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SEMIREGULAR FACTORIZATIONS OF REGULAR MULTIGRAPHS

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Abstract. An $(r, r+1)$-*factor* of a graph G is a spanning subgraph H such that $d_H(v) \in \{r, r+1\}$ for all vertices $v \in \mathcal{V}(G)$. If G is expressed as the union of edge-disjoint $(r, r+1)$-factors, then this expression is an $(r, r+1)$-*factorization* of G. Let $\mu(r)$ be the smallest integer with the property that if G is a regular loopless multigraph of degree d with $d \geq \mu(r)$, then G has an $(r, r+1)$-factorization. It is shown that $\mu(r) \leq \frac{3}{2}r^2 + 3r + 1$ if r is even. The proof employs a novel list-coloring approach. Together with known results, this shows that $\mu(r) = r^2 + 1$ if r is odd and $\frac{3}{2}(r^2 - 2r) \leq \mu(r) \leq \min\{2r^2 - 3r, \frac{3}{2}r^2 + 3r + 1\}$ if r is even.

§1. *Introduction.* In this note, *simple graphs* have no loops or multiple edges, *multigraphs* may have multiple edges but cannot have loops, and *pseudographs* may have multiple edges and multiple loops. When we refer to a graph, it could be any of the above. The vertex-set and edge-set of a graph G are denoted by $\mathcal{V}(G)$ and $\mathcal{E}(G)$, respectively.

For a non-negative integer r, a graph G is an $(r, r+1)$-graph provided that $d_G(v) \in \{r, r+1\}$ for every $v \in \mathcal{V}(G)$. Such a graph is said to be *semiregular*. Given a graph G, a spanning $(r, r+1)$-subgraph of G is called an $(r, r+1)$-*factor*. If G can be expressed as the edge-disjoint union of $(r, r+1)$-factors, then this expression is called an $(r, r+1)$-*factorization* of G. We let $\sigma(r)$, $\mu(r)$ and $\psi(r)$ be, respectively, the smallest integers such that each regular simple graph, multigraph and pseudograph with degree $d \geq \sigma(r)$, $d \geq \mu(r)$ and $d \geq \psi(r)$, respectively, has an $(r, r+1)$-factorization. In the case where there is no such smallest integer $\sigma(r)$, $\mu(r)$ or $\psi(r)$, we put the parameter equal to ∞.

The combined efforts of Era [4] in 1984 and Egawa [3] in 1986 showed that

$$\sigma(r) = \begin{cases} r^2 & \text{if } r \text{ is even,} \\ r^2 + 1 & \text{if } r \text{ is odd.} \end{cases}$$

In 2008, the second author [7] studied this further for multigraphs and pseudographs. In particular, he pointed out that

$$\psi(r) = \infty$$

and that partial results for the evaluation of $\mu(r)$ could be extracted from Era's paper [4]. These partial results were proved in [4] without being explicitly stated.

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For instance, proven but not stated was the fact that

$$\mu(r) = r^2 + 1 \quad \text{if } r \text{ is odd.}$$

If we look at the work done on the case where r is even, we find that Era also gave a result, [4, Proposition 1(1)], from which it follows quickly that if d and r are both even, then every loopless d-regular multigraph with $d \geq \frac{3}{2}r^2 - 3r$ is $(r, r+1)$-factorable. (It was stated misleadingly by the second author in [7, Theorem 25] that Era's result showed that this holds if $d \geq \frac{3}{2}r^2 - 2r - 2$.) An example given by the second author in [7] for the case where r is even and d is odd shows that $\mu(r) \geq \frac{3}{2}r^2 - 2r$.

To establish the correct value of $\mu(r)$ when r is even, it is necessary to improve on the existing result for the case of r even and d odd. The existing result follows from a result of Era, [4, Proposition 1(2)], and shows that $\mu(r) \leq 2r^2 - 3r$. The main result in our paper is the following.

THEOREM 1. *Let r be an even positive integer and let d be an odd positive integer. If $d \geq \frac{3}{2}r^2 + 3r + 1$, then a d-regular loopless multigraph G has an $(r, r+1)$-factorization.*

Thus, when r is even we have that

$$\tfrac{3}{2}r^2 - 2r \leq \mu(r) \leq \tfrac{3}{2}r^2 + 3r + 1.$$

This new upper bound is unlikely to be the actual upper bound, but we do not have particular confidence in the lower bound either. The example given in [7] of a multigraph with degree $d = \frac{3}{2}r^2 - 2r - 1$ which is not $(r, r+1)$-factorable is as follows.

The example has six vertices, a, b, c, a', b' and c'; c and c' are joined by one edge while a and c are joined by $\frac{3}{4}r^2 - r - 1$ edges, as are the pairs $\{b, c\}$, $\{a', c'\}$ and $\{b', c'\}$ of vertices. Finally, a and b, and similarly a' and b', are joined by $\frac{3}{4}r^2 - r$ edges. The argument given in [7] showing that this graph is not $(r, r+1)$-factorable is a little complicated, so we do not repeat it here.

Putting this together with the results already mentioned above, we obtain the following theorem.

THEOREM 2. *Let r be a positive integer. Then*

$$\mu(r) = r^2 + 1 \quad \textit{if } r \textit{ is odd,}$$
$$\tfrac{3}{2}r^2 - 2r \leq \mu(r) \leq \min\{2r^2 - 3r,\ \tfrac{3}{2}r^2 + 3r + 1\} \quad \textit{if } r \textit{ is even.}$$

Note that $\frac{3}{2}r^2 + 3r + 1 \leq 2r^2 - 3r$ whenever $r \geq 14$ and that, moreover, $\mu(r) = \frac{3}{2}r^2 + \mathcal{O}(r)$.

The contrast between the bound of approximately $\frac{3}{2}r^2$ when r is even and approximately r^2 when r is odd in Theorem 2 is remarkable.

§2. *List colorings and other preliminaries.* Given a graph G and a set $\mathcal{C}$ of colors, a *list assignment* of the edges of G is a map $L : \mathcal{E}(G) \longrightarrow \mathcal{P}(\mathcal{C})$ which assigns a subset of colors in $\mathcal{C}$ to each edge of G (here $\mathcal{P}(\mathcal{C})$ is the set of all subsets of $\mathcal{C}$). A *list coloring* (or *edge-list coloring*) of G is a map

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$\phi : \mathcal{E}(G) \longrightarrow \mathcal{C}$ such that $\phi(e) \in L(e)$ for each edge $e \in \mathcal{E}(G)$. If $\phi(e_1) \neq \phi(e_2)$ whenever e_1 and e_2 are both incident with a common vertex, then the list coloring is called a *proper list coloring* (or *proper edge-list coloring*).

Generalizing this notion, if instead there is an integer $k \geq 1$ such that the list coloring has the property that at each vertex $v \in \mathcal{V}(G)$ the number of edges incident with v which are colored the same is at most k, then the list coloring is called a *k-improper list coloring* (or a *k-improper edge-list coloring*). Thus a 1-improper list coloring is just a proper list coloring.

The *choice index* $c'(G)$ of a graph G is the smallest number x such that if $|L(e)| \geq x$ for each edge $e \in \mathcal{E}(G)$, then G has a proper edge-list coloring. The *kth choice index* $c'_k(G)$ of G is the smallest number x such that if $|L(e)| \geq x$ for each $e \in \mathcal{E}(G)$, then G has a k-improper edge-list coloring ϕ with $\phi(e) \in L(e)$ for all $e \in \mathcal{E}(G)$. If the lists are all constrained to be the same, then we have the *chromatic index* $\chi'(G)$ or the *kth improper chromatic index* $\chi'_k(G)$ of G. Clearly, $c'(G) = c'_1(G)$ and $\chi'(G) = \chi'_1(G)$.

The list-coloring conjecture (or edge-list-coloring conjecture), attributed to various authors but primarily to Vizing [16], is the following statement.

CONJECTURE 1. *For any graph G,*

$$c'(G) = \chi'(G).$$

In 1995, Galvin [5] proved the following celebrated result.

THEOREM 3 (Galvin). *For a bipartite multigraph G,*

$$c'(G) = \chi'(G) = \Delta(G),$$

where $\Delta(G)$ *is the maximum degree of the vertices of G.*

For each $k \geq 1$, in 1996 the second author proposed in [6] the following analogue of Conjecture 1.

CONJECTURE 2. *For each integer* $k \geq 1$ *and for each graph G,*

$$c'_k(G) = \chi'_k(G).$$

In fact, it is not too difficult to see that Conjectures 1 and 2 are equivalent.

In [6], the second author used Galvin's result (Theorem 3) to prove the following.

THEOREM 4. *If* $k \geq 2$ *is an even integer, then, for any graph G,*

$$c'_k(G) = \chi'_k(G) = \left\lceil \frac{\Delta(G)}{k} \right\rceil.$$

We shall use Theorem 4 in our proof of Theorem 2. The start of the proof of Theorem 2 follows Era's development, and, as Era's work mainly involved intricate applications of Petersen's theorem [15], this paper is heavily dependent on Petersen's theorem as well.

THEOREM 5 (Petersen). *A regular graph of even degree is the union of edge-disjoint 2-factors.*

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Various people have shown that any regular graph of degree d has an $(r, r-1)$-factor for each value of r with $1 \leq r \leq d$. For a short proof that works for pseudographs see Thomassen [15].

§3. *The upper bound for $\mu(r)$ when r is even: a proof of Theorem 1.* In this section we will prove Theorem 1. Our proof is, to an extent, based on Era's approach in [4, Proposition 1(2)]. However, after a similar start, the argument uses improper list colorings. Let r be an even positive integer and suppose that G is a regular loopless multigraph of degree d, where d is odd and $d \geq \frac{3}{2}r^2 + 3r + 1$. Let F be a minimal $(1, 2)$-factor of G; such a $(1, 2)$-factor of G exists (see Thomassen [15] or Era [4]). As remarked by Era in [4], minimal means that F consists of disjoint two and three vertex paths. Let $H = G - \mathcal{E}(F)$; then H is a $(d-2, d-1)$-factor of G and $\Delta(H) = d-1$. Embed H into a $(d-1)$-regular graph $\tilde{H}$ so that H is an induced subgraph of $\tilde{H}$. Since $d-1$ is even, $d-1 = \frac{3}{2}(r+x)r + pr + 2y$ for some integral values of x, p and y such that $0 \leq y \leq r/2 - 1$, x is even with $x \geq 2$, and $p \in \{0, 1, 2\}$.

We 2-factorize $\tilde{H}$ and take collections of $r/2$ 2-factors to form $\frac{3}{2}(r+x) + p$ edge-disjoint r-factors $\tilde{H}_1, \tilde{H}_2, \ldots, \tilde{H}_{3(r+x)/2+p}$. Using these r-factors $\tilde{H}_i$, we then let

$$\tilde{R} = \tilde{H} \backslash (\tilde{H}_1 \cup \cdots \cup \tilde{H}_{3(r+x)/2+p}).$$

Finally, let $H_i = \tilde{H}_i \cap H$ and $R = \tilde{R} \cap H$. Then the H_i are $(r-1, r)$-factors of H and R is a $(2y-1, 2y)$-factor of H. We note also that each vertex v of H_i or of R has degree r or $2y$, respectively, unless v is the mid-point of a P_3 of F, for in this case the degree of v is r and $2y$ in $H_1, \ldots, H_{3(r+x)/2+p}$ and R, respectively, with at most one exception in which the degree of v is $r-1$ or $2y-1$, respectively. We color the edges of H_i with the ith color c_i for $1 \leq i \leq \frac{3}{2}(r+x) + p$. We color the two vertex paths in F with color c_1. If v is a vertex of H_i which has degree $r-1$ in H_i, then v is incident with the mid-point of one of the three vertex paths of F. In this case, color the edges of that P_3 with the color c_i. If a P_3 in F has the property that the mid-point v has degree $2y-1$ in R, then color one edge of the P_3 with color c_1 and the other edge with color c_2.

At this point, all the edges of G have a color except those of R, as no edge of R has yet been colored. Each vertex has either r or $r+1$ edges of each color, and it remains only to color the edges of R.

Supply each edge e of R with a list $L(e)$ of colors from $\kappa_1, \kappa_2, \ldots, \kappa_{(r+x)/2}$ in the following way. If $e = ab \in \mathcal{E}(R)$, let $\mathcal{C}(e)$ be the set of colors from $\{c_1, c_2, \ldots, c_{3(r+x)/2}\}$ which were used on the edges of F incident with the end vertices a and b of e. (We do not include in $\mathcal{C}(e)$ either of the colors $c_{3(r+x)/2+1}$ or $c_{3(r+x)/2+2}$.) Then $1 \leq |\mathcal{C}(e)| \leq 3$. For $1 \leq i \leq \frac{1}{2}(r+x)$ we let

$$\kappa_i = \{c_{3(i-1)+1}, c_{3(i-1)+2}, c_{3(i-1)+3}\}.$$

Let

$$\kappa(e) = \{\kappa_i : 1 \leq i \leq \tfrac{1}{2}(r+x) \text{ and } \exists c_j \in \kappa_i \text{ with } c_j \in \mathcal{C}(e)\}.$$

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Then $1 \leq |\kappa(e)| \leq 2$. Let $L(e) = \{\kappa_1, \kappa_2, \ldots, \kappa_{(r+x)/2}\} \backslash \kappa(e)$. Then $|L(e)| \geq \frac{1}{2}(r+x) - 2 \geq r/2 - 1$ since $x \geq 2$. By Petersen's theorem applied to $\tilde{R}$, $\tilde{R}$ is the union of $y \leq r/2 - 1$ edge-disjoint 2-factors, so $\chi_2'(\tilde{R}) = y \leq r/2 - 1$, and it follows from Theorem 4 that since $|L(e)| \geq r/2 - 1$, R has a 2-improper L-edge coloring. The colors involved in this L-edge coloring are $\kappa_1, \kappa_2, \ldots, \kappa_{(r+x)/2}$. Let the 2-improper color classes be denoted by $F_1, F_2, \ldots, F_{(r+x)/2}$. Each F_i consists of disjoint paths and cycles. We properly color the edges of F_i with colors $c_{3(i-1)+1}$, $c_{3(i-1)+2}$ and $c_{3(i-1)+3}$.

Finally, we let G_i be H_i together with those edges of R and F that are colored c_i. Then $G_1, G_2, \ldots, G_{3(r+x)/2+p}$ is the required $(r, r+1)$-factorization of G. □

§4. *Final remarks.* More general but closely related questions for multigraphs were studied by Akiyama and Kano [1] and Cai [2]; the most recent survey paper on this topic is that of Plummer [14]. See also works by Kano [11] and Kano and Saito [12]. For simple graphs, more general exact results were obtained by the second author and Wojciechowski in [10] and later by the second author alone in [9]. Finally, more general results concerning pseudographs due to the second author can be found in [8].

References

1. J. Akiyama and M. Kano, Almost-regular factorization of graphs. *J. Graph Theory* **9**(1) (1985), 123–128.
2. M.-C. Cai, $[a, b]$-factorizations of graphs. *J. Graph Theory* **15**(3) (1991), 283–301.
3. Y. Egawa, Era's conjecture on $[k, k+1]$-factorizations of regular graphs. *Ars Combin.* **21** (1986), 217–220.
4. H. Era, Semiregular factorizations of regular graphs. In *Graphs and Applications: Proceedings of the First Colorado Symposium on Graph Theory* (eds F. Harary and J. Maybee), Wiley (New York, 1984), 101–116.
5. F. Galvin, The list chromatic index of a bipartite multigraph. *J. Combin. Theory Ser. B* **63**(1) (1995), 153–158.
6. A. J. W. Hilton, Some improper list colouring theorems. *Congr. Numer.* **113** (1996), 171–178.
7. A. J. W. Hilton, $(r, r+1)$-factorizations of $(d, d+1)$-regular graphs. *Discrete Math.* **308**(5–6) (2008), 645–669.
8. A. J. W. Hilton, Degree-bounded factorizations of bipartite multigraphs and of pseudographs. *Discrete Math.* **310**(2) (2010), 288–302.
9. A. J. W. Hilton, On the number of $(r, r+1)$-factors in an $(r, r+1)$-factorization of a simple graph. *J. Graph Theory* **60** (2009), 257–268.
10. A. J. W. Hilton and J. Wojciechowski, Semiregular factorization of simple graphs. *AKCE Int. J. Graphs Comb.* **2**(1) (2005), 57–62.
11. M. Kano, $[a, b]$-factorization of a graph. *J. Graph Theory* **9**(1) (1985), 129–146.
12. M. Kano and A. Saito, $[a, b]$-factorizations of graphs. *Discrete Math.* **47**(1) (1983), 113–116.
13. J. Petersen, Die Theorie der regulären Graphen. *Acta Math.* **15**(1) (1891), 193–220.
14. M. J. Plummer, Graph factors and factorization: 1985–2003: a survey. *Discrete Math.* **307**(7–8) (2007), 791–821.
15. C. Thomassen, A remark on the factor theorems of Lováz and Tutte. *J. Graph Theory* **5**(4) (1981), 441–442.
16. V. G. Vizing, Coloring the vertices of a graph in prescribed colors. *Diskret. Analiz No. 29, Metody Diskret. Anal. v Teorii Kodov i Shem* **101** (1976), 3–10 (in Russian).

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