SPACE-TIME ON THE ROTATING DISK

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Part I

1. *Introduction.* In a recent paper Clark (1) has dealt with the problem of the rotating disk, the material of which is such that the waves of dilatation in this particular material travel with the velocity of light. The material of the disk is supposed to be under an isotropic stress p when in a strained state, and the relation between stress p and the dilatation Δ is found to be connected by an expression

$$p = ac^2\Delta(1+\Delta)^{-1}, \tag{1·1}$$

where a = density in the unstrained state, and Δ is given by

$$1+\Delta = \left|\delta_{ij}+\frac{\partial U_i}{\partial x^j}\right|. \tag{1·2}$$

U_i $(i = 1, 2, 3)$ are the components of the strain.

According to Clark it is possible to think of incompressible material as one in which the waves of dilatation travel with the velocity of light. Under these assumptions he proves that for a disk, composed of this material, the relativity contraction is exactly balanced by the classical expansion, and no change of radius will take place. The problem of the rotating disk has also been attacked by others; among them are Lorentz, Eddington, Einstein and Berenda (2).

Our aim in this paper is to deal with some of those aspects of the problem which have not been considered by the previous investigators. The metric of an observer at rest on the rotating disk is obtained from the field equations of the theory of general relativity. In this process no particular properties are to be assigned to the material composing the disk, apart from the assumptions that the disk is under a certain stress when in a strained state and that the material has a certain degree of non-breakability when the disk is rotating about its axis. The results of various methods of distance measurements on the rotating disk will be considered, but no attempt is made to relate the radii of the rotating and non-rotating disks.

In relativistic mechanics elastic properties of a finite continuum (e.g. a rigid body) must be such that any disturbance set up at a point of the body will be transmitted to another point with a velocity less than that of light, and the state of the body composed of the particles can only be determined by an infinite number of variables (5).

On the other hand, in Newtonian rigid-body mechanics, the whole process required only six coordinates which were enough to fix the position and the orientation of the body, namely, a rigid body was defined as one whose parts would act together as a whole.

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The new mechanics completely rejects this idea and requires another definition of a rigid body which will be consistent with its foundations. In the relativistic treatment of a rigid body one has to make assumptions regarding the internal forces of matter, and the compatibility of these assumptions with the theory opens the way for some delicate questions that are rather difficult to answer. It is not, for instance, conceivable to think of rigid solid bodies and of an incompressible material, both of which are perfect media, for the elastic waves to propagate with an infinite speed of propagation; this is contrary to the foundations of the theory of relativity, for the upper bound of any type of velocity is c, the velocity of light *in vacuo*.

2. *The metric.* We now proceed to obtain the metric with the use of the field equations of general relativity

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = -8\pi T_{\alpha\beta}. \tag{2·1}$$

(*a*) The disk is rotating with a constant angular velocity about its axis, with respect to some inertial system of coordinates. The metric for an inertial system in cylindrical coordinates $(\bar{r}, \bar{\theta}, \bar{z})$ is of the form

$$ds^2 = c^2 d\bar{t}^2 - d\bar{r}^2 - \bar{r}^2 d\bar{\theta}^2 - d\bar{z}^2. \tag{2·2}$$

(*b*) All parts of the rotating disk are alike, and there is a rotational symmetry in space so that all metrical coefficients are functions of r alone.

Under the assumptions (*a*) and (*b*) the most general form for the metric is thus

$$ds^2 = -dr^2 - F d\theta^2 - dz^2 + G dt^2 - 2H d\theta\, dt, \tag{2·3}$$

where F, G, H are functions of r alone.

This metric can be brought into a more familiar form by making the following time transformation for a fixed value of r (because of (*b*) no generality will be lost in fixing r):

$$dt' = \phi\left(dt - \frac{H}{G} d\theta\right), \tag{2·4}$$

where ϕ is an integrating factor which will make the right-hand side of (2·4) an exact differential form. On squaring both sides of (2·4) and substituting in (2·3) for

$$G dt^2 - 2H d\theta\, dt = \frac{G dt'^2}{\phi^2} - \frac{H^2}{G} d\theta^2,$$

our metric takes the form

$$ds^2 = -dr^2 - \left(F + \frac{H}{G}\right) d\theta^2 - dz^2 + \frac{G}{\phi^2} dt'^2. \tag{2·5}$$

From the theory of integrating factors it can easily be shown that we can choose ϕ such that it is independent of θ and t.

Now putting $F + \dfrac{H^2}{G} = e^{\mu}$, $\dfrac{G}{\phi^2} = e^{\nu}$, we write the metric in the simplest form as

$$ds^2 = e^{\nu} dt'^2 - dr^2 - e^{\mu} d\theta^2 - dz^2, \tag{2·6}$$

where μ and ν are functions of r alone.

When the disk is at rest, i.e. when $\omega = 0$, we have

$$e^{\nu} = c^2 \quad \text{and} \quad e^{\mu} = r^2.$$

Hence, when $\omega = 0$, ν and μ are given by

$$\nu = \log c^2, \quad \mu = \log r^2. \tag{2·7}$$

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These two conditions (2·7) for μ and ν are to be used throughout in the solutions of the differential equations that we shall obtain from the field equations.

Now using the metric (2·6) we find that the only non-vanishing Christoffel symbols are

$$\Gamma^1_{22} = -\tfrac{1}{2}\mu' e^{\mu}, \quad \Gamma^1_{44} = \tfrac{1}{2}\nu' e^{\nu}, \quad \Gamma^2_{12} = \tfrac{1}{2}\mu', \quad \Gamma^4_{14} = \tfrac{1}{2}\nu', \tag{2·8}$$

where the Christoffel symbols are obtained from

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta} = \tfrac{1}{2}g^{\alpha\delta}\left(\frac{\partial g_{\beta\delta}}{\partial x^{\gamma}} + \frac{\partial g_{\gamma\delta}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}}\right).$$

In (2·8) dashes imply differentiation with respect to r.

For the components of the energy-momentum tensor $T^{\alpha}_{\beta} = g_{\gamma\beta}T^{\alpha\gamma}$, we shall choose

$$T^1_1 = p, \quad T^2_2 = p, \quad T^3_3 = -p, \quad T^{44} = (\rho c^2 - p)\left(\frac{dt}{ds}\right)^2 + e^{-\nu}p, \tag{2·9}$$

which leads to some interesting results as will be seen in the following work.

Where $$T^{\mu\nu} = (\rho c^2 - p)\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + g^{\mu\nu}p,$$

all other components of this tensor vanish. After the calculation of the components of Ricci tensor $R_{\alpha\beta}$ and substituting in (2·1), we have the field equations as

$$\left.\begin{aligned} &\text{(A)} \quad 8\pi p = -\tfrac{1}{4}\mu'\nu', \\ &\text{(B)} \quad 8\pi p = -\tfrac{1}{2}\nu'' - \tfrac{1}{4}\nu'^2, \\ &\text{(C)} \quad 8\pi p = \tfrac{1}{2}\mu'' + \tfrac{1}{2}\nu'' + \tfrac{1}{4}\mu'^2 + \tfrac{1}{4}\nu'^2 + \tfrac{1}{4}\mu'\nu'. \end{aligned}\right\} \tag{2·10}$$

From (A), (B) and (C) we obtain the differential equations

$$\text{(A')} \quad \nu'' + \tfrac{1}{2}\nu'^2 - \tfrac{1}{2}\mu'\nu' = 0, \qquad \text{(B')} \quad \mu'' + \tfrac{1}{2}\mu'^2 + \tfrac{3}{2}\mu'\nu' = 0.$$

From the conditions (2·7) the solutions of these equations are

$$\nu = \log(c^2 + \tfrac{1}{2}Acr^2), \quad \mu = \log\left(\frac{r^2c^2}{c^2 + \tfrac{1}{2}Acr^2}\right),$$

where A is a constant of integration. If we put tentatively

$$\tfrac{1}{2}Ac = -\omega^2,$$

we shall see that ω will reveal itself as the angular velocity of the disk about its axis, as we proceed.

Hence the final forms of μ and ν are

$$\nu = \log(c^2 - \omega^2r^2), \quad \mu = \log\left(\frac{r^2c^2}{c^2 - r^2\omega^2}\right), \tag{2·11}$$

and the metric becomes, on dropping the dash from t,

$$ds^2 = (c^2 - \omega^2r^2)\,dt^2 - dr^2 - \frac{r^2}{1 - \dfrac{\omega^2r^2}{c^2}}\,d\theta^2 - dz^2. \tag{2·12}$$

We may also, in passing, write down p in terms of r as

$$p = \frac{\omega^2}{8\pi(c^2 - r^2\omega^2)^2}. \tag{2·13}$$

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3. *Intrinsic geometry of an observer at rest on the rotating disk.* The spatial part of the metric is given in this case by $dt = 0$, i.e.

$$-dl'^2 = -dr^2 - \frac{r^2}{1-\frac{\omega^2 r^2}{c^2}}\,d\theta^2 - dz^2. \tag{3·1}$$

Considering the two-dimensional case, we have for the first fundamental form to the surface of the rotating disk the expression

$$dl^2 = dr^2 + \frac{r^2}{1-\frac{\omega^2 r^2}{c^2}}\,d\theta^2. \tag{3·2}$$

$\theta = \text{const.}$ and $r = \text{const.}$ are the coordinate curves on this surface. $\theta = \text{const.}$ are geodesics and $r = \text{const.}$ are the orthogonal trajectories of these geodesics. This can easily be seen by taking the general form

$$dl^2 = g_{pq}\,du^p\,du^q \quad (p, q = 1, 2)$$

as the first fundamental form of a surface; then the orthogonal trajectories of $u^2 = \text{const.}$ are given by

$$g_{12}du^1 + g_{22}du^2 = 0.$$

Each curve $r = \text{const.}$ is the locus of a point at a constant distance from the centre of the disk; this distance is measured along the geodesics through the centre. The curves $r = \text{const.}$ are geodesic circles. The arc length of a curve $r = \text{const.}$ between two geodesics $\theta = 0$, $\theta = \theta_0$ is given by

$$\int_0^{\theta} \sqrt{g_{22}}\,d\theta,$$

and the limit of this is zero as r tends to zero.

The angle at the centre between the two geodesics is obtained from

$$\lim_{r\to 0}\frac{\int_0^{\theta_0}\sqrt{g_{22}}\,d\theta}{r} = \lim_{r\to 0}\int_0^{\theta_0}\frac{\partial\sqrt{g_{22}}}{\partial r}\,d\theta.$$

Hence a necessary and sufficient condition that the coordinate θ be the angle made at the centre with the geodesic $\theta = 0$ by the coordinate curve $\theta = \text{const.}$ is that

$$\left(\frac{\partial\sqrt{g_{22}}}{\partial r}\right)_{r=0} = 1.$$

If, in addition to this, the condition $(g_{22})_{r=0} = 0$ is satisfied, then these coordinate nets are called geodesic polar coordinates. These two conditions are satisfied for the surface of the rotating disk.

The Gaussian curvature $K = -\frac{1}{2}R$ comes from

$$K = -\frac{1}{\sqrt{g_{22}}}\frac{\partial\sqrt{g_{22}}}{\partial r^2} = -\frac{3\omega^2}{c^2}\left(1-\frac{\omega^2 r^2}{c^2}\right)^{-2}; \tag{3·3}$$

since the Gaussian curvature of the rotating disk does not vanish the surface of the rotating disk is not isometric with a plane. Hence the intrinsic geometry of an observer at rest on the rotating disk is non-Euclidean.

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If a' is the radius of the rotating disk measured by an observer on the rotating disk the area will follow from

$$\mathscr{A} = \text{Area} = \int_0^{a'}\int_0^{2\pi} \sqrt{g_{22}}\, dr\, d\theta = \int_0^{a'}\int_0^{2\pi} \frac{r\, d\theta\, dr}{\sqrt{\left(1-\frac{\omega^2 r^2}{c^2}\right)}} = \frac{2\pi c^2}{\omega^2}\left\{1-\sqrt{\left(1-\frac{\omega^2 a'^2}{c^2}\right)}\right\},$$

so

$$\mathscr{A} = \frac{2\pi c^2}{\omega^2}\left\{1-\sqrt{\left(1-\frac{\omega^2 a'^2}{c^2}\right)}\right\}.$$

If $\omega a' \ll c$, expanding the bracket we get

$$\mathscr{A} \doteqdot \pi a'^2\left(1+\frac{\omega^2 a'^2}{4c^2}\right);$$

the remaining terms are neglected.

If $\mathscr{A}_0 = \pi a^2$ is the area of the non-rotating disk with radius a, then assuming that $\mathscr{A}$ and $\mathscr{A}_0$ are equal we get

$$a^2 = a'^2\left(1+\frac{\omega^2 a'^2}{4c^2}\right).$$

Hence

$$a'^2 \doteqdot a^2\left(1+\frac{\omega^2 a^2}{4c^2}\right)^{-1},$$

where a' in the bracket is replaced by a approximately, so that taking the square root of both sides we obtain

$$a' = a\left(1-\frac{1}{8}\frac{\omega^2 a^2}{c^2}\right). \tag{3·4}$$

This is Eddington's result, who got (3·4) from physical considerations of the material structure of the disk, whereas we obtained it by assuming $\mathscr{A}_0 = \mathscr{A}$. Now returning to (3·3) and plotting K against r we obtain the variation of curvature with distance from the centre O as indicated in the figure, so that the maximum curvature occurs at $r = 0$.

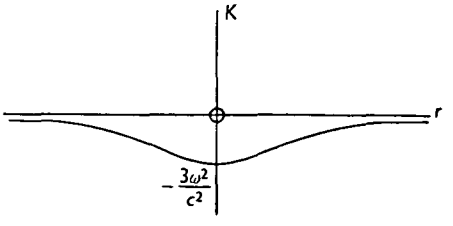


Fig. 1.

A surface at all points of which $K < 0$ is a surface of negative curvature, and hence it is a locus of paraboloidal points.

4. *A brief outline of time measurement.* When the time transformation

$$dt' = dt - \frac{\omega r^2}{c^2 - r^2\omega^2}\, d\theta$$

is made in the original metric (2·12) we get

$$ds^2 = -dr^2 - r^2 d\theta^2 - dz^2 + (c^2 - r^2\omega^2)\, dt^2 - 2\omega r^2 d\theta\, dt. \tag{4·1}$$

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This form of the metric was the one used by the previous investigators. This metric is in fact obtained by making the coordinate transformation

$$d\bar{r} = dr, \quad d\bar{\theta} = d\theta + \omega\, dt, \quad d\bar{z} = dz, \quad d\bar{t} = dt \tag{4·2}$$

in (2·2).

The spatial part of our metric was obtained by putting

$$dt' = dt - \frac{\omega r^2}{c^2 - r^2\omega^2}\, d\theta = 0,$$

namely,

$$dt = \frac{\omega r^2}{c^2 - r^2\omega^2}\, d\theta. \tag{4·3}$$

Hence the intrinsic geometry to the rotating disk was induced by a space-time, the transformation equations of which could be obtained in differential form by substituting (4·3) in (4·2) which gives

$$d\bar{r} = dr, \quad d\bar{\theta} = \frac{d\theta}{1 - \dfrac{\omega^2 r^2}{c^2}}, \quad d\bar{z} = dz, \quad d\bar{t} = \frac{\omega r^2/c^2}{1 - \dfrac{r^2\omega^2}{c^2}}\, d\theta. \tag{4·4}$$

Again, if in (2·2) we put

$$d\bar{r} = dr, \quad \bar{r}\, d\bar{\theta} = \frac{r\, d\theta}{\sqrt{\left(1 - \dfrac{\omega^2 r^2}{c^2}\right)}}, \quad d\bar{z} = dz, \quad c\, d\bar{t} = \sqrt{(c^2 - \omega^2 r^2)}\, dt, \tag{4·5}$$

we again have our original metric (2·12) so that (4·5) are the transformation equations. From the last equation of (4·5) we have

$$d\bar{t} = D\, dt, \tag{4·6}$$

where

$$D = \sqrt{\left(1 - \frac{r^2\omega^2}{c^2}\right)} = \text{Doppler shift.}$$

From (4·6) it follows that

$$d\bar{t} < dt, \tag{4·7}$$

so that the clocks at rest on the disk go slow relative to the clocks at rest with respect to some inertial system of coordinates.

5. *Equation of the light rays.* The velocity of light in space-time represented by the metric (2·12) can be obtained by setting the interval $ds^2 = 0$.

For the light rays travelling radially outwards from the centre of the disk we have

$$ds^2 = 0 = e^{\nu} dt^2 - dr^2.$$

Thus the velocity of light is

$$\frac{dr}{dt} = \sqrt{(c^2 - r^2\omega^2)}. \tag{5·1}$$

For a photon emitted from the centre of the disk at time t_E, and received by an observer at distance a' from the centre, at time $t_\odot$ it follows from (5·1) that

$$t_\odot - t_E = \frac{1}{c}\int_0^{a'} \frac{dr}{\sqrt{\left(1 - \dfrac{r^2\omega^2}{c^2}\right)}},$$

or, on integrating,

$$t_\odot - t_E = \frac{1}{\omega} \arcsin \frac{a'\omega}{c}. \tag{5·2}$$

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If we assume that $(t_\odot - t_E)\,c = a$ approximately, then (5·2) gives

$$a = \frac{c}{\omega}\arcsin\frac{a'\omega}{c},$$

or
$$\frac{\omega a'}{c} = \sin\frac{\omega a}{c} \doteqdot \frac{\omega a}{c} - \frac{1}{6}\frac{\omega^3 a^3}{c^3}.$$

Hence
$$a' = a\left(1 - \frac{1}{6}\frac{\omega^2 a^2}{c^2}\right). \tag{5·3}$$

Thus it is seen that (5·3) does not agree with the result (3·4), which was obtained from a different assumption. This discrepancy between different results obtained by using different methods is obviously due to the curvature of the space-time.

Part 2

6. *Application of the measurement of distance in general relativity.* In special relativity measurement of distance is one with which we are familiar in our everyday life. The intersection of a Galilean space-time by $t = \text{const.}$ is an Euclidean space, so that an observer at any instant may perceive in his immediate neighbourhood an instantaneous Euclidean three-dimensional space, which at $t = 0$ consists of a three-dimensional set of points $(x_0^\mu, y_0^\mu, z_0^\mu)$ that he can regard as simultaneous. In Galilean space-time the light tracks are straight lines in this space. If an observer O at rest at the point (x_0, y_0, z_0) relative to a given rectangular axis receives at time t_0 a light signal emitted by a source S at the point (x_1, y_1, z_1), then the distance OS, as measured by O, is given by

$$r^2 = \sum_1^3 (x_1 - x_0)^2.$$

On the other hand, in a curved space-time, the concept of simultaneity over a finite extent for world events is no longer possible for a particular observer. The spatial distance in general theory of relativity may be defined as a function of two world events in space-time, and it exists whenever the two world points can be joined by a null geodesic which represents the history of a light-pulse in space-time. The idea of simultaneity can, however, be considered by introducing n observers, $O_1, O_2, \ldots, O_n$ and their local Galilean systems along the null geodesic joining O to S, and by supposing that each observer is in the local Galilean space of the succeeding one, e.g. O_p and O_{p+1}, may observe a world event simultaneously in their local Galilean space-time. Thus the distances O_pO_{p+1} in the local Galilean space-time may be determined by the observer O_{p+1}. If we let n tend to infinity, then the sum (the integral) of the infinitesimal distances which are projected lengths of the null geodesics in the infinitesimal instantaneous Euclidean spaces of O_{p+1}'s will give the required spatial distance OS. Different methods of measurement of distance will all give the same result in Euclidean space, but in a non-Euclidean space if 'the observer does not make any allowances for space curvature' they will lead to slightly different results. In what follows these differences will be examined (4).

Definition of distance in a curved space has been first treated by Whittaker and later by McCrea, Walker, Ruse, Etherington and Temple. In the following we

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shall adopt the definition of distance given in a paper by Kermack, McCrea and Whittaker (3).

If Γ is a null geodesic in space-time, then the contravariant null vector η^ρ along this geodesic is defined by

$$\eta^\rho = \frac{dx^\rho}{d\lambda}, \tag{6·1}$$

where λ satisfies

$$\left.\begin{aligned} \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \\ \text{and} \qquad g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0. \end{aligned}\right\} \tag{6·2}$$

Along Γ the expression $\eta_\rho dx^\rho$ is an invariant.

If we use the metric (2·12) it follows from the theory (3) that we must solve the differential equation

$$\frac{d^2x^0}{d\lambda^2} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0,$$

or

$$\frac{d^2t}{d\lambda^2} + \frac{d\nu}{dr}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0,$$

a first integral of which is

$$\frac{dt}{d\lambda} = k e^{-\nu} = \frac{k}{c^2 - r^2\omega^2}. \tag{6·3}$$

We are assuming that the light signals are emitted from the centre of the rotating disk, namely, that the signals are emitted from a local Galilean space-time and received by an observer at the point

$$\lim_{n\to\infty} O_n = a-.$$

The constant k in (6·3) is determined from the fact that when observer O_1 is very near to O then the distance

$$OO_1 = (d\lambda)_0 = c(dt)_0.$$

Hence we write

$$\left(\frac{dt}{c\,dt}\right)_0 = k(e^{-\nu})_0 = \frac{k}{c^2}.$$

Therefore

$$k = c$$

and

$$\frac{dt}{d\lambda} = c e^{-\nu} = \frac{c}{c^2 - r^2\omega^2}. \tag{6·4}$$

Along a null geodesic for which $\theta = \text{const.}$ and $z = \text{const.}$, we have, using the condition $ds = 0$,

$$dt = e^{-\frac{1}{2}\nu}dr = \frac{dr}{\sqrt{(c^2 - r^2\omega^2)}}. \tag{6·5}$$

Thus (6·4) becomes

$$d\lambda = \frac{1}{c}e^{\frac{1}{2}\nu}dr.$$

Integrating in $(O, a-)$ we get the distance Δ_1 which is the radius of the disk as measured by an observer at $r = a-$, i.e.

$$\Delta_1 = \int_{r=0}^{r=a-} d\lambda = \frac{1}{c}\int_0^{a-}\sqrt{(c^2 - r^2\omega^2)}\,dr = \frac{1}{2}\frac{c}{\omega}\left[\arcsin\frac{a\omega}{c} + \frac{a\omega}{c}\sqrt{\left(1 - \frac{a^2\omega^2}{c^2}\right)}\right],$$

or approximately

$$\Delta_1 = a\left(1 - \frac{a^2\omega^2}{6c^2}\right). \tag{6·6}$$

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Next we apply the same process by interchanging the positions of the light-source and the observer. The constant k in (6·3) is defined in a similar way as follows:

When observer O is at an infinitesimal distance from S the distance OS as measured by O will be

$d\lambda$ = velocity of light (at $r = a-$)

× time interval between the two world events = $\sqrt{(c^2-a^2\omega^2)}\,dt$.

Hence $$\left(\frac{dt}{d\lambda}\right)_s = \left(\frac{dt}{\sqrt{(c^2-a^2\omega^2)}\,dt}\right)_s = k\left(\frac{1}{c^2-r^2\omega^2}\right)_s = k\frac{1}{c^2-a^2\omega^2};$$

therefore $k = \sqrt{(c^2-a^2\omega^2)}$, and (6·3) becomes

$$\frac{dt}{d\lambda} = \frac{\sqrt{c^2-a^2\omega^2}}{c^2-r^2\omega^2}. \tag{6·7}$$

Now, as before, $dt = dr(c^2-r^2\omega^2)^{-\frac{1}{2}}$ along the light-path. Thus, integrating (6·7), we have

$$\Delta_2 = \int_0^{a-} d\lambda = \frac{1}{\sqrt{\left(1-\frac{a^2\omega^2}{c^2}\right)}}\int_0^{a-}\sqrt{\left(1-\frac{r^2\omega^2}{c^2}\right)}\,dr. \tag{6·8}$$

Hence $$\Delta_2 = \frac{\Delta_1}{\sqrt{\left(1-\frac{a^2\omega^2}{c^2}\right)}}. \tag{6·9}$$

We may also note that $\dfrac{\Delta_1}{\Delta_2} = \sqrt{\left(1-\dfrac{a^2\omega^2}{c^2}\right)}$ = Doppler shift at $r = a$.

7. *Doppler effect.* The gravitational fields and the relative motions of the observer and the source give rise to a Doppler shift in the measured wave-lengths of light. Since our metrical coefficients are independent of t and depend on r alone (a statical space-time) the world-lines of observers at rest are given by

$$r = \text{const.}, \quad \theta = \text{const.}, \quad z = \text{const.},$$

and the Doppler shift is determined as follows:

$$D = \frac{\text{freq. (atom)}}{\text{freq. (observ.)}} = \frac{\text{period (observ.)}}{\text{period (atom)}} = \frac{(\sqrt{g_{44}})\,\text{observ.}}{(\sqrt{g_{44}})\,\text{atom}}. \tag{7·1}$$

(i) Let A be an observer at distance r from the light-source at the centre of the rotating disk. Then from (7·1) it follows that the observer at A will observe a Doppler shift D_A given by

$$D_A = \frac{\lambda+\delta\lambda}{\lambda} = \sqrt{\left(1-\frac{r^2\omega^2}{c^2}\right)}, \tag{7·2}$$

where λ is the wave-length of light.

(ii) Now interchanging the positions of light-source and observer, then a shift observed by an observer at the centre of the rotating disk will be

$$D_C = \frac{\lambda+\delta\lambda}{\lambda} = \frac{1}{\sqrt{\left(1-\frac{r^2\omega^2}{c^2}\right)}}. \tag{7·3}$$

From (7·2) and (7·3) we infer that in case (i) the observer at rest at distance $a-$ from the centre will observe a decrease in the radius of the disk, and in case (ii) the observer at

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the centre will observe an increase in the radius proportional to the shifts D_A and D_B. In (i) $\delta\lambda < 0$ and in (ii) $\delta\lambda > 0$.

From (6·8) we note that the ratio Δ_2/Δ_1 is equal to the Doppler shift observed at the centre of the disk for the light waves proceeding from the source S at distance $a-$ from the centre. The inverse of the same ratio Δ_1/Δ_2 gives the Doppler shift that an observer would find for the light waves, coming from the centre which is at a distance $a-$ from him.

8. *Distance by parallax.* We shall now derive a new distance formula in general relativity and apply the result to the rotating disk.

Let S be the source of light signals at a distance $a-$ from the centre of the disk and let O_1, O_2 be two observers very near the centre O such that the line SO in Galilean space-time of the non-rotating disk bisects O_1O_2 perpendicularly. By using the definition of the 'geodesic deviation' in a Riemannian space we may derive a parallax formula. Consider the two null geodesics $\Gamma(SO_1)$ and $\Gamma(SO_2)$. As before let $\eta^\alpha = dx^\alpha/d\lambda$ be the transport vector along Γ, where dx^α corresponds to a displacement along Γ, where λ satisfies the equations

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \quad \text{and} \quad g_{\alpha\beta}\eta^\alpha\eta^\beta = 0. \tag{8·1}$$

Now the singly infinite family of null geodesics through S form a two-dimensional Riemannian space V_2, namely, the null-cone at the point S. λ is a parameter varying along each geodesic of the family. If O_1', O_2' are any two observers on Γ and Γ' equidistant from S, i.e. λ is the same for both, and if a^α is a unit vector in the direction of the deviation $O_1'O_2'$, we may write

$$\mu = \eta_\alpha d\xi^\alpha = \text{const. for } \Gamma \text{ and } \Gamma', \tag{8·2}$$

where $d\xi^\alpha$ represents the infinitesimal distance $O_1'O_2'$.

The new parameter μ defined in (8·2) is constant along each null geodesic of the family, but varying as we pass from one null geodesic to another. The equation of the null-cone may be written

$$x^\alpha = x^\alpha(\lambda, \mu), \tag{8·3}$$

which is a solution of $g_{\alpha\beta}dx^\alpha dx^\beta = 0$.

Now, the apparent direction of the source in the instantaneous space of O_1' is specified by the projection of η^α in the space, i.e. by the vector

$$p'^\alpha = \eta^\alpha - a^\alpha(\eta_\alpha a^\alpha) = \eta^\alpha - ka^\alpha,$$

where

$$k = \eta_\alpha a^\alpha$$

and

$$p'_\alpha p'^\alpha = -(k^2).$$

Hence, the vector p^α defined by $p^\alpha = p'^\alpha/\sqrt{(-k^2)}$ is a unit vector along the apparent direction of the source in the instantaneous space of O_1'.

Since the curves $\mu = \text{const.}$ are null geodesics in space-time we have all over the null-cone

$$\frac{\delta p^\alpha}{\delta\lambda} = 0 = \frac{dp^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma}p^\beta\frac{dx^\gamma}{d\lambda}, \tag{8·4}$$

where p^α is the field of unit tangent vector to the projected null-geodesic in the instantaneous space of O_1' (6).

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Consider two adjacent null-geodesics Γ and Γ' with parameters μ and $\mu+d\mu$. The infinitesimal vector ξ^α can be expressed by

$$\xi^\alpha = \frac{\partial x^\alpha}{\partial \mu} d\mu. \tag{8·5}$$

The change in ξ^α as we pass along Γ will be given by

$$\frac{\delta \xi^\alpha}{\delta \lambda} = \frac{\delta}{\delta \lambda}\left(\frac{\partial x^\alpha}{\partial \mu}\right) d\mu, \tag{8·6}$$

since $d\mu$ is a constant for the pair of geodesics Γ and Γ'. Now

$$\frac{\delta}{\delta \lambda}\frac{\partial x^\alpha}{\partial \mu} = \frac{\partial x^\alpha}{\partial \mu \partial \lambda} + \Gamma^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial \mu}\frac{\partial x^\gamma}{\partial \lambda} = \frac{\delta}{\delta \mu}\frac{\partial x^\alpha}{\partial \lambda}. \tag{8·7}$$

Therefore

$$\frac{\delta \xi^\alpha}{\delta \lambda} = \frac{\delta p^\alpha}{\delta \mu} d\mu. \tag{8·8}$$

We take the absolute derivative of this equation with respect to λ and use (8·6). Thus we obtain

$$\frac{\delta^2 \xi^\alpha}{\delta \lambda^2} = \frac{\delta}{\delta \lambda}\left(\frac{\delta p^\alpha}{\delta \mu}\right) d\mu = \left(\frac{\delta}{\delta \mu}\frac{\delta p^\alpha}{\delta \lambda} + R^\alpha_{.\beta\gamma\epsilon} p^\beta p^\gamma \frac{\partial x^\epsilon}{\partial \mu}\right) d\mu. \tag{8·9}$$

where in obtaining (8·9) we have used the relation

$$\frac{\delta^2 p^\alpha}{\delta \lambda \delta \mu} - \frac{\delta^2 p^\alpha}{\delta \mu \delta \lambda} = R^\alpha_{.\beta\gamma\epsilon} p^\beta \frac{\partial x^\gamma}{\partial \lambda}\frac{\partial x^\epsilon}{\partial \mu}.$$

The first term on the right-hand side because of (8·4) vanishes. Rearranging the terms we get the equation of 'geodesic deviation' in space-time as

$$\frac{\delta^2 \xi^\alpha}{\delta \lambda^2} + R^\alpha_{.\beta\gamma\epsilon} p^\beta \xi^\gamma p^\epsilon = 0. \tag{8·10}$$

These four equations determine ξ^α as functions of λ if the initial values of ξ^α and $\delta\xi^\alpha/\delta\lambda$ are given (6).

Now for small ω, we may take

$$K = -\frac{3\omega^2}{c^2}\left(1 - \frac{r^2\omega^2}{c^2}\right)^{-2} = -\frac{3\omega^2}{c^2}. \tag{8·11}$$

To this order space-time is of constant curvature and the tensor can be written as

$$R^\alpha_{.\beta\gamma\epsilon} = K(\delta^\alpha_\gamma g_{\beta\epsilon} - \delta^\alpha_\epsilon g_{\beta\gamma}), \tag{8·12}$$

since the unit tangent vector p^α is perpendicular to the deviation ξ^α, i.e. $g_{\alpha\beta} p^\alpha \xi^\beta = 0$. The equation (8·10) becomes

$$\frac{\delta^2 \xi^\alpha}{\delta \lambda^2} + K\xi^\alpha = 0$$

or

$$\frac{\delta^2 \xi^\alpha}{\delta \lambda^2} - \frac{3\omega^2}{c^2}\xi^\alpha = 0. \tag{8·13}$$

Introducing the unit vector field b_α propagated parallel to and along the geodesic (i.e. $\delta b_\alpha/\delta\lambda = 0$) and multiplying (8·13) by b_α, we find that the resulting equation is

$$\frac{d^2(b_\alpha \xi^\alpha)}{d\lambda^2} - \frac{3\omega^2}{c^2}(b_\alpha \xi^\alpha) = 0. \tag{8·14}$$

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The complete integral of this equation has the form

$$b_\alpha \xi^\alpha = A \sinh \frac{\sqrt{3}\,\omega}{c}\lambda + B \cosh \frac{\sqrt{3}\,\omega}{c}\lambda.$$

Since for $\lambda = 0$, $\xi^\alpha = 0$, then $B = 0$, and the solution becomes

$$b_\alpha \xi^\alpha = A \sinh \frac{\sqrt{3}\,\omega}{c}\lambda. \tag{8·15}$$

Now taking an orthogonal set of n unit vectors $b^{(i)}_\alpha$ propagated parallel to and (where i is not a tensor index) along the geodesic, we have

$$g_{\alpha\beta} b^\alpha_{(i)} b^\beta_{(j)} = \delta^i_j. \tag{8·16}$$

Since the vectors $b^{(i)}_\alpha$ are independent the determinant $|\, b^\alpha_{(i)} \,|$ does not vanish in the region considered. Hence, solving equations (8·16), we get

$$g_{\alpha\beta} = b^{(i)}_\alpha b^{(i)}_\beta, \tag{8·17}$$

$$g_{\alpha\beta}\xi^\alpha \xi^\beta = (b^{(i)}_\alpha \xi^\alpha)(b^{(i)}_\beta \xi^\beta) = A^2 \sinh^2 \frac{\sqrt{3}\,\omega}{c}\lambda.$$

Hence

$$\xi = A \sinh \frac{\sqrt{3}\,\omega}{c}\lambda, \tag{8·18}$$

where A is a constant and ξ is the magnitude of the small geodesic deviation. From (8·18) it follows that the small angle at which the two adjacent geodesics separate at S is

$$\theta = \left(\frac{d\xi}{d\lambda}\right)_{\lambda=0} = A\frac{\sqrt{3}\,\omega}{c}.$$

Therefore, putting $\xi/\theta = \Delta$, $\lambda = a$, we obtain from (8·18) the distance by parallax

$$\Delta = \frac{c}{\sqrt{3}\,\omega}\sinh\frac{a\sqrt{3}\,\omega}{c},$$

which gives

$$\Delta = a\left(1 + \frac{1}{2}\frac{a^2\omega^2}{c^2}\right) \quad \text{(approx.)}. \tag{8·19}$$

Because, to the degree of accuracy considered, the curvature of space-time is constant, the result (8·19) will hold for both positions (the centre and the rim) of the source S.

For the measurement of apparent size of a small object on the circumference we again have

$$\Delta' = a\left(1 + \frac{a^2\omega^2}{2c^2}\right) = \Delta.$$

We now sum up the various results, which are obtained by different methods, in the following table:

Signals emitted at	Radius of the rotating disk from				Observer at
	The projected length of the null-geodesics	The Doppler shift	The light-time	The parallax	
C	Δ_1	aD_A	Δ_1	Δ'	A
A	$D_C\Delta_1$	aD_C	Δ_1	Δ'	C

Here C denotes the centre of the disk, and A denotes a point at distance $a-$ from C.

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Conclusion. Space-time on the rotating disk is treated without reference to the material composing the disk, and the metric is obtained from the field equations of the general relativity.

The diversity of the results of the measured radius with different methods justifies the assertion that for an observer on a rotating disk there is no way of escape from a curved space-time.

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