta^2/4]. \tag{6.12}$$

This cross section is dramatically different from the previous ones, it has a Gaussian damping for $q\beta \gtrsim 1$.

We show in Fig. 6.2 the differential cross sections given by these profiles, $G_i^2(q)$, for a common "diffuseness parameter", $\beta = 0.2$ fm. From the fact that the constant and quartic profiles both yield an oscillatory pattern, it is tempting to conclude that this is due to the quartic distribution also having some kind of an edge or shoulder.

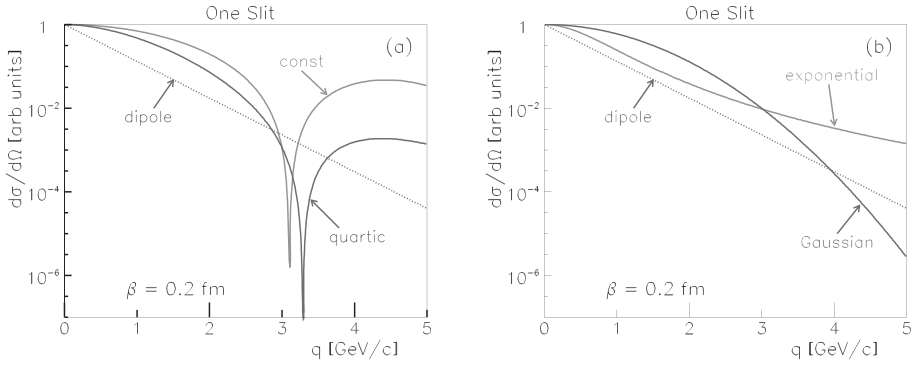


Figure 6.2 Differential cross sections (arbitrary normalization) for scattering from a single slit. (a) solid: constant profile, given by $G_1(q)$ and quartic profile, given by $G_3(q)$; dashed: dipole profile, given by $G_2(q)$. (b) solid: exponential profile given by $G_4(q)$ and Gaussian profile, given by $G_5(q)$; dashed: same (dipole) reference as in (a).

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6.2 Two Slits

Scattering from two slits introduces a new scale, namely their separation. Let us consider scattering from two parallel slits, located at $x = \pm R$. Each will contribute a (conveniently normalized) scattering amplitude

$$F_{\pm}(q) = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x e^{iq(\pm R+x)} A(x). \tag{6.13}$$

The total amplitude is given by

$$F(q) = F_{-}(q) + F_{+}(q). \tag{6.14}$$

With two identical slits, the amplitude will factorize,

$$F(q) = \cos(qR)G(q), \tag{6.15}$$

where $G(q)$ is defined by Eq. (6.1) and could be one of the amplitudes given above. This function $G(q)$ represents damping with increasing momentum transfer, and possibly also modulation. We show in Fig. 6.3(a) differential cross sections corresponding to the above profile transparency distributions.

We note that whereas the period of the rapid oscillations is determined by the separation between the two slits, the envelope is determined by the internal transparency distribution of the slits.

This factorization Eq. (6.15) is a result of having identical (and well separated) slits, and leads to zeros in the cross section. If the two slits were different, the amplitude would still have had the same oscillatory pattern, but without zeros. This can be seen as follows. Let

$$\begin{aligned} F(q) &= \tfrac{1}{2}[e^{-iqR}G_i(q) + e^{iqR}G_j(q)] \\ &= \cos(qR)[G_i(q) + G_j(q)] - i\sin(qR)[G_i(q) - G_j(q)]. \end{aligned} \tag{6.16}$$

Then,

$$\begin{aligned} |F(q)|^2 &= \tfrac{1}{4}\{\cos^2(qR)[G_i(q) + G_j(q)]^2 + \sin^2(qR)[G_i(q) - G_j(q)]^2\} \\ &= \tfrac{1}{4}[G_i(q) - G_j(q)]^2 + \cos^2(qR)G_i(q)G_j(q). \end{aligned} \tag{6.17}$$

The regular periodic oscillations are now offset from zero by a q-dependent amount determined by the difference between the two profiles, $\frac{1}{4}[G_i(q) - G_j(q)]^2$. As an example, we show in Fig. 6.3(b) the scattering from the interference of an exponential and a Gaussian slit,

$$F(q) = \tfrac{1}{2}[e^{-iqR}G_4(q) + e^{iqR}G_5(q)]. \tag{6.18}$$

The overall period of oscillation is of course unchanged, but a more complicated pattern is generated. The resulting cross section has minima offset from zero, and values intermediate between the two "pure" distributions.

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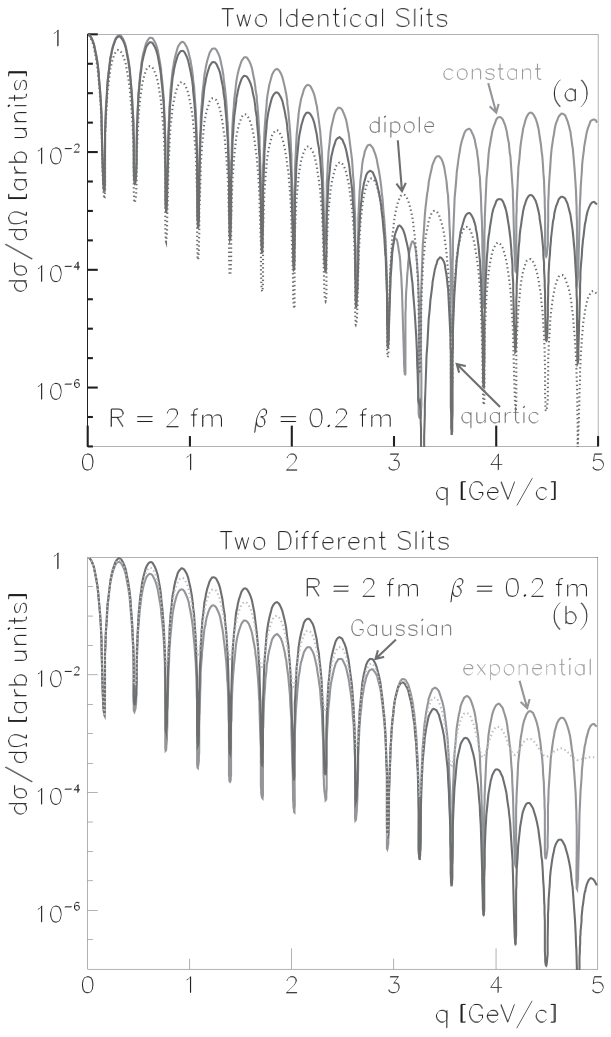


Figure 6.3 Differential cross sections (arbitrary normalization) for scattering from two slits. Different slit transparency profiles are compared. (a) solid: constant profile and quartic profile, Eq. (6.5); dashed: dipole profile, Eq. (6.3). (b) solid: exponential profile, Eq. (6.9); and Gaussian profile, Eq. (6.11); dashed: interference of two different slits, Eq. (6.18).

6.3 Three Slits

Let us first consider three identical and equidistant slits, located at $x = -R$, $x = 0$ and $x = R$. The scattering amplitude (conveniently normalized) can then be expressed as

$$F(q) = \tfrac{1}{3}[F_-(q) + F_0(q) + F_+(q)]. \tag{6.19}$$

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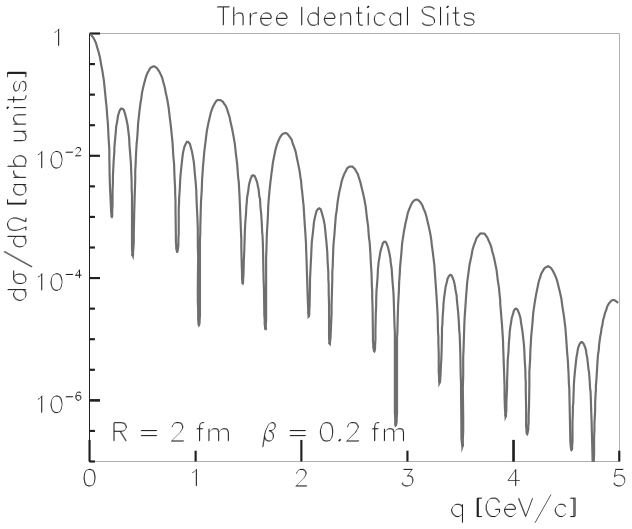


Figure 6.4 Differential cross sections (arbitrary normalization) for scattering from three identical slits. Equidistant slits located at $-2, 0$ and $+2$ fm. The slit profile is given by $G_3(q)$.

It can be taken over from the two-slit results above, with the substitution

$$\cos(qR) \to \tfrac{1}{3}[1 + 2\cos(qR)], \tag{6.20}$$

leading to

$$\begin{aligned} \frac{|F(q)|^2}{G^2(q)} &= \tfrac{1}{9}[4\cos^2(qR) + 4\cos(qR) + 1] \\ &= \tfrac{1}{9}[2\cos(2qR) + 4\cos(qR) + 3]. \end{aligned} \tag{6.21}$$

Note that the inter-slit separation is now R, not $2R$ as in the two-slit case.

The differential cross section is shown in Fig. 6.4, for the case $R = 2$ fm, and the slit profile given by $G_2(q)$. A regular oscillatory pattern is seen, displaying these two periods. Apart from the modulation provided by $G_2^2(q)$, successive maxima alternate in magnitude by a factor of 9 or $1/9$.

In the more general case of three identical slits positioned at x_1, x_2, and x_3, the amplitude can be written as

$$\begin{aligned} F(q) &= \tfrac{1}{3}[e^{iqx_1} + e^{iqx_2} + e^{iqx_3}]G(q) \\ &= \tfrac{1}{3}e^{iq(x_1+x_2+x_3)/3}[e^{iq(2x_1-x_2-x_3)/3} + e^{iq(2x_2-x_1-x_3)/3} \\ &\quad + e^{iq(2x_3-x_1-x_2)/3}]G(q). \end{aligned} \tag{6.22}$$

Apart from an overall (irrelevant) phase, each term contains two factors of the kind $\exp[iz_{ij}/3]$, where

$$z_{ij} = q(x_i - x_j). \tag{6.23}$$

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The squared modulus can be written out in terms of trigonometric functions of these arguments, yielding

$$\frac{|F(q)|^2}{G^2(q)} = \tfrac{1}{9}\{2[\cos(z_{12}) + \cos(z_{13}) + \cos(z_{23})] + 3\}. \tag{6.24}$$

In the limit of

$$|z_{12}| = |z_{23}| = qR, \quad |z_{13}| = 2qR, \tag{6.25}$$

we recover the result of Eq. (6.21).

We show in Fig. 6.5 two examples of scattering from nonequidistant slits. In the first example (Fig. 6.5(a)) the three arguments of the trigonometric functions are

$$|z_{12}| = q, \quad |z_{23}| = 3q, \quad \text{and } |z_{13}| = 4q. \tag{6.26}$$

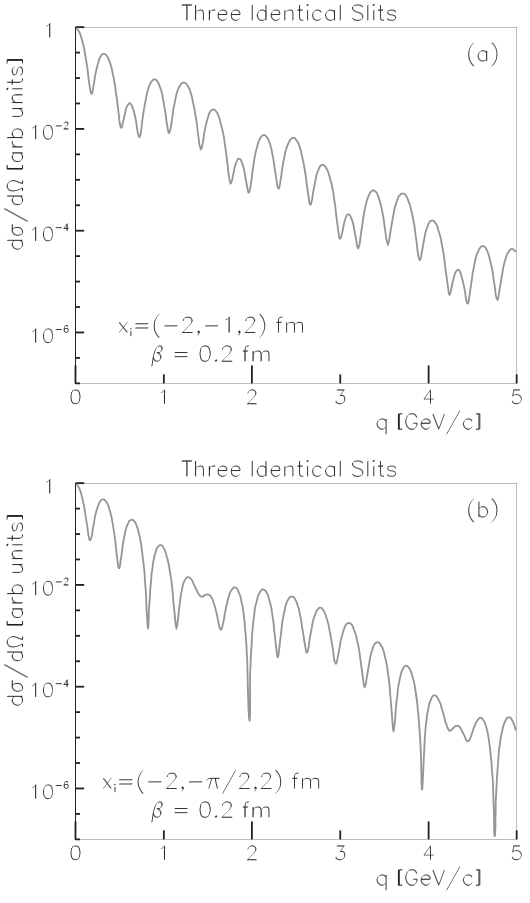


Figure 6.5 Differential cross sections (arbitrary normalization) for scattering from three identical slits. (a) Slits located at -2, -1 and $+2$ fm. (b) Slits located at -2, $-\pi/2$ and $+2$ fm. The slit profile is the dipole one, given by $G_2(q)$.

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Since the slit separations are related by a rational fraction, $|z_{12}|:|z_{23}| = 1:3$ the pattern is repeated.

In the second example (Fig. 6.5(b)) the corresponding arguments of the trigonometric functions are

$$|z_{12}| = \tfrac{1}{2}(4-\pi)q, \quad |z_{23}| = \tfrac{1}{2}(4+\pi)q, \quad \text{and} \quad |z_{13}| = 4q. \tag{6.27}$$

Now, the slit separations are related by an irrational fraction, $|z_{12}|:|z_{23}| = (4-\pi):(4+\pi)$. Thus, there is no repetition of the pattern. However, the short-period oscillation, given by $|z_{13}| = 4q$ is common in these two examples, as is reflected in the illustrations.

The examples shown in this chapter illustrate how multiple scales can lead to more involved cross sections. However, in some cases, the pattern may be sufficiently systematic that it can be decoded in terms of an underlying geometry.

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