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CYCLICAL TRENDS IN CONTINUOUS TIME MODELS

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It is undoubtedly desirable that econometric models capture the dynamic behavior, like trends and cycles, observed in many economic processes. Building models with such capabilities has been an important objective in the continuous time econometrics literature, for instance, the cyclical growth models of Bergstrom (1966); the economy-wide macroeconometric models of, for example, Bergstrom and Wymer (1976); unobserved stochastic trends of Harvey and Stock (1988 and 1993) and Bergstrom (1997); and differential-difference equations of Chambers and McGarry (2002). This paper considers continuous time cyclical trends, which complement the trend-plus-cycle models in the unobserved components literature but could also be incorporated into Bergstrom type systems of differential equations, as were stochastic trends in Bergstrom (1997).

1. INTRODUCTION

One aim of Rex Bergstrom's research was to model the relationships between economic variables using continuous time dynamic systems. His models were specified as linear systems of stochastic differential equations making intensive use of economic theory to provide a parsimonious parameterization. Economic cycles are therefore an integral part of these systems and are revealed via the steady state solution of the model. Thus, the stability of growth and evidence of cycles are observed through examination of the eigenvalues of the structural parameter matrix, and sensitivity analysis provides, among other things, policy directions for reducing instability or cycle amplitudes. Examples include the early trade cycle model of Bergstrom (1966), which revealed stable growth and a damped eight-year cycle, then later, Bergstrom and Wymer (1976) became the prototype for many continuous time macro-models. With a further development to include second-order derivatives, Bergstrom, Nowman, and Wymer (1992) also revealed stable growth and damped cycles.

The forecasting performance of these models, however, was affected by the presence of deterministic time trends for factors like technical progress. With the advent of influential work on unobserved stochastic trends, as in Harvey and Stock

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(1988), Bergstrom (1997) incorporated such trends within a system of differential equations. To illustrate, the system could be expressed as $dx(t) = Ax(t)dt + \zeta(dt)$, where the vector $x(t)$ contains the levels and derivatives of observable economic variables as well as $\mu(t)$, an unobservable trend component given by $d\mu(t) = \lambda dt + \eta(dt)$ for λ a constant drift parameter ($\zeta(dt)$ and $\eta(dt)$ are random measures). Hence, the trends are embedded within, and thus help to drive, the structure of the model that generates the economic process. The empirical results in Bergstrom and Nowman (2007), where three such trends account for the growth of productivity, labor supply, and the use of plastic money, show that the model forecasts well and produces plausible long-run behavior with cycles of length 9 and 40 years.

The cyclical trends presented in this paper could ultimately be used to extend the system in Bergstrom (1997) because factors like technical progress and labor supply may also be cyclical, influenced perhaps by whether the economy is in a period of boom or slump. Hence, Bergstrom's models may be further developed by explicitly modeling this cyclical behavior with a combination of unobserved stochastic trend and cycle. As a further example, Bergstrom's models feature inventories that are also cyclical as firms try to smooth their production by dipping into and then replenishing their inventory levels and, as they are difficult to measure, may be effectively treated with an unobserved stochastic cycle. While cyclical behavior is integral to Bergstrom's models, it is not treated as an unobserved component as would be required if one wanted to model effects like those just outlined. Therefore, one could envisage extensions to Bergstrom (1997) in which vector $x(t)$ also includes a cyclical component $\phi(t)$ as given in, say, equation (3) below. This could be added to a stochastic trend for a trend-plus-cycle specification, or it could be incorporated as a cyclical trend as in (1) below (or akin with the Bergstrom constant drift specification, as $d\mu(t) = (\lambda + \phi(t))\,dt + \eta(dt)$). The cyclical properties of these factors would also be realizable and potentially important from a policy perspective. How practitioners combine the cycles and trends will be important, as the trend-plus-cycle specification may not allow sufficient interaction between the components. The cyclical trend proposed here (as far as the author is aware, this is the first investigation of cyclical trends in continuous time) provides a greater dependence by placing the cycle in the trend equation, allowing for periodicity in growth rates. Indeed, if one believes that technical progress is affected by the business cycle, then a cyclical trend may be able to account for this more naturally than a trend-plus-cycle.

The incorporation of combinations of unobserved stochastic trends and cycles into systems of differential equations would provide a synthesis of two fairly distinct modeling procedures: the Bergstrom approach and the unobserved components (UC) framework. The former uses economic theory to specify the relationships between variables, but the latter specifically models the time series features observed in real data, with little use of economic theory. Examples of additive UC models, where an economic process $y(t)$ is decomposed as $y(t) = \mu(t) + \phi(t)$ with a trend and cyclical component, respectively, can be found in

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Harvey and Stock (1993), where the cycle is specified as a stochastic differential equation (SDE) as in (3) below, and Chambers and McGarry (2002), where it is a differential-difference equation (DDE) as in (4) below. A cyclical trend may also be preferable to a trend-plus-cycle in such a framework. There is limited empirical evidence in discrete time UC models to suggest which is more suitable for economic variables or if other specifications are appropriate, e.g., higher-order stochastic cycles.[1] Harvey (1985), though, does find evidence of a discrete time cyclical trend in four out of five U.S. macro time series.

The models proposed in this paper therefore have a dual purpose. Firstly, they complement Harvey and Stock (1993) and Chambers and McGarry (2002) by broadening the range of models available in the continuous time UC framework. Secondly, the cyclical trend could be incorporated into Bergstrom type systems of equations as described above with the potential to bring further improvements to the forecasting ability of Bergstrom's models. However, the cyclical trend is introduced here in the simplest of environments to illustrate its potential in future research.

2. THE CYCLICAL TREND MODELS

The cyclical trend component $\mu(t)$ to be considered here includes a stochastic drift term $\beta(t)$ as well as the cyclical term $\phi(t)$, such that

$$d\mu(t) = (\beta(t) + \phi(t))\,dt + \eta(dt) \tag{1}$$

$$d\beta(t) = \zeta(dt), \tag{2}$$

and here, $\phi(t)$ can be specified *either* as an SDE (as in Harvey and Stock, 1993), i.e.,

$$d\begin{bmatrix}\phi(t)\\ \phi^*(t)\end{bmatrix} = \begin{bmatrix}\ln\rho & \omega\\ -\omega & \ln\rho\end{bmatrix}\begin{bmatrix}\phi(t)\\ \phi^*(t)\end{bmatrix}dt + \begin{bmatrix}\kappa(dt)\\ \kappa^*(dt)\end{bmatrix} \tag{3}$$

or *alternatively* as a DDE (as in Chambers and McGarry, 2002), i.e.,

$$d\phi(t) = (a_0\phi(t) + a_1\phi(t-l))\,dt + \kappa(dt). \tag{4}$$

For example, in the UC framework one might specify a continuous time economic process $y(t)$ as a cyclical trend, i.e., $y(t) = \mu(t)$. The random measures $\eta(dt)$ and $\zeta(dt)$ have zero mean, respective variances $\sigma_\eta^2 dt$ and $\sigma_\zeta^2 dt$, and are serially and mutually uncorrelated. The trend term has a stochastic drift element, becoming a constant drift if $\sigma_\zeta^2 = 0$. Harvey and Jaeger (1993) argue in favor of the local linear trend for UC models because the possibility of a zero variance in the *level* produces a smoother trend, which suits some economic time series. Bergstrom and Nowman (2007), however, are satisfied that a constant drift is appropriate for their purposes; otherwise, technical progress would almost certainly exceed any bound at some future point in time.

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2.1. The SDE cycle: In (3), ρ is a damping parameter and ω represents the angular frequency of the cycle, where $\rho > 0$ and $0 \leq \omega < \pi$ for identification purposes and for a stationary cycle $\rho < 1$. The random measures $\kappa(dt)$ and $\kappa^*(dt)$ have zero mean, common variance $\sigma_\kappa^2 dt$, and are serially and mutually uncorrelated. The exact discrete time components model would be $y_t = \mu_t + \epsilon_t$, where

$$\mu_t = \mu_{t-1} + \beta_{t-1} + \int_0^1 \rho^s \cos(\omega s)\, ds \phi_{t-1} + \int_0^1 \rho^s \sin(\omega s)\, ds \phi^*_{t-1} + \eta_t, \qquad \textbf{(5)}$$

the irregular term[2] ϵ_t has variance σ_ϵ^2, and the remaining equations are as in an additive model, i.e., $\beta_t = \beta_{t-1} + \zeta_t$, $\phi_t = \rho \cos\omega\phi_{t-1} + \rho \sin\omega\phi^*_{t-1} + \kappa_t$, and $\phi^*_t = -\rho \sin\omega\phi_{t-1} + \rho \cos\omega\phi^*_{t-1} + \kappa^*_t$, where stock variables are given, for example, by $\mu_t = \mu(t)$ and flows by $\mu_t = \int_{t-1}^t \mu(r)\, dr$.[3] The unknown parameters are $\Theta = \{\rho, \omega\}$ and $\sigma_\eta^2, \sigma_\zeta^2, \sigma_\kappa^2$, and σ_ϵ^2.

2.2. The DDE cycle: In (4), the cycle duration is calculated indirectly from a complicated function of the parameters a_0, a_1, and the lag parameter l. The random measure $\kappa(dt)$ again has zero mean and variance $\sigma_\kappa^2 dt$. The estimable parameters are $\Xi = \{a_0, a_1, l\}$ and the variances. At present, no convenient way has been found to express exactly the process generated by a DDE in discrete time, i.e., a model analogous to (5). To overcome this, the frequency domain methods of Ercolani and Chambers (2006) are used.

It has been assumed that there is no correlation between the different component disturbances. This is a common feature of models of this kind and is imposed for identification purposes, but it can be quite restrictive. Bergstrom and Nowman (1999), for example, consider a model in which interest rates are a function of two unobservable variables for short and long-term economic news, which are modeled as a bivariate system of differential equations with correlated innovations. The discrete time model for interest rates is an ARMA(2,1), and there are too few autocorrelations of this moving average error to identify the continuous-time innovation variances unless the correlation parameter is fixed. Bergstrom and Nowman set this parameter to zero, and hence estimation requires the restrictive assumption that the news streams are uncorrelated processes. Similar problems occur with the models presented here. If one models a single economic process as a cyclical trend, i.e., $y(t) = \mu(t)$ with (1), (2), and (3) above, the discrete time model $y_t = \mu_t + \epsilon_t$ can be expressed as an ARIMA(2,2,4). Thus, correlation between the innovations in the trend and cyclical components, i.e., $\sigma_{\eta\kappa}$ and $\sigma_{\zeta\kappa}$, causes an identification problem for the variance parameters unless $\sigma_{\eta\kappa}$ and $\sigma_{\zeta\kappa}$ are fixed. We consequently follow Bergstrom and Nowman (1999) and set these parameters to zero. This is not too restrictive, since there is always dependence between the trend and cycle (given (1)), but in a trend-plus-cycle model, such restrictions imply independent components, which may be inappropriate for some processes. Identification problems in continuous time models can arise from other sources. The aliasing phenomenon permeates most relevant continuous time

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econometric models. It is most problematic in the presence of complex eigenvalues and hence may occur in models of cyclical behavior. Therefore it is acknowledged by the author that some identifiability issues may exist in the models above and that the simulation results that follow must be interpreted in light of this.

Two alternative cyclical specifications have been provided in equations (3) and (4) above, and it is appropriate to give some comparative discussion. The SDE is certainly the more intuitive, as its parameters relate directly to the cycle properties of frequency and damping, and it benefits from the ease with which forecasts can be made and cyclical and trend behavior extracted. However, in his motivation for the use of continuous time models, Gandolfo (1981) states that the treatment of dynamic economic processes would be better handled with DDEs, which are essentially generalizations of SDEs. The adjustment mechanism of some processes requires the presence of lagged variables, maybe lagged derivatives, and hence, Bergstrom type differential equation systems are not appropriate. For example, DDEs arise in the time-to-build literature where the gestation period of investment projects plays a major role in generating cyclical behavior and is incorporated using a lag. Unfortunately, the current absence of an exact discrete time model limits the econometric usefulness of DDEs. Identification problems may be a feature, but a proper investigation of how this manifests itself is limited without an exact discrete time model. It also makes signal extraction problematic and it is not obvious how to execute such activities without the use of an approximate discretization. The data generated in the simulations below are from an ARMA approximation of the DDE, observed at short sub-intervals of the unit time interval, which may provide a partial resolution to the problem of extracting components. Specifically, the data are generated using $\phi(t)-\phi(t-h)=[a_0\phi(t-h)-a_1\phi(t-l-h)]h+\sigma_\kappa\sqrt{h}\kappa(t)$ for $\kappa(t)\sim N(0,\sigma_\kappa^2)$ and h the length of the sub-interval, where stocks are constructed by taking every $1/h$th observation and flows are an average over every $1/h$ observations. The finer the sub-interval, the closer one would be to generating a continuous time series and the better the approximation.

A more comprehensive simulation study is desirable but would require an exact discrete model for the DDE as well as a resolution to the identification problems that may pervade these models. Hence, the results, while suggestive, must be treated as preliminary. Subject to these caveats, the exercise attempts to assess the effects of cycle specification, sample size T, cycle duration, cycle damping, and sampling scheme. The free parameters are those that determine cycle durations, i.e., a_0, a_1, l, and ω, and are chosen to give similar spectral peaks between SDE and DDE specifications for cycles of length 5, 10, and 40 (the latter two matching those found in Bergstrom and Nowman, 2007). For example, high peak spectra (prominent cycles) of length 10 are produced by $\Xi=\{1,-1.1722,0.675\}$ and $\Phi=\{0.9,\pi/5\}$ and low peaks by $\Xi=\{0.5,-0.7423,1\}$ and $\Phi=\{0.85,\pi/5\}$. Damping values of 0.9 and 0.85 seem appropriate empirically, and although numerically close, the spectral difference is quite marked. Each experiment generates 10,000 replications of data, and $T=64,128,256,$

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and 512. The results in Table 1 are mean squared errors (MSEs) multiplied by 10,000.

As is expected, the results improve uniformly with sample size, and the large sample results are very good. With the SDE, the MSE in many cases reduces quite dramatically with sample size, particularly for low damping (high peak),

TABLE 1. Simulation results

		High peak				Low peak			
Cycle length	Sampling scheme	$T = 64$	128	256	512	$T = 64$	128	256	512
a. Parameter ω in the SDE Model									
5	Stocks	1,324	31	13	6	3,140	1,767	31	15
	Flows	1,119	31	13	6	6,824	591	32	15
10	Stocks	621	63	15	7	7,391	2,705	40	19
	Flows	1,083	38	15	7	3,946	1,084	39	19
40	Stocks	4,974	382	33	17	5,205	4,715	47	15
	Flows	128	57	33	17	2,407	89	47	16
b. Parameters a_0, a_1, and l in the DDE Model									
5	Stocks								
	a_0	2,332	1,021	349	18	3,637	1,557	365	4
	a_1	1,158	507	168	11	1,646	467	133	19
	l	410	179	63	4	1,216	624	154	11
	Flows								
	a_0	2,274	1,037	353	16	3,397	1,463	356	5
	a_1	1,107	511	169	9	1,662	442	131	20
	l	423	189	66	4	1,194	596	149	12
10	Stocks								
	a_0	1,886	920	209	2	1,322	605	123	6
	a_1	1,429	740	173	2	705	348	84	8
	l	1,275	466	98	3	3,584	1,546	278	5
	Flows								
	a_0	1,838	887	197	2	1,290	593	114	6
	a_1	1,363	707	161	3	666	338	80	7
	l	1,332	474	96	3	3,755	1,580	269	5
40	Stocks								
	a_0	842	411	140	8	14,567	1,281	266	39
	a_1	721	372	128	7	15,551	1,272	403	26
	l	2,498	936	282	14	2,090	633	17	1
	Flows								
	a_0	839	415	136	8	11,394	1,361	268	40
	a_1	716	376	125	8	14,042	1,226	251	27
	l	2,577	970	289	14	2,568	468	10	1

Note: The entries are mean squared errors multiplied by 10^4.

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where very accurate results are obtained at $T = 128$. For the DDE cycle, there appears to be very little difference between stocks and flows, whereas the SDE cycle does display sampling scheme differences for small sample sizes but with no discernible pattern. The cycle damping effect is evident for low T in the SDE model, reflecting that it may be more difficult to estimate less prominent cycles. Combined with a long cycle length, damping has a similar effect for the DDE, but at shorter cycle lengths the low peak combinations appear to produce lower MSEs for the a_0 and a_1 parameters. One would expect difficulties in estimating long-length cycles with few observations, given that very few cycles will be completed within the time interval. However, in some cases this is not observed in the results. For the DDE, the lag parameter certainly has a lower MSE for cycle 5 than 40, but the reverse is true for a_0 and a_1, and estimating ω with flow variables in the SDE seems to improve with cycle length.

The results show that estimating cycles consistent with those found in Bergstrom and Nowman (2007) will require fairly large samples, and there appear to be advantages in using the SDE cycle specification to do this if the cycles are prominent, with damping factors of about 0.9 or higher. With 30 years of quarterly data, fairly accurate estimates should be obtained in the presence of a cycle of 2–3 years or about 10 years (length 10 and 40, respectively), irrespective of whether the data are stocks or flows, although the prominence of the cycles may have an effect.

3. CONCLUDING COMMENTS

This paper has introduced continuous time cyclical trends with the purpose not only of widening the choice of model available in the UC framework, but also as a potential alternative to modeling the unobserved stochastic components that feature in Bergstrom and Nowman (2007). If it is believed that technical progress, etc., are cyclical, as well as trending processes, then one way to capture this is to follow Bergstrom (1997) and treat them as unobserved components in an economic system of differential equations. The extra generality afforded by incorporating cyclical trends may further improve the forecasting ability of Bergstrom's macroeconomic models. Of course, it remains for future research to derive the exact discrete model of such a system, analogous to Bergstrom (1997), but this paper has at least introduced the specification, albeit in a simple setting, and discussed the many issues involved. These are serious issues that must be treated in future research and must be borne in mind when interpreting the simulation results. Inherent small sample problems will occur with cyclical models, as both long duration and heavy damping may result in cyclical parameters that are hard to estimate, although the simulations here provide mixed results in this respect. Without a discrete time equivalent, the DDE cycle is not as flexible a specification as the SDE, and when reading the simulation results, one has to bear in mind that the DDE data are generated from an approximation. Identification problems may

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exist in some form, as in many continuous time models, and this remains one of the greatest problems in continuous time econometrics.

NOTES

1. In Harvey and Trimbur (2003) and Trimbur (2006), the cyclical components have periodic innovations.

2. See Chambers and McGarry (2002) for an explanation of why this is appropriate.

3. Chambers and McGarry (2002) show how to derive the error variance and autocovariance matrices.

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