Ergod. Th. & Dynam. Sys. (2007), **27**, 1819–1837 © *2007 Cambridge University Press*
doi: 10.1017/S0143385707000296 *Printed in the United Kingdom*

The structure of tame minimal dynamical systems

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(*Received* 18 *September* 2006 *and accepted in revised form* 5 *March* 2007)

Abstract. A dynamical version of the Bourgain–Fremlin–Talagrand dichotomy shows that the enveloping semigroup of a dynamical system is either very large and contains a topological copy of $\beta\mathbb{N}$, or it is a 'tame' topological space whose topology is determined by the convergence of sequences. In the latter case, the dynamical system is said to be tame. We use the structure theory of minimal dynamical systems to show that, when the acting group is Abelian, a tame metric minimal dynamical system (i) is almost automorphic (i.e. it is an almost one-to-one extension of an equicontinuous system), and (ii) admits a unique invariant probability measure such that the corresponding measure-preserving system is measure-theoretically isomorphic to the Haar measure system on the maximal equicontinuous factor.

0. *Introduction*

In this work a dynamical system is a pair (X, Γ), where X is a compact Hausdorff space and Γ an abstract group acting as a group of homeomorphisms of the space X. That is, we are given a homomorphism (not necessarily an isomorphism) of Γ into $\mathrm{Homeo}(X)$. For $\gamma \in \Gamma$ and $x \in X$ we write γx for the image of x under the homeomorphism which corresponds to γ. We will often abuse this notation and consider γ as a homeomorphism of X.

The *enveloping semigroup* $E(X, \Gamma)$ of the dynamical system (X, Γ) is defined as the closure of the image of Γ in the product space X^X. It is not hard to check that, under composition of maps, $E(X, \Gamma)$ is a compact *right topological semigroup*, i.e. for each $q \in E(X, \Gamma)$ the map $R_q : p \mapsto pq$ is continuous. In fact, the canonical homomorphism of Γ into $E(X, \Gamma)$ is a *right topological semigroup compactification* of Γ, i.e. it has a dense range and for each $\gamma \in \Gamma$ multiplication on the left $L_\gamma : p \mapsto \gamma p$ is continuous on $E(X, \Gamma)$. This left multiplication by elements of Γ makes $(E(X, \Gamma), \Gamma)$ a dynamical system.

The enveloping semigroup was introduced by Robert Ellis in 1960 [**5**] and became an indispensable tool in abstract topological dynamics. However, explicit computations of

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enveloping semigroups are quite rare. One reason for this is that often $E(X, G)$ is non-metrizable.

Following an idea of Köhler [**19**], Glasner and Megrelishvili [**14**] proved the following dynamical version of the Bourgain–Fremlin–Talagrand (BFT) dichotomy theorem [**2**].

THEOREM 0.1. (A dynamical BFT dichotomy) *Let (X, G) be a metric dynamical system and let $E(X)$ be its enveloping semigroup. We have the following dichotomy. Either*

(i) *$E(X)$ is separable Rosenthal compact, hence with cardinality* card $E(X) \leq 2^{\aleph_0}$*; or*

(ii) *the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$, hence* card $E(X) = 2^{2^{\aleph_0}}$.

A dynamical system is said to be *tame* if the first alternative occurs, i.e. $E(X)$ is Rosenthal compact. Recently, dynamical characterizations of both tame dynamical systems and dynamical systems whose enveloping groups are metrizable were obtained by Glasner, Megrelishvili and Uspenskij in [**14**] and [**15**].

THEOREM 0.2. *A compact metric dynamical system (G, X) is tame if and only if every element of $E(X)$ is a Baire 1 function from X to itself.*

THEOREM 0.3. *Let X be a compact metric G-space. The following conditions are equivalent:*

(i) *the dynamical system (G, X) is hereditarily almost equicontinuous (HAE);*

(ii) *the dynamical system (G, X) is RN, that is, it admits a proper representation on a Radon–Nikodým Banach space;*

(iii) *the enveloping semigroup $E(X)$ is metrizable.*

For the definitions of HAE systems and the other undefined notions which appear in these theorems, as well as for some further motivation and examples, we refer the reader to the papers [**13, 14**] and [**15**].

In [**13**] it has been shown that a *minimal* metrizable tame dynamical system with a commutative acting group is PI (see §1 below) and has zero topological entropy. Recently, Huang [**16**], and independently Kerr and Li [**18**], improved these results to show that under the same conditions a minimal tame system is an almost one-to-one extension of its maximal equicontinuous factor and is uniquely ergodic (see also [**17**]). In these works the authors make heavy use of the structure theory of minimal dynamical systems, as developed by Ellis, Veech, Ellis, Glasner and Shapiro, McMahon and van der Woude (see e.g. the survey [**12**] and the references therein). However, the main tool in both works (of Huang and Kerr and Li) is the combinatorial notion of independence and the various related notions of independence n-tuples. In fact, Kerr and Li in their work [**18**], use independence to unify the theory of these various notions and, in particular, they are able to characterize tame systems (which they call regular) as those systems that (in some precise sense) do not admit infinite independence sets [**18**, Proposition 6.4.2]. In turn they use this characterization to define a notion of relative regularity and develop the whole theory in the relative set-up.

In the present work, which can be regarded as a continuation of [**13**], we pursue purely structure theoretical methods to recover the results of Huang and Kerr and Li mentioned above, avoiding the combinatorial treatment altogether. The key tool used in the proof,

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here as well as in [**13**], is a proposition about diffused measures (Proposition 3.3 below), which first appeared in [**11**].

Section 1 is a brief review of the structure theory of minimal dynamical systems. In §2, an analogue of an old theorem of Ditor and Eifler [**4**] is proved, which may have some independent interest. It shows that when a continuous surjection $\pi : X \to Y$, with X and Y compact metrics, is semi-open then so is the induced map $\pi_* : \mathfrak{M}(X) \to \mathfrak{M}(Y)$ on the spaces of probability measures equipped with the weak* topology. In §3, the idea of diffused measures, first used in [**11**], is pursued and the key Proposition 3.3 proved. Section 4 develops the theory of tame systems using and extending results from [**13**]. In the final §5, the main theorem is proved.

Except for the introductory §1, the group Γ is assumed to be Abelian. For simplicity, only the case where the dynamical system is metrizable is considered and only the absolute (and not the more general relative) case.

1. *A brief survey of abstract topological dynamics*

This section is a brief review of the structure theory of minimal dynamical systems. We will emphasize some aspects which will be relevant in the present work. For full details the reader is referred to the books [**1, 6, 10**] and [**22**] and the review articles [**12**] and [**21**].

A *topological dynamical system* or in short a system is a pair (X, Γ), where X is a compact Hausdorff space and Γ is an abstract infinite group which acts on X as a group of homeomorphisms. When there is no room for confusion we write X for the system (X, Γ). A *subsystem* of (X, Γ) is a closed invariant subset $Y \subset X$ with the restricted action. For a point $x \in X$, we let $\mathcal{O}_\Gamma(x) = \{\gamma x : \gamma \in \Gamma\}$, and $\overline{\mathcal{O}}_\Gamma(x) = \mathrm{cls}\{\gamma x : \gamma \in \Gamma\}$. These subsets of X are called the *orbit* and *orbit closure* of x, respectively. We say that (X, Γ) is *point transitive* if there exists a point $x \in X$ with a dense orbit. In that case, x is called a *transitive point*. If every point is transitive we say that (X, Γ) is a *minimal system*. We say that $x \in X$ is an *almost periodic* or a *minimal* point if $\overline{\mathcal{O}}_\Gamma(x)$ is a minimal system.

The dynamical system (X, Γ) is *topologically transitive* if for any two non-empty open subsets U and V of X there exists some $\gamma \in \Gamma$ with $\gamma U \cap V \neq \emptyset$. Clearly a point transitive system is topologically transitive and when X is metrizable the converse holds as well: in a metrizable topologically transitive system the set of transitive points is a dense G_δ subset of X.

The system (X, Γ) is *weakly mixing* if the product system $(X \times X, \Gamma)$ (where $\gamma(x, x') = (\gamma x, \gamma x'),\ x, x' \in X,\ \gamma \in \Gamma$) is topologically transitive.

If (Y, Γ) is another system, then a continuous onto map $\pi : X \to Y$ satisfying $\gamma \circ \pi = \pi \circ \gamma$ for every $\gamma \in \Gamma$ is called a *homomorphism* of dynamical systems. In this case we say that (Y, Γ) is a *factor* of (X, Γ) and also that (X, Γ) is an *extension* of (Y, Γ). With the system (X, Γ) we associate the induced action (the *hyper system* associated with (X, Γ)) on the compact space 2^X of closed subsets of X equipped with the Vietoris topology. A subsystem Y of $(2^X, \Gamma)$ is a *quasifactor* of (X, Γ) if $\bigcup\{A : A \in Y\} = X$.

The system (X, Γ) can always be considered as a quasifactor of (X, Γ) by identifying x with $\{x\}$. Recall that if $(X, \Gamma) \xrightarrow{\pi} (Y, \Gamma)$ is a homomorphism then in general $\pi^{-1} : Y \to 2^X$ is an uppersemicontinuous map and that $\pi : X \to Y$ is open if and only if $\pi^{-1} : Y \to 2^X$ is continuous, if and only if $\{\pi^{-1}(y) : y \in Y\}$ is a quasifactor of (X, Γ) .

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We assume for simplicity that our acting group Γ is a discrete group. $\beta\Gamma$ will denote the Stone–Čech compactification of Γ. The universal properties of $\beta\Gamma$ make it:

- a compact semigroup with right continuous multiplication (for a fixed $p \in \beta\Gamma$ the map $q \mapsto qp,\ q \in \beta\Gamma$ is continuous), and left continuous multiplication by elements of Γ, considered as elements of $\beta\Gamma$ (for a fixed $\gamma \in \Gamma$ the map $q \mapsto \gamma q,\ q \in \beta\Gamma$ is continuous);
- a dynamical system $(\beta\Gamma, \Gamma)$ under left multiplication by elements of Γ.

The system $(\beta\Gamma, \Gamma)$ is the universal point transitive Γ-system, i.e. for every point transitive system (X, Γ) and a point $x \in X$ with dense orbit, there exists a homomorphism of systems $(\beta\Gamma, \Gamma) \to (X, \Gamma)$ which sends e, the identity element of Γ, onto x. For $p \in \beta\Gamma$ we let px denote the image of p under this homomorphism. This defines an 'action' of the semigroup $\beta\Gamma$ on every dynamical system. In fact, by universality there exists a unique homomorphism $(\beta\Gamma, \Gamma) \to (E(X, \Gamma), \Gamma)$ onto the enveloping semigroup $E(X, \Gamma)$ which is also a semigroup homomorphism and we can interpret, and often do, the $\beta\Gamma$ action on X via this homomorphism.

When dealing with the hyper system $(2^X, \Gamma)$ we write $p \circ A$ for the image of the closed subset $A \subset X$ under $p \in \beta\Gamma$ to distinguish it from the (usually non-closed) subset $pA = \{px : x \in A\}$. If p is the limit of a net γ_i in Γ then

$$p \circ A = \{x \in X : \text{there are a subnet } \gamma_{i_j} \text{ and a net } x_j \in A \text{ with } x = \lim_j \gamma_{i_j} x_j\}.$$

We always have $pA \subset p \circ A$.

The compact semigroup $\beta\Gamma$ has a rich algebraic structure. For instance, for countable Γ there are 2^c minimal left (necessarily closed) ideals in $\beta\Gamma$ which are all isomorphic as systems, with each serving as a universal minimal system. Each such minimal ideal, say M, has a subset J of 2^c idempotents such that $\{vM : v \in J\}$ is a partition of M into disjoint isomorphic (non-closed) subgroups. An idempotent in $\beta\Gamma$ is said to be *minimal* if it belongs to some minimal ideal. A point x in a dynamical system (X, Γ) is a minimal point if and only if there is some minimal idempotent v in $\beta\Gamma$ with $vx = x$, if and only if there exists some $v \in J$ with $vx = x$.

The group of dynamical system automorphisms of (M, Γ), $G = \mathrm{Aut}(M, \Gamma)$ can be identified with any one of the groups vM as follows: with $\alpha \in vM$ we associate the automorphism $\hat{\alpha} : (M, \Gamma) \to (M, \Gamma)$ given by right multiplication $\hat{\alpha}(p) = p\alpha,\ \ p \in M$. The group G plays a central role in the algebraic theory. It carries a natural T_1 compact topology, called by Ellis the τ-*topology* (see e.g. [**6**]), which is weaker than the relative topology induced on $G = uM$ as a subset of M. The τ-closure of a subset A of G consists of those $\beta \in G$ for which the set $\mathrm{graph}(\beta) = \{(p, p\beta) : p \in M\}$ is a subset of the closure in $M \times M$ of the set $\bigcup\{\mathrm{graph}(\alpha) : \alpha \in A\}$. Both right and left multiplication on G are τ continuous and so is inversion.

It is convenient to fix a minimal left ideal M in $\beta\Gamma$ and an idempotent $u \in M$. As explained above we identify G with uM and it follows that, for any subset $A \subset G$,

$$\mathrm{cls}_\tau A = u(u \circ A) = G \cap (u \circ A).$$

Also in this way we can consider the 'action' of G on every system (X, Γ) via the action of $\beta\Gamma$ on X. With every minimal system (X, Γ) and a point $x_0 \in uX = \{x \in X : ux = x\}$

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we associate a τ-closed subgroup

$$\mathfrak{G}(X, x_0) = \{\alpha \in G : \alpha x_0 = x_0\},$$

the *Ellis group* of the pointed system (X, x_0). The quotient space $G/\mathfrak{G}(X, x_0)$ can be identified with the subset $uX \subset X$ via the map $\alpha \mapsto \alpha x_0$ and the induced quotient τ-topology is called the *τ-topology* on uX. Again the τ-topology is weaker than the relative topology induced on uX as a subset of X, it is T_1 and compact, and the closure operation is given by

$$\mathrm{cls}_\tau A = u(u \circ A) = uX \cap (u \circ A), \quad A \subset uX.$$

For a homomorphism $\pi : X \to Y$ with $\pi(x_0) = y_0$ we have

$$\mathfrak{G}(X, x_0) \subset \mathfrak{G}(Y, y_0).$$

For a τ-closed subgroup F of G the *derived group* F' is given by

$$F' := \bigcap \{\mathrm{cls}_\tau O : O \text{ a } \tau\text{-open neighborhood of } u \text{ in } F\}. \tag{1.1}$$

F' is a τ-closed normal (in fact, characteristic) subgroup of F and it is characterized as the smallest τ-closed subgroup H of F such that F/H is a compact Hausdorff topological group. In particular, for an Abelian Γ, the topological group G/G' is the Bohr compactification of Γ.

A pair of points $(x, x') \in X \times X$ for a system (X, Γ) is said to be *proximal* if there exists a net $\gamma_i \in \Gamma$ and a point $z \in X$ such that $\lim \gamma_i x = \lim \gamma_i x' = z$ (if and only if there exists $p \in \beta\Gamma$ with $px = px'$). We denote by P the set of proximal pairs in $X \times X$. We have

$$P = \bigcap \{\Gamma V : V \text{ a neighborhood of the diagonal in } X \times X\}.$$

A system (X, Γ) is said to be *proximal* when $P = X \times X$ and *distal* when $P = \Delta$, the diagonal in $X \times X$. It is said to be *strongly proximal* when the following much stronger condition holds: the dynamical system $(\mathfrak{M}(X), \Gamma)$, induced on the compact space $\mathfrak{M}(X)$ of probability measures on X, is proximal. A minimal system (X, Γ) is said to be *point distal* if there exists a point $x \in X$ such that if x, x' is a proximal pair then $x = x'$.

The *regionally proximal relation* on X is defined by

$$Q = \bigcap \{\overline{\Gamma V} : V \text{ a neighborhood of } \Delta \text{ in } X \times X\}.$$

It is easy to verify that Q is trivial—i.e. equals Δ—if and only if the system is equicontinuous.

An extension $(X, \Gamma) \xrightarrow{\pi} (Y, \Gamma)$ of minimal systems is called a *proximal extension* if the relation $R_\pi = \{(x, x') : \pi(x) = \pi(x')\}$ satisfies $R_\pi \subset P$ and a *distal extension* when $R_\pi \cap P = \Delta$. One can show that every distal extension is open. π is a *highly proximal (HP) extension* if, for every closed subset A of X with $\pi(A) = Y$, necessarily $A = X$. It is easy to see that a HP extension is proximal. In the metric case an extension $(X, \Gamma) \xrightarrow{\pi} (Y, \Gamma)$ of minimal systems is HP if and only if it is an *almost one-to-one extension*, that is, the set $\{y \in Y : \pi^{-1}(y) \text{ is a singleton}\}$ is a dense G_δ subset of Y. The map π is *strongly proximal* if for every $y \in Y$ and every probability measure ν with $\mathrm{supp}\, \nu \subset \pi^{-1}(y)$, there

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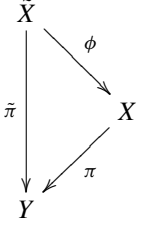
exists a net $\gamma_i \in \Gamma$ and a point $x \in X$ such that $\lim_i \gamma_i \nu = \delta_x$ in the weak* topology on the space $\mathfrak{M}(X)$ of probability measures on X. The extension π is called an *equicontinuous extension* if for every ϵ, a neighborhood of the diagonal $\Delta = \{(x, x) : x \in X\} \subset X \times X$, there exists a neighborhood of the diagonal δ such that $\gamma(\delta \cap R_\pi) \subset \epsilon$ for every $\gamma \in \Gamma$. In the metric case an equicontinuous extension is also called an *isometric extension*. The extension π is a *weakly mixing extension* when R_π as a subsystem of the product system $(X \times X, \Gamma)$ is topologically transitive.

The algebraic language is particularly suitable for dealing with such notions. For example, an extension $(X, \Gamma) \xrightarrow{\pi} (Y, \Gamma)$ of minimal systems is a proximal extension if and only if the Ellis groups $\mathfrak{G}(X, x_0) = A$ and $\mathfrak{G}(Y, y_0) = F$ coincide. It is distal if and only if, for every $y \in Y$ and $x \in \pi^{-1}(y)$, $\pi^{-1}(y) = \mathfrak{G}(Y, y)x$, if and only if for every $y = py_0 \in Y$, where p is an element of M, $\pi^{-1}(y) = p\pi^{-1}(y_0) = pFx_0$, where $F = \mathfrak{G}(Y, y_0)$.

In particular, (X, Γ) is distal if and only if $Gx = X$ for some (hence every) $x \in X$. The extension π is an equicontinuous extension if and only if it is a distal extension and, denoting $\mathfrak{G}(X, x_0) = A$ and $\mathfrak{G}(Y, y_0) = F$,

$$F' \subset A.$$

In this case, setting $A_0 = \bigcap_{g \in F} gAg^{-1}$, the group F/A_0 is the group of the *group extension* $\tilde{\pi}$ associated with the equicontinuous extension π. More precisely, there exists a minimal dynamical system $(\tilde{X}, \Gamma)$, with $\mathfrak{G}(\tilde{X}, \tilde{x}_0) = A_0$, on which the compact Hausdorff topological group $K = F/A_0$ acts as a group of automorphisms and we have the following commutative diagram



where $\tilde{\pi} : \tilde{X} \to Y \cong \tilde{X}/K$ is a group extension and so is the extension $\phi : \tilde{X} \to X \cong \tilde{X}/L$ with $L = A/A_0 \subset F/A_0 = K$. ($\tilde{X} = X$ if and only if A is a normal subgroup of F.)

A minimal system (X, Γ) is said to be *incontractible* if the union of minimal subsets is dense in every product system (X^n, Γ). This is the case if and only if $p \circ Gx = X$ for some (hence every) $x \in X$ and $p \in M$. When Γ is Abelian Gx is always dense in X so that every minimal system is incontractible. However, the following relative notion is an important tool even when Γ is Abelian.

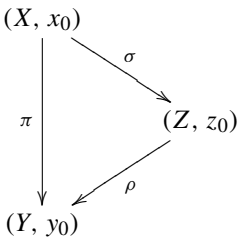
We say that $(X, \Gamma) \xrightarrow{\pi} (Y, \Gamma)$ is a *relatively incontractible (RIC) extension* if: for every $y = py_0 \in Y$, where p is an element of M, $\pi^{-1}(y) = p \circ u\pi^{-1}(y_0) = p \circ Fx_0$, where $F = \mathfrak{G}(Y, y_0)$.

One can show that every RIC extension is open and that every distal extension is RIC. It then follows that every distal extension is open.

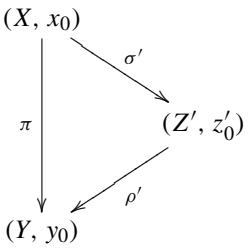
We have the following theorem from [**7**] about the interpolation of equicontinuous extensions. For a proof see [**11**, Theorem X.2.1].

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THEOREM 1.1. *Let $\pi : X \to Y$ be a RIC extension of minimal systems. Fix a point $x_0 \in X$ with $ux_0 = x_0$ and let $y_0 = \pi(x_0)$. Let $A = \mathfrak{G}(X, x_0)$ and $F = \mathfrak{G}(Y, y_0)$. Then there exists a commutative diagram of pointed systems*

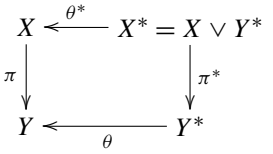


such that ρ is an equicontinuous extension with Ellis group $\mathfrak{G}(Z, z_0) = AF'$ and the extension ρ is an isomorphism if and only if $AF' = F$. Moreover, if



is another such diagram with ρ' an equicontinuous extension, then there exists a homomorphism $(Z, z_0) \to (Z', z_0')$.

Given a homomorphism $\pi : (X, \Gamma) \to (Y, \Gamma)$ of minimal metric systems, there are several standard constructions of associated 'shadow diagrams'. In the *O shadow diagram*



the map π^* is open and the maps θ and θ^* are almost one-to-one. The explicit construction is as follows. The set-valued map $\pi^{-1} : Y \to 2^X$ (where the latter is the compact space of closed subsets of X, equipped with the Hausdorff, or Vietoris, topology) is uppersemicontinuous and we let $Y_0 \subset Y$ be the set of continuity points of this map. Set $Y^* = \mathrm{cls}\{\pi^{-1}(y) : y \in Y_0\} \subset 2^X$, and $X^* = X \vee Y^* = \mathrm{cls}\{(x, \pi^{-1}(y)) : y \in Y_0,\ \pi(x) = y\} \subset X \times Y^*$. By the uppersemicontinuity of π^{-1}, every $y^* \in Y^*$ is contained in a fiber $\pi^{-1}(y)$ for some $y \in Y$ and we let $\theta(y^*) = y$. The maps π^* and θ^* are the restriction to X^* of the coordinate projections on X and Y^*, respectively. One then shows that $X^* = \{(x, y^*) : x \in y^* \in Y^*\}$ and that indeed, π^* is open and the maps θ and θ^* are highly proximal. The O shadow diagram collapses, i.e. $Y = Y^*$, $X = X^*$ and $\pi = \pi^*$, if and only if $\pi : X \to Y$ is an open map, if and only if the map $\pi^{-1} : Y \to 2^X$ is continuous.

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In the *RIC-shadow diagram*

$$\begin{array}{ccc} X & \xleftarrow{\theta^*} & X^* = X \vee Y^* \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi^*} \\ Y & \xleftarrow[\theta]{} & Y^* \end{array}$$

π^* is RIC and θ, θ^* are proximal (thus we still have $A = \mathfrak{G}(X, x_0) = \mathfrak{G}(X^*, x_0^*)$ and $F = \mathfrak{G}(Y, y_0) = \mathfrak{G}(Y^*, y_0^*)$). The concrete description of these objects uses quasifactors and the circle operation:

$$Y^* = \{p \circ Fx_0 : p \in M\} \subset 2^X, \quad X^* = \{(x, y^*) : x \in y^* \in Y^*\} \subset X \times Y^*$$

and

$$\theta(p \circ Fx_0) = py_0, \quad \theta^*(x, y^*) = x, \quad \pi^*(x, y^*) = y^* \quad (p \in M),$$

where $F = \mathfrak{G}(Y, y_0)$. The map θ is an isomorphism (hence $\pi = \pi^*$) when and only when π is already RIC.

Finally, we say that $\pi : (X, \Gamma) \to (Y, \Gamma)$ has a *relatively invariant measure (RIM)* if there exists a projection $P : C(X) \to C(Y)$ such that:

(i) $P(f) \geq 0$ for $f \geq 0$ in $C(X)$;

(ii) $P(\mathbf{1}) = 1$;

(iii) $P(h \circ \pi) = h$ for every $h \in C(Y)$;

(iv) $P(f \circ \gamma) = P(f) \circ \gamma$ for every $f \in C(X)$ and $\gamma \in \Gamma$.

This property is equivalent to the existence of a continuous *section*, i.e. a continuous Γ equivariant map $y \mapsto \lambda_y$ from Y into $\mathfrak{M}(X)$ such that $\pi(\lambda_y) = \delta_y$ for every $y \in Y$. Here and in the following we use the same letter π to denote the induced map $\pi : \mathfrak{M}(X) \to \mathfrak{M}(Y)$ on the spaces of probability measures. Sometimes, however, we will write π_* for the induced map.

In the *RIM shadow diagram*

$$\begin{array}{ccc} X & \xleftarrow{\tilde{\theta}} & \tilde{X} = X \vee \tilde{Y} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ Y & \xleftarrow[\theta]{} & \tilde{Y} \end{array}$$

the map $\tilde{\pi}$ has a RIM and the maps θ and $\tilde{\theta}$ are strongly proximal. It can be shown that every isometric extension has a RIM and is open. See [**9**] for more details; also a treatment of SPI (*strongly proximal isometric*) systems can be found in [**11**].

We say that a minimal system (X, Γ) is a *strictly PI-system* if there is an ordinal η (which is countable when X is metrizable) and a family of systems $\{(W_\iota, w_\iota)\}_{\iota \leq \eta}$ such that: (i) W_0 is the trivial system; (ii) for every $\iota < \eta$ there exists a homomorphism $\phi_\iota : W_{\iota+1} \to W_\iota$ which is either proximal or equicontinuous (isometric when X is metrizable); (iii) for a limit ordinal $\nu \leq \eta$ the system W_ν is the inverse limit of the systems $\{W_\iota\}_{\iota < \nu}$; and (iv) $W_\eta = X$. We say that (X, Γ) is a *PI-system* (*proximal isometric*) if there exists a strictly PI-system $\tilde{X}$ and a proximal homomorphism $\theta : \tilde{X} \to X$.

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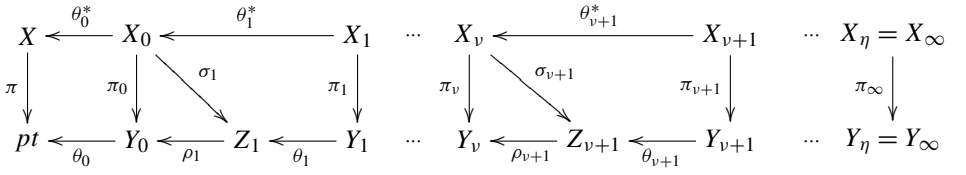
If in the definition of PI-systems we replace proximal extensions by HP extensions (almost one-to-one extensions in the metric case) we get the notion of *HPI* (*highly proximal isometric*; *AI-systems* (*almost one-to-one isometric*) in the metric case). If we replace the proximal extensions by trivial extensions (i.e. we do not allow proximal extensions at all) we have *I-systems*. In this terminology, the structure theorem for distal systems (Furstenberg [**8**]), and the Veech–Ellis structure theorem for point distal systems (Veech [**21**] and Ellis [**5**]), can be stated as follows.

THEOREM 1.2. *A metric minimal system is distal if and only if it is an I-system.*

THEOREM 1.3. *A metric minimal dynamical system is point distal if and only if it is an AI-system.*

The structure theorem for the general minimal system is proved in [**7**] and [**20**] (see also [**21**]) and it asserts that every minimal system admits a canonically defined proximal extension which is a weakly mixing RIC extension of a strictly PI-system. Both the Furstenberg and the Veech–Ellis structure theorems are corollaries of this general structure theorem.

THEOREM 1.4. (Structure theorem for minimal systems) *Given a minimal system* (X, Γ)*, there exists an ordinal* η *(countable when* X *is metrizable) and a canonically defined commutative diagram (the canonical PI-tower)*



where, for each $\nu \leq \eta$, π_ν *is RIC,* ρ_ν *is isometric,* θ_ν, θ_ν^* *are proximal and* π_∞ *is RIC and weakly mixing. For a limit ordinal* ν, X_ν, Y_ν, π_ν, *etc are the inverse limits (or joins) of* X_ι, Y_ι, π_ι, *etc for* $\iota < \nu$. *Thus* X_∞ *is a proximal extension of* X *and a RIC weakly mixing extension of the strictly PI-system* Y_∞. *The homomorphism* π_∞ *is an isomorphism (so that* $X_\infty = Y_\infty$*) if and only if* X *is a PI-system.*

Two further corollaries of this theorem are the theorems of Bronstein on the structure of PI systems [**3**], and of van der Woude on HPI systems [**23**]. Here we will use the latter which we will now proceed to describe.

A homomorphism $\pi : (X, \Gamma) \to (Y, \Gamma)$ is said to be *semi-open* if the interior of $\pi(U)$ is non-empty for every non-empty open subset U of X. When X is minimal every $\pi : (X, \Gamma) \to (Y, \Gamma)$ is semi-open (see Lemma 2.4 below). We will say that a subset $W \subset X \times X$ is a *S-set* if it is closed invariant topologically transitive and the restriction to W of the projection maps are semi-open.

THEOREM 1.5. (van der Woude) *A minimal system* (X, Γ) *is HPI if and only if every S-set in* $X \times X$ *is minimal.*

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2. *On semi-open maps*

A result of Ditor and Eifler from 1972 [**4**] asserts that a continuous surjection $\pi : X \to Y$ between compact Hausdorff spaces X and Y is open if and only if the induced map $\pi_* : \mathfrak{M}(X) \to \mathfrak{M}(Y)$ is an open surjection. In the course of the proof of our main theorem (Theorem 5.1) we will need an analogous result (in the metric case) for semi-open maps (Theorem 2.3), which we now proceed to establish. First we need two preliminary lemmas.

LEMMA 2.1. *Let $\pi : X \to Y$ be a continuous surjection between compact Hausdorff spaces. The conditions* (i) *and* (ii) *below are equivalent. If X is metrizable then the three conditions are equivalent:*

(i) *π is semi-open;*

(ii) *the preimage of every dense subset in Y is dense in X;*

(iii) *the set*

$$X_0 = \{x \in X : \text{ the set-valued map } \pi^{-1} : Y \to 2^X \text{ is continuous at } \pi(x)\}$$

is dense in X.

Proof. The equivalence of (i) and (ii) is straightforward. For any continuous surjection $\pi : X \to Y$ the corresponding set map $\pi^{-1} : Y \to 2^X$ is uppersemicontinuous and, when X is metrizable, this implies that it has a dense G_δ subset $Y_0 \subset Y$ of continuity points. Assuming (ii) we conclude that $X_0 = \pi^{-1}(Y_0)$ is a dense G_δ subset of X. Conversely if (iii) is valid and $U \subset X$ is open and non-empty, then $U \cap X_0 \neq \emptyset$ and if x_0 is any point in this intersection then π is open at x_0, so that $\pi(U)$ is a neighborhood of $\pi(x_0)$ and we conclude that $\text{int}(\pi(U)) \neq \emptyset$. □

LEMMA 2.2. *Let $\pi : X \to Y$ be a continuous surjection between compact metric spaces. Let $f : X \to \mathbb{R}$ be a continuous function and define $f^* : Y \to \mathbb{R}$ by*

$$f^*(y) = \sup\{f(x) : \pi(x) = y\}.$$

Then f^* is continuous at every point of the set

$$Y_0 = \{y \in Y : \text{ the set-valued map } \pi^{-1} : Y \to 2^X \text{ is continuous at } y\}.$$

Proof. Fix $y \in Y_0$ and suppose $y_n \to y$ is a convergent sequence. For each n let $x_n \in X$ satisfy $\pi(x_n) = y_n$ and $f^*(y_n) = f(x_n)$. Let $x_{n_k} \to x$ be a convergent subsequence. Then $\pi(x) = y$ and hence

$$f^*(y) \geq f(x) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f^*(y_{n_k}).$$

Since this is true for every partial limit of $f^*(y_n)$ we conclude that

$$f^*(y) \geq \limsup_{n\to\infty} f^*(y_n).$$

On the other hand, if $f^*(y) = f(x)$ with $\pi(x) = y$ then, since π is open at x, we can find a sequence x_n with $\pi(x_n) = y_n$ and $x_n \to x$, so that

$$f^*(y) = f(x) = \lim_{n\to\infty} f(x_n) \leq \liminf_{n\to\infty} f^*(y_n).$$ □

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THEOREM 2.3. *Let $\pi : X \to Y$ be a continuous surjection between compact metric spaces which is semi-open. Then the induced map $\pi_* : \mathfrak{M}(X) \to \mathfrak{M}(Y)$ is a semi-open surjection.*

Proof. Let

$$Y_0 = \{y \in Y : \text{ the set-valued map } \pi^{-1} : Y \to 2^X \text{ is continuous at } y\}$$

and

$$X_0 = \{x \in X : \text{ the set-valued map } \pi^{-1} : Y \to 2^X \text{ is continuous at } \pi(x)\}.$$

Then $X_0 = \pi^{-1}(Y_0)$ and, by Lemma 2.1, Y_0 and X_0 are dense G_δ subsets of Y and X, respectively. Let $\mathfrak{M}_0(X)$ be the collection of measures of the form $\sum_{i=1}^n c_i \delta_{x_i}$, where $0 \le c_i \le 1$, $\sum_{i=1}^n c_i = 1$. If, in addition, we require that each x_i is in X_0 we obtain the smaller collection $\mathfrak{M}_{00}(X)$. Clearly $\mathfrak{M}_{00}(X)$, and hence also $\mathfrak{M}_0(X)$, are dense in $\mathfrak{M}(X)$. We define $\mathfrak{M}_0(Y)$ analogously as the collection of measures in $\mathfrak{M}(Y)$ of the form $\nu = \sum_{i=1}^n c_i \delta_{y_i}$. Again $\mathfrak{M}_0(Y)$ is dense in $\mathfrak{M}(Y)$.

Let $\mathcal{U} \subset \mathfrak{M}(X)$ be a closed set with non-empty interior. We have to show that $\operatorname{int}(\pi_*(\mathcal{U})) \neq \emptyset$. Suppose to the contrary that $\operatorname{int}(\pi_*(\mathcal{U})) = \emptyset$. Fix $\mu_0 = \sum_{i=1}^m c_i \delta_{x_i} \in \mathfrak{M}_{00}(X) \cap \operatorname{int}(\mathcal{U})$ and let $\nu_0 = \pi_*(\mu_0) = \sum_{i=1}^m c_i \delta_{y_i}$, where $y_i = \pi(x_i)$. Let $\nu_j \in \mathfrak{M}_0(Y) \setminus \pi_*(\mathcal{U})$ be a sequence which converges to ν_0. Set $Q_j = \pi_*^{-1}(\nu_j)$. Each Q_j is a closed and convex subset of $\mathfrak{M}(X)$ and, with no loss of generality, we assume that $Q = \lim_{j\to\infty} Q_j$ exists in $2^{\mathfrak{M}(X)}$. Then Q is a compact convex subset of $\mathfrak{M}(X)$ with $\pi_*(\theta) = \nu_0$ for every $\theta \in Q$.

If $\mu_0 \in Q$ then eventually $Q_j \cap \mathcal{U} \neq \emptyset$, hence $\nu_j \in \pi_*(\mathcal{U})$, contradicting our assumption. Thus we have $\mu_0 \notin Q$ and by the separation theorem there exist a function $f \in C(X)$ and $\epsilon > 0$ with

$$\mu_0(f) \ge \theta(f) + \epsilon \quad \text{for every } \theta \in Q. \tag{2.1}$$

Define the associated function

$$f^*(y) = \sup\{f(x) : \pi(x) = y\}.$$

Each measure ν_j has the form $\sum_{i=1}^{m_j} c_{j,i} \delta_{y_{j,i}}$ and we choose points $x_{j,i}$ with $\pi(x_{j,i}) = y_{j,i}$ such that $f^*(y_{j,i}) = f(x_{j,i})$. Now form the measures $\mu_j = \sum_{i=1}^{m_j} c_{j,i} \delta_{x_{j,i}}$ and assume, with no loss of generality, that $\mu = \lim_{j\to\infty} \mu_j$ exists in $\mathfrak{M}(X)$. Since $\mu_j \in Q_j$ for each j, we have $\mu \in Q$.

Note that by our construction $\nu_j(f^*) = \mu_j(f)$ for every j. By assumption the set $\operatorname{supp} \nu_0 = \{y_1, \ldots, y_m\}$ is a subset of Y_0 and therefore, by Lemma 2.2, each y_i is a continuity point for f^*. From this fact it is easy to deduce that

$$\lim_{j\to\infty} \nu_j(f^*) = \nu_0(f^*).$$

It then follows that

$$\mu(f) = \lim_{j\to\infty} \mu_j(f) = \lim_{j\to\infty} \nu_j(f^*) = \nu_0(f^*) = \sum_{i=1}^m c_i f^*(y_i) = \mu_0(f^* \circ \pi) \ge \mu_0(f).$$

This contradicts (2.1) and our proof is complete. □

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Recall the following well-known result; for completeness we include a proof.

LEMMA 2.4. *Let $\pi : (X, \Gamma) \to (Y, \Gamma)$ be a homomorphism between minimal systems. Then π is semi-open.*

Proof. Let $W \subset X$ be a closed set with non-empty interior. By minimality of (X, Γ) there is a finite set $\{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ with $X = \bigcup_{i=1}^n \gamma_i W$. Therefore, $Y = \bigcup_{i=1}^n \pi(\gamma_i W)$ and it follows that for some i the interior of the closed set $\pi(\gamma_i W) = \gamma_i \pi(W)$ is non-empty. Thus, as required, also $\mathrm{int}(\pi(W)) \neq \emptyset$. □

3. *A key proposition on diffused measures*

As explained above, we fix a minimal ideal M in $\beta\Gamma$ and let u be an idempotent in M. We denote the subgroup uM of M by G and identify it with the group of automorphisms of the universal Γ-minimal system (M, Γ), where for $\alpha \in G$ the corresponding automorphism $R_\alpha : M \to M$ is given by right multiplication $p \mapsto p\alpha$. For an Abelian Γ each subgroup $vM \subset M$, where v is an idempotent in M, is dense in M and it follows that the G-dynamical system (M, G) (where G acts by right multiplication) is minimal. From now on we always assume that Γ is an Abelian group.

LEMMA 3.1. *Let (Y^*, Γ) be a metric minimal system and $\theta : (Y^*, \Gamma) \to (Y, \Gamma)$ its maximal equicontinuous factor. Suppose further that θ is an almost one-to-one extension. Let O be a non-empty open subset of Y^*.*

(i) *There is a non-empty open subset $V \subset O$ such that $\theta^{-1}(\theta(V)) \subset O$.*

(ii) *There is a non-empty open subset $W \subset O$ such that $\mathrm{cls}_\tau(W \cap uY^*) \subset O$.*

Proof. (i) Suppose $O \subset Y^*$ is a non-empty open set for which the statement of the lemma fails. Choose a point $y^* \in O$ such that $\theta^{-1}(\theta(y^*)) = \{y^*\}$ and let $V_n \subset O$ be a sequence of open balls centered at y^* with $\mathrm{diam}(V_n) \searrow 0$. By assumption, there are pairs of points $z_n \in V_n$ and $z_n' \notin O$ with $\theta(z_n) = \theta(z_n')$. However, as $\lim_{n\to\infty} \theta^{-1}(\theta(z_n)) = \{y^*\}$, we have $z_n' \to y^*$ in contradiction of the fact that O is a neighborhood of y^*.

(ii) Since Y is equicontinuous its τ-topology coincides with its compact Hausdorff group topology. Since θ is an almost one-to-one map, the restriction $\theta \restriction uY^* : uY^* \to Y$ is a homeomorphism of uY^*, equipped with the τ topology, onto Y. Let $O \subset Y^*$ be a non-empty open set. Let $V \subset O$ be as in part (i), and let W be a non-empty open subset such that $\overline{W} \subset V$. Now

$$\begin{aligned}\mathrm{cls}_\tau(W \cap uY^*) &= \theta^{-1}(\overline{\theta(W)}) \cap uY^* \\ &\subset \theta^{-1}(\overline{\theta(W)}) = \theta^{-1}(\theta(\overline{W})) \\ &\subset \theta^{-1}(\theta(V)) \subset O.\end{aligned}$$

□

LEMMA 3.2. *Let (X, Γ) be a minimal metric system and let $\phi : (X, \Gamma) \to (Y, \Gamma)$ be its maximal equicontinuous factor. Suppose further that Y is infinite and that we have the following diagram:*

$$(X, \Gamma) \xrightarrow{\pi} (Y^*, \Gamma) \xrightarrow{\theta} (Y, \Gamma),$$

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where π is an isometric extension, θ is an almost one-to-one extension and $\phi = \theta \circ \pi$. Let U be a non-empty open subset of X. Fix a minimal ideal $M \subset \beta\Gamma$ and an idempotent $u \in M$. Then

$$\mathrm{cls}_\tau(U \cap uX) \supset \pi^{-1}(\pi(U)) \cap uX.$$

Proof. Fix a point $x_0 \in X$ with $ux_0 = x_0$ and let y_0 and y_0^* be its images in Y and Y^*, respectively. Let

$$A = \mathfrak{G}(X, x_0) = \{\alpha \in G : \alpha x_0 = x_0\}, \quad \text{and}$$
$$F = \mathfrak{G}(Y, y_0) = \mathfrak{G}(Y^*, y_0^*) = \{\alpha \in G : \alpha y_0 = y_0\} = \{\alpha \in G : \alpha y_0^* = y_0^*\}.$$

The assumption that π is an isometric extension implies that $B = F/A$ is a homogeneous space of a Hausdorff compact topological group. The fact that (Y, Γ) is the maximal equicontinuous factor of (X, Γ) implies that $F \supset G'$ and that $G'A = AG' = F$. Let $\sigma : M \to X$ denote the evaluation map $p \mapsto px_0$.

Let U be a non-empty open subset of X. Set $\tilde{U} = \sigma^{-1}(U) = \{p \in M : px_0 \in U\}$. Then $\tilde{U}$ is a non-empty open subset of M and, by minimality of the G-system (M, G), the collection $\{\tilde{U}\alpha : \alpha \in G\}$ is an open cover of M. Choose a finite subcover, say $\{\tilde{U}\alpha_i : i = 1, 2, \ldots, n\}$. Now

$$\bigcup_{i=1}^{n} \mathrm{cls}_\tau(\tilde{U}\alpha_i \cap G) = \bigcup_{i=1}^{n} \mathrm{cls}_\tau(\tilde{U} \cap G)\alpha_i = G;$$

hence $\mathrm{cls}_\tau(\tilde{U} \cap G)$ has a non-empty τ-interior. Since $\mathrm{cls}_\tau(\tilde{U} \cap G)$ is also τ-closed, it must contain a left translate of G', say $\beta G'$ for some $\beta \in G$ (this follows from the definition of G', see equation (1.1)). Projecting back to X via σ we get

$$\begin{aligned}\mathrm{cls}_\tau(U \cap uX) &= \sigma(\mathrm{cls}_\tau(\tilde{U} \cap G)) \\ &\supset \beta G'x_0 = \beta G'Ax_0 = \beta Fx_0 \\ &= (\beta F\beta^{-1})\beta x_0 = \pi^{-1}(\pi(\beta x_0)).\end{aligned}$$

Let $O = \pi(U)$; then O is a non-empty open subset of Y^* and by Lemma 3.1 there is a non-empty open subset $W \subset O$ such that $\mathrm{cls}_\tau(W \cap uY^*) \subset O$. Set $U_1 = \pi^{-1}(W) \cap U$. Then U_1 is a non-empty open subset of X and by the above argument there exists $\beta_1 \in G$ with $\pi^{-1}(\pi(\beta_1 x_0)) \subset \mathrm{cls}_\tau(U_1 \cap uX) \subset \mathrm{cls}_\tau(U \cap uX)$. Now

$$\begin{aligned}\pi(\beta_1 x_0) &\in \pi(\mathrm{cls}_\tau(U_1 \cap uX)) \\ &= \mathrm{cls}_\tau(\pi(U_1) \cap uY^*) \\ &= \mathrm{cls}_\tau(W \cap uY^*) \subset O = \pi(U).\end{aligned}$$

Thus we have shown that, for every non-empty open subset $U \subset X$, the set $\mathrm{cls}_\tau(U \cap uX)$ contains a full fiber $\pi^{-1}(y^*)$ for some $y^* \in \pi(U)$. Since π is an open map we conclude that

$$\mathrm{cls}_\tau(U \cap uX) \supset \pi^{-1}(\pi(U)) \cap uX$$

as required. □

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PROPOSITION 3.3. *Let (X, Γ) be a minimal metric system and let $\phi : (X, \Gamma) \to (Y, \Gamma)$ be its maximal equicontinuous factor. Suppose further that Y is infinite and that we have the following diagram:*

$$(X, \Gamma) \xrightarrow{\pi} (Y^*, \Gamma) \xrightarrow{\theta} (Y, \Gamma),$$

where π is an isometric extension, θ is an almost one-to-one extension and $\phi = \theta \circ \pi$. For every $y^* \in Y^*$ the fiber $\pi^{-1}(y^*)$ has the structure of a homogeneous space of a compact Hausdorff topological group and we let λ_{y^*} be the corresponding Haar measure on this fiber. Thus π is a RIM and open extension and $y^* \mapsto \lambda_{y^*}$, $Y^* \to \mathfrak{M}(X)$, is the corresponding section. Let $\Lambda : \mathfrak{M}(Y^*) \to \mathfrak{M}(X)$, defined by

$$\Lambda(\nu) = \int_{Y^*} \lambda_{y^*} \, d\nu(y^*),$$

be the associated affine injection. Set $\mathfrak{M}_m(X) = \{\Lambda(\nu) : \nu \in \mathfrak{M}(Y^*)\}$. Then the set

$$R = \{\nu \in \mathfrak{M}(X) : \text{the orbit closure of } \nu \text{ meets } \mathfrak{M}_m(X)\}$$

is a dense G_δ subset of $\mathfrak{M}(X)$.

Proof. Fix a compatible metric d on $\mathfrak{M}(X)$ and let $\kappa \in \mathfrak{M}(X)$ and $\epsilon, \eta > 0$ be given. Find an atomic measure $\lambda = (1/n) \sum_{i=1}^n \delta_{x_i}$ such that $d(\kappa, \lambda) < \epsilon/2$. Since Y is infinite and π is an open map we can also assume that $\pi(x_i) \neq \pi(x_j)$ for $i \neq j$. Choose open disjoint neighborhoods U_i of x_i so small that every measure of the form $\mu = (1/n) \sum_{i=1}^n \mu_i$, $\mu_i \in \mathfrak{M}(X)$ with $\operatorname{supp} \mu_i \subset U_i$, will satisfy $d(\mu, \lambda) < \epsilon/2$, and hence also $d(\mu, \kappa) < \epsilon$.

Set $\nu = (1/n) \sum_{i=1}^n \lambda_{y_i^*}$ with $y_i^* = \pi(x_i)$. For each y_i^* choose points $\{x'_{i,j}\}_{j=1}^k \in \pi^{-1}(y_i^*)$ so that $d(\mu', \nu) < \epsilon/2$, where

$$\mu' = \frac{1}{nk} \sum_{i=1}^n \sum_{j=1}^k \delta_{x'_{i,j}}.$$

By Lemma 3.2,

$$\begin{aligned} u \circ \left(\bigcup_{i=1}^n U_i \right) &\supset uX \cap u \circ \left(uX \cap \bigcup_{i=1}^n U_i \right) \\ &= \operatorname{cls}_\tau \left(uX \cap \bigcup_{i=1}^n U_i \right) \\ &\supset uX \cap \left(\bigcup_{i=1}^n \pi^{-1}(\pi(U_i)) \right). \end{aligned}$$

Therefore, there exist an element $\gamma \in \Gamma$ and for each i a set $\{x_{i,j}\}_{j=1}^k \subset U_i$, such that $d(\gamma x_{i,j}, x'_{i,j})$ is so small that the inequality $d(\gamma\mu, \mu') < \eta/2$ is satisfied, with

$$\mu = \frac{1}{nk} \sum_{i=1}^n \sum_{j=1}^k \delta_{x_{i,j}}.$$

Thus, $d(\gamma\mu, \nu) < \eta$. Since we also have $d(\mu, \kappa) < \epsilon$ and $\epsilon > 0$ is arbitrary, we have shown that the open set

$$R_\eta = \{\mu \in \mathfrak{M}(X) : \text{ there exists } \gamma \in \Gamma \text{ with } d(\gamma\mu, \mathfrak{M}_m(X)) < \eta\}$$

is dense in $\mathfrak{M}(X)$. Clearly $R = \bigcap\{R_\eta : \eta > 0\}$ is the required dense G_δ subset of $\mathfrak{M}(X)$. □

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4. *Some properties of tame minimal systems*

THEOREM 4.1. (See [**13**]) *Let (X, Γ) be a metric tame dynamical system. Let $\mathfrak{M}(X)$ denote the compact convex set of probability measures on X (with the weak* topology). Then each element $p \in E(X, \Gamma)$ defines an element $p_* \in E(\mathfrak{M}(X), \Gamma)$ and the map $p \mapsto p_*$ is both a dynamical system and a semigroup isomorphism of $E(X, \Gamma)$ onto $E(\mathfrak{M}(X), \Gamma)$.*

Proof. Since $E(X, \Gamma)$ is Fréchet we have for every $p \in E$ a sequence $\gamma_i \to p$ of elements of Γ converging to p. Now for every $f \in C(X)$ and every probability measure $\nu \in \mathfrak{M}(X)$ we get by the Riesz representation theorem and Lebesgue's dominated convergence theorem

$$\gamma_i \nu(f) = \nu(f \circ \gamma_i) \to \nu(f \circ p) := p_* \nu(f).$$

Since the Baire class 1 function $f \circ p$ is well defined and does not depend upon the choice of the convergent sequence $\gamma_i \to p$, this defines the map $p \mapsto p_*$ uniquely. It is easy to see that this map is an isomorphism of dynamical systems, and hence a semigroup isomorphism. Finally as Γ is dense in both enveloping semigroups, it follows that this isomorphism is onto. □

As we have seen, when (X, Γ) is a metrizable tame system the enveloping semigroup $E(X, \Gamma)$ is a separable Fréchet space. Therefore, each element $p \in E$ is a limit of a sequence of elements of Γ, $p = \lim_{n\to\infty} \gamma_n$. It follows that the subset $C(p)$ of continuity points of each $p \in E$ is a dense G_δ subset of X. More generally, if $Y \subset X$ is any closed subset then the set $C_Y(p)$ of continuity points of the map $p \restriction Y : Y \to X$ is a dense G_δ subset of Y. For an idempotent $v = v^2 \in E$ we write C_v for $C_{\overline{vX}}(v)$.

LEMMA 4.2. *Let (X, Γ) be a metrizable tame dynamical system, and $E = E(X, \Gamma)$ its enveloping semigroup.*

(i) *For every $p \in E$ the set $C(p) \subset X$ is a dense G_δ subset of X.*

(ii) *For every idempotent $v \in E$, we have $C_v \subset vX$.*

(iii) *If $v \in E$ is an idempotent such that $\overline{vX} = X$ then $C(v) \subset vX$.*

(iv) *When Γ is commutative and (X, Γ) is minimal we have $C(v) \subset vX$ for every idempotent $v \in E$.*

Proof. (i) See the remark above.

(ii) Given $x \in C_v$ choose a sequence $x_n \in vX$ with $\lim_{n\to\infty} x_n = x$. We then have $vx = \lim_{n\to\infty} vx_n = \lim_{n\to\infty} x_n = x$, and hence $C_v \subset vX$.

(iii) For such v we have $C(v) = C_v \subset vX$ by part (ii).

(iv) When Γ is Abelian, $\gamma p = p\gamma$ for every $\gamma \in \Gamma$ and $p \in E$. In particular, the subset vX is Γ invariant and hence dense in X. □

PROPOSITION 4.3. *Let (X, Γ) be a metric tame dynamical system. Then $p_*\mu = \mu$ for every $p \in E(X, \Gamma)$ and every Γ invariant measure $\mu \in \mathfrak{M}(X)$.*

Proof. Let $p = \lim_{n\to\infty} \gamma_n$. Then for every $f \in C(X)$ and $x \in X$ we have $\lim_{n\to\infty} f \circ \gamma_n(x) = f(px)$, and hence by the Lebesgue dominated convergence theorem

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$$\int f(x)\,d\mu(x) = \lim_{n\to\infty} \int f(\gamma_n x)\,d\mu(x)$$
$$= \int f(px)\,d\mu(x)$$
$$= \int f(x)\,dp_*\mu(x). \qquad \square$$

PROPOSITION 4.4. *Let Γ be an Abelian group. Then any metric tame minimal system (X, Γ) is point distal.*

Proof. We will prove that the condition in Theorem 1.5 holds, i.e. that every S-set in $X \times X$ is minimal. So let $W \subset X \times X$ be an S-set. Let $v = v^2$ be some minimal idempotent in $E(X, \Gamma)$. By Lemma 4.2(iii) the set $C(v)$ of continuity points of the map $v : X \to X$ is a dense G_δ subset of X and moreover $C(v) \subset vX$, so that vX is residual in X. Since by assumption the projection maps $\pi_i : W \to X$ are semi-open, it follows that the sets $\pi_i^{-1}(vX)$ are residual in W. Since W_{tr}, the set of transitive points in W, is a dense G_δ subset of W, we conclude that the set $(\pi_1^{-1}(vX) \cap \pi_2^{-1}(vX)) \cap W_{tr} = (vX \times vX) \cap W_{tr}$ is residual in W and, in particular, it is non-empty. Now if (x, x') is any point in this intersection then $v(x, x') = (vx, vx') = (x, x')$ and (x, x') is a minimal point. Therefore, $W = \overline{\mathcal{O}}_\Gamma(x, x')$ is minimal. $\square$

5. *Minimal tame systems are almost automorphic and uniquely ergodic*

THEOREM 5.1. *Let Γ be an Abelian group and (X, Γ) a metric tame minimal system. Then we have the following.*

(i) *The system (X, Γ) is almost automorphic. Thus, there exist:*
  (a) *a compact topological group Y with Haar measure η, and a group homomorphism $\kappa : \Gamma \to Y$ with dense image;*
  (b) *a homomorphism $\pi : (X, \Gamma) \to (Y, \Gamma)$, where the Γ action on Y is via κ;*
  (c) *the sets $X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$ and $Y_0 = \pi(X_0)$ are dense G_δ subsets of X and Y, respectively.*

(ii) *The system (X, Γ) is uniquely ergodic with unique invariant measure μ such that $\pi_*(\mu) = \eta$, and $\pi : (X, \mu, \Gamma) \to (Y, \eta, \Gamma)$ is a measure-theoretical isomorphism of the corresponding measure-preserving systems.*

Proof. (1) By Proposition 4.4 the system (X, Γ) is point distal. By the general theory of minimal point distal systems, if (X, Γ) is non-trivial, it admits a unique, non-trivial, maximal equicontinuous factor $\pi : (X, \Gamma) \to (Y, \Gamma)$. The system (Y, Γ) is thus of the form stated in the theorem. If Y is a finite group it follows from structure theory that $X = Y$. Thus, we now assume that Y is infinite.

Let

$$\begin{array}{ccc} X & \xleftarrow{\theta^*} & X^* = X \vee Y^* \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi^*} \\ Y & \xleftarrow[\theta]{} & Y^* \end{array}$$

be the associated O shadow diagram. Then (X^*, Γ) is also a minimal point distal system and (Y, Γ) is its maximal equicontinuous factor. Again, by the theory of point

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distal systems the extension π^* is RIC. Now, either π^* is an isomorphism—in which case $X = X^* = Y^*$ and X is an almost one-to-one extension of Y, and hence almost automorphic—or there exists a maximal intermediate isometric extension

$$X^* \xrightarrow{\phi} \hat{X} \xrightarrow{\hat{\pi}} Y^* \xrightarrow{\theta} Y,$$

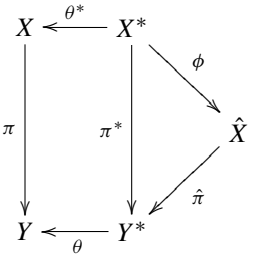
with $\hat{\pi} : \hat{X} \to Y^*$ a non-trivial isometric extension. Thus we now assume that $\hat{\pi}$ is a non-trivial isometric extension. As was shown in Proposition 3.3 (see also [**11**]), $\hat{\pi}$ has a RIM and we let $y^* \mapsto \lambda_{y^*}$ be the corresponding section. Associated with this section we also have the affine injection $\Lambda : \mathfrak{M}(Y^*) \to \mathfrak{M}(\hat{X})$ defined by

$$\Lambda(\nu) = \int_{Y^*} \lambda_{y^*}\, d\nu(y^*),$$

and we set $\mathfrak{M}_m(\hat{X}) := \{\Lambda(\nu) : \nu \in \mathfrak{M}(Y^*)\}$.

Being isometric, $\hat{\pi}$ is also open and thus Proposition 3.3 applies (with $\hat{X}$ and $\hat{\pi}$ in the roles of X and π, respectively). We conclude that the set $\hat{R}$ of measures in $\mathfrak{M}(\hat{X})$ whose orbit closure meets $\mathfrak{M}_m(\hat{X}) = \Lambda(\mathfrak{M}(Y^*))$ is a dense G_δ subset of $\mathfrak{M}(\hat{X})$.

(ii) We now have the following diagram



and we set

$$\mathfrak{M}_m(X^*) := \phi^{-1}(\mathfrak{M}_m(\hat{X})) \quad \text{and}$$
$$\mathfrak{M}_m(X) := \theta^*(\mathfrak{M}_m(X^*)) = \theta^*(\phi^{-1}(\mathfrak{M}_m(\hat{X}))).$$

Let

$$R^* := \phi^{-1}(\hat{R}) = \{\xi \in \mathfrak{M}(\hat{X}) : \text{the orbit closure of } \xi \text{ meets } \mathfrak{M}_m(\hat{X})\}$$

and

$$R' := \theta^*(R^*) \subset R = \{\xi \in \mathfrak{M}(X) : \text{the orbit closure of } \xi \text{ meets } \mathfrak{M}_m(X)\}. \tag{5.1}$$

By Lemma 2.4 the map $\phi : X^* \to \hat{X}$ is semi-open and, as $\hat{R}$ is a dense subset of $\mathfrak{M}(\hat{X})$, Theorem 2.3 implies that R^* is a dense subset of $\mathfrak{M}(X^*)$. Therefore, R' is a dense subset of $\mathfrak{M}(X)$. From the definition of R in (5.1) it is easy to deduce that it is a G_δ set and, because it contains R', it is in fact a dense G_δ subset of $\mathfrak{M}(X)$.

(iii) Recall that the system (X, Γ) is tame and, by Theorem 4.1, so is $(\mathfrak{M}(X), \Gamma)$. Moreover, we have $E(X, \Gamma) = E(\mathfrak{M}(X), \Gamma)$. In particular, $u \in E(\mathfrak{M}(X), \Gamma)$, as a Baire class 1 function, has a dense G_δ set of continuity points, $C_{\mathfrak{M}(X)}(u) \subset \mathfrak{M}(X)$. Therefore, $S := C_{\mathfrak{M}(X)}(u) \cap R$ is a dense G_δ subset of $\mathfrak{M}(X)$.

Since uX is a dense subset of X it follows that the collection of finite convex combinations of point masses picked from uX forms a dense subset of $\mathfrak{M}(X)$. This implies that $u\mathfrak{M}(X)$ is dense in $\mathfrak{M}(X)$ and by Lemma 4.2(iii) we have $C_{\mathfrak{M}(X)}(u) \subset u\mathfrak{M}(X)$.

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Thus also $S \subset u\mathfrak{M}(X)$. Now if $\nu \in S$ then $u\nu = \nu$ and, u being a minimal idempotent, the closure of the Γ orbit of ν in $\mathfrak{M}(X)$ is a minimal set; hence this entire orbit closure is contained in $\mathfrak{M}_m(X)$. In particular, $\nu \in \mathfrak{M}_m(X)$ and we conclude that

$$S = C_{\mathfrak{M}(X)}(u) \cap R \subset \mathfrak{M}_m(X).$$

Therefore, S is dense in $\mathfrak{M}_m(X)$ and, in turn, this implies the equality

$$\mathfrak{M}_m(X) = \mathfrak{M}(X). \tag{5.2}$$

(iv) Given a point $x \in X$, the corresponding point mass $\delta_x \in \mathfrak{M}(X)$ must have, by (5.2), a preimage in $\mathfrak{M}_m(X^*)$, say $\theta^*(\xi) = \delta_x$ with $\xi \in \mathfrak{M}_m(X^*)$. In particular, for x with $\theta^{*-1}(x) = \{x^*\}$ a singleton, we must have $\xi = \delta_{x^*} \in \mathfrak{M}_m(X^*)$ and therefore $\phi_*(\delta_{x^*}) = \delta_{\hat{x}} \in \mathfrak{M}_m(\hat{X})$ with $\hat{x} = \phi(x^*)$. By the definition of $\mathfrak{M}_m(\hat{X})$ there exists a measure $\rho \in \mathfrak{M}(Y^*)$ with

$$\delta_{\hat{x}} = \Lambda(\rho) = \int_{Y^*} \lambda_{y^*}\, d\rho(y^*).$$

This clearly implies that the measure ρ is a point mass, say $\rho = \delta_{y^*}$, and that the measure λ_{y^*}, which is the Haar measure on the homogeneous space which forms the fiber $\hat{\pi}^{-1}(y^*) \subset \hat{X}$, is also a degenerate point mass. That is, the isometric extension $\hat{\pi}$ is in fact an isomorphism. Now the collapse of $\hat{\pi}$ implies the collapse of the entire AI-tower, so that in fact $Y^* = X^* = X$, and we have shown that X is indeed an almost one-to-one extension of Y.

(v) Suppose that μ_1 and μ_2 are two invariant probability measures on X. Then, (X, Γ) being tame, by Proposition 4.3, $u_*\mu_i = \mu_i$ and we conclude that $\mu_i(uX) = 1$, for $i = 1, 2$. Since π is a proximal extension, for every $y \in Y = uY$ the fiber $\pi^{-1}(y)$ intersects uX at exactly one point: $\pi^{-1}(y) \cap uX = \{x\}$. Now by disintegrating each μ_i over η, inside the set uX, we conclude that $\mu_1 = \mu_2$. This proves the unique ergodicity of (X, Γ). It is also clear from the proof that the map $\pi : (X, \mu, \Gamma) \to (Y, \eta, \Gamma)$, where μ is the unique invariant measure on X, is an isomorphism of measure-preserving systems. □

Remark 5.2. The set $X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$ is a dense G_δ and Γ-invariant subset of X and thus has μ measure either zero or one. In [**18**, §11] Kerr and Li construct a minimal Toeplitz system which is tame and not null. Since in this construction the growth of the sequence $\{n_1 < n_2 < \cdots\}$ is arbitrary, it follows that the resulting Toeplitz system can be made not regular in the sense that the densities of the periodic parts converge to $d < 1$. For such non-regular systems $\mu(X_0) = 0$. This shows that the unique invariant measure of a minimal tame system need not be supported by the set X_0 where π is one-to-one.

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