THE ENRIQUES THREEFOLD

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It is a classical and well-established result that a unirational algebraic variety of dimension $\leqslant 2$ is necessarily rational. It is also generally agreed, though not so well established, that this result is no longer true for varieties of dimension three. In this connexion, the critical example has been the Enriques threefold E, asserted by Roth (1) to be unirational but not rational. The purpose of this note is to point out a fallacy in the proof of the non-rationality of E and, incidentally, to resolve a difficulty raised by Serre (2) in which E would appear to feature as a counter-example to a well-established general theorem. As far as the author is aware, there are no other examples of non-rational unirational threefolds, so that the question of their existence is still open.

The Enriques threefold E is a sextic primal of four-dimensional projective space having an equation of the form

$$x_1x_2x_3x_4\{x_0^2+2x_0(lx_1+mx_2+nx_3+px_4)+(a,b,c,\ldots,w\chi x_1,x_2,x_3,x_4)^2\}$$
$$+(A,B,C,\ldots,W\chi x_2x_3x_4,x_3x_4x_1,x_4x_1x_2,x_1x_2x_3)^2=0. \quad (1)$$

The vertex X_0 of the simplex of reference $X_0X_1X_2X_3X_4$ is an ordinary quadruple point of E, the four edges X_0X_i are triple lines of E, and the six planes $X_0X_iX_j$ constitute the double surface on E.

E is certainly unirational (cf. Roth (1), p. 97) and its non-singular model E' has been thought to possess Severi torsion (thus apparently establishing the non-rationality of E). Indeed, if γ_1, γ_2 denote the *double-planes* $X_0X_1X_2$, $X_0X_3X_4$ on E (a double-plane being thought of as an implicit rational surface on E), then it is possible to find a linear system of quartic surfaces on E containing $2\gamma_1$ and $2\gamma_2$ as particular members; the non-equivalence of γ_1 and γ_2 on E, together with the statement that E has ordinary singularities, would then indicate the existence of torsion on E'. (This is the point of disagreement with Serre (2), where it is shown that a non-singular unirational variety can possess no torsion.)

A consequent re-examination of E has revealed the existence of eight hitherto unnoticed quadruple points, two on each triple line, *which are not ordinary singularities*. In terms of equation (1), these may readily be located at $(-l\pm(l^2-a)^{\frac{1}{2}}, 1, 0, 0, 0)$ and at six other points on the remaining three triple lines. The cones of tangent lines at these quadruple points are irreducible quartic (threefold) cones of the type obtained by joining a Steiner monoidal quartic surface to a point outside its ambient three-space. Owing to the presence of these non-ordinary singularities, the arguments for the existence of torsion on E'—and hence also the arguments for the non-rationality of E—are invalidated.

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On examination of the non-singular model E', with γ_1', γ_2' denoting the proper transforms of γ_1, γ_2, one finds that $2(\gamma_1' - \gamma_2')$ is equivalent on E' to a linear combination of surfaces which 'arise out of' the eight new quadruple points of E, this linear combination having coefficients ± 1. Since this does not indicate torsion on E', there seems no further reason to suppose that E' is a counter-example to Serre's theorem. It still remains, of course, to decide whether the threefold is rational or not.

REFERENCES

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(2) Serre, J.-P. On the fundamental group of a unirational variety. *J. Lond. Math. Soc.* 34 (1959), 481–4.

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A CRYSTAL DISLOCATION PROBLEM

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The problem considered in this note arises in the theory of perfect crystal dislocations and I am indebted to Prof. F. R. N. Nabarro for drawing my attention to it.

A vector $\mathbf{p}$ of a three-dimensional lattice Λ will be called a *slip vector* if $\mathbf{t}.(\mathbf{p}-\mathbf{t}) \leqslant 0$ for all $\mathbf{t} \in \Lambda$. Following Frank and Nicholls (1), we shall call a slip vector $\mathbf{p}$ *stable* if $\mathbf{t}.(\mathbf{p}-\mathbf{t}) < 0$ for all $\mathbf{t} \in \Lambda$ other than $\mathbf{0}$ and $\mathbf{p}$. Frank and Nicholls have enumerated the slip vectors for the simple cubic, face-centred cubic, diamond, hexagonal close-packed and body-centred cubic lattices, and have shown that there are only a finite number of slip vectors in each of these cases. In this note we show that, for any lattice, there are only a finite number of slip vectors and that they can be found by examining those points of the lattice that lie inside or on a certain sphere. More precisely, we prove the

Theorem. *Let the lattice Λ be generated by the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and let $\mathbf{p}$ be a point of Λ. If $\mathbf{p}^2 > \mathbf{a}_1^2+\mathbf{a}_2^2+\mathbf{a}_3^2$, there exists a vector $\mathbf{t} \in \Lambda$ for which $\mathbf{t}.(\mathbf{p}-\mathbf{t}) > 0$; if $\mathbf{p}^2 = \mathbf{a}_1^2+\mathbf{a}_2^2+\mathbf{a}_3^2$, there exists a vector $\mathbf{t} \in \Lambda$, with $\mathbf{t} \neq \mathbf{0}, \mathbf{p}$, for which $\mathbf{t}.(\mathbf{p}-\mathbf{t}) \geqslant 0$.*

It follows that, if $\mathbf{p}$ is a slip vector, then $\mathbf{p}^2 \leqslant \mathbf{a}_1^2+\mathbf{a}_2^2+\mathbf{a}_3^2$; further, if $\mathbf{p}$ is stable then $\mathbf{p}^2 < \mathbf{a}_1^2+\mathbf{a}_2^2+\mathbf{a}_3^2$.

Proof. Suppose that $\mathbf{p} = n_1\mathbf{a}_1+n_2\mathbf{a}_2+n_3\mathbf{a}_3$, where n_1, n_2 and n_3 are integers. For $\nu = 1, 2, 3$ take ϵ_ν to be 0 or 1 according as n_ν is even or odd. Write

$$\mathbf{t}_0 = \tfrac{1}{2}\{(n_1+\epsilon_1)\,\mathbf{a}_1+(n_2+\epsilon_2)\,\mathbf{a}_2+(n_3+\epsilon_3)\,\mathbf{a}_3\}.$$

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