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# COINTEGRATING REGRESSIONS WITH TIME VARYING COEFFICIENTS

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This paper considers cointegrating regressions with time varying coefficients. The coefficients are modeled as smooth functions evolving over time. It is shown that they can be estimated nonparametrically, using suitably modified series estimators. Presented is the efficient method of estimation, which relies on simple prefiltering of the data and preestimation of the model. The test for the adequacy of model specification is also developed. Our model and statistical methods are applied to analyze the U.S. automobile demand function.

## 1. INTRODUCTION

The notion of cointegration has gained great popularity since the publication of an influential paper by Engle and Granger (1987). Cointegration provides an effective and appealing conceptual framework to model economic long-run relationships and interpret models that include nonstationary variables with unit roots. It has, however, been much less successful and fruitful in the practical domain. In many of the published empirical papers (perhaps more in unpublished ones), what we see is the evidence for the absence, not the presence, of cointegration.

The observed frustrating evidence may be attributed to the misspecification of empirical models, most notably due to the existence of omitted variables and/or parameter instability. Of these two prominent sources for model misspecification, the latter seems particularly relevant here. The parameter inconstancy can especially be important in cointegrated models, because they describe long-run relationships and are estimated using data whose time span is usually rather long. Needless to say, any relationship is likely to change if investigated over a long time span. Recently, it has indeed been investigated by Hansen (1992) and Quintos and Phillips (1993), who considered Lagrange multiplier (LM) tests for parameter constancy in cointegrating regressions.

This paper considers cointegrating relationships that vary or evolve smoothly over time. More specifically, we look at cointegrating regressions with time varying coefficients that are assumed to be generated by a smooth function. Such regressions seem to be very useful for modeling long-run economic relationships

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that are believed to slowly evolve with time. Long-run demands for many commodities may exhibit such characteristics, possibly as a result of habit persistency among other things. We indeed show in the paper that the U.S. automobile demand can be suitably modeled using a cointegrating regression with coefficients smoothly changing over time. The empirical results from our application clearly demonstrate the empirical relevancy of our model and methods.

It is shown that the coefficients in our models can be estimated nonparametrically using series estimators. The resulting estimator is consistent, if we let the number of series functions used to approximate the unknown coefficient generating function increase appropriately as the sample size grows. When the regressors are strictly exogenous, the estimator is efficient and has mixed normal limiting distribution. This extends the result in Park and Phillips (1988) for cointegrating regressions with fixed coefficients. For more general models, we may use the approach by Park (1992) to obtain an estimator that is efficient and asymptotically Gaussian. The method is based on the regressions transformed through prefiltering of the data and preestimation of the model. We also provide methods to test for the adequacy of our model specification, against two most relevant alternative hypotheses: the cointegrating regression model with fixed coefficients and a spurious regression with time varying coefficients.

The remainder of the paper is organized as follows. Section 2 introduces the model, assumptions and some preliminary results useful for the development of our theory. The basic statistical theories are given in Section 3. In particular, the consistency of our estimator is established, and its limiting distribution is shown to be mixed normal for the regression with strictly exogenous regressors. Consistent estimation of the error variance is also discussed. Section 4 considers more general models and develops efficient methods of estimation and testing. Asymptotic mixed normality of the estimator is shown to be extended to the general models. We present the methods and theories of testing for the adequacy of model specification in Section 5. In Section 6, an empirical application of our method to the U.S. automobile demand equation is presented. Section 7 concludes the paper, and the mathematical proofs are given in the Appendix.

It is convenient to define here some notation that will be used frequently throughout the paper. As usual, $|\cdot|$ denotes the modulus. By convention, if applied to a vector or matrix, the notation denotes the maximum of the moduli of their components. For a vector $x = (x_i)$, $\|\cdot\|$ signifies the Euclidean norm, i.e., $\|x\|^2 = \sum x_i^2$. It is also used to denote the standard operator norm for a matrix $A$. Therefore, $\|A\| = \sup \|Ax\|/\|x\|$. It is well known that $\|A\|^2$ is dominated by the maximum eigenvalue of $A'A$. Moreover, $\|A\|^2 \leq \mathrm{tr}(A'A) = \sum a_{ij}^2$ for $A = (a_{ij})$. The same notation, $\|\cdot\|$, is used to denote the supremum norm for functions on $C[0,1]$, the space of continuous functions on the unit interval $[0,1]$. This should cause no confusion. For functions that are vector valued, the notation signifies the maximum of the supremum norms of the component functions. Standard notations such as $o_p$ and $O_p$ for stochastic orders and $\to_p$ and $\to_d$ for convergences of random sequences are used repeatedly without any reference. Finally, equality in distribution is denoted by $=_d$.

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## 2. THE MODEL AND PRELIMINARY RESULTS

We consider the model given by

$$y_t = \alpha_t' x_t + u_t, \tag{1}$$

where the regressor $(x_t)$ is an $m$-dimensional integrated process of order one and the coefficient $(\alpha_t)$ is an $m$-vector of parameters that are allowed to change over time. We denote by $n$ the sample size and let

$$\alpha_t = \alpha\left(\frac{t}{n}\right), \tag{2}$$

where $\alpha$ is a smooth function defined on $[0,1]$. Note that $\alpha_t$ is dependent upon $n$, though we suppress the subscript $n$ for notational simplicity. The $(u_t)$ are stationary errors. More precise assumptions on our model (1) with (2) will be introduced subsequently.

We specify $v_t = \Delta x_t$, where $\Delta$ is the usual difference operator, and define

$$w_t = (u_t, v_t')'. \tag{3}$$

We will make the following assumption.

Assumption 1. Let

$$w_t = \sum_{i=0}^{\infty} \Phi_i \varepsilon_{t-i},$$

and we assume that $\Phi(1)$ is nonsingular and $\sum i|\Phi_i| < \infty$ and that $(\varepsilon_t)$ are independent and identically distributed (i.i.d.) with $\Psi = \mathbf{E}\varepsilon_t \varepsilon_t' > 0$ and $\mathbf{E}|\varepsilon_t|^p < \infty$ for some $p \geq 4$.

The process $(w_t)$ is thus assumed to be a general linear process. The conditions on the summability of $(\Phi_i)$ and the moments of $(\varepsilon_t)$ are standard, and comparable assumptions are routinely imposed in time series literature.

For the subsequent development of our theory, it is useful to write

$$w_t = w_t^{\bullet} + (\widetilde{w}_{t-1} - \widetilde{w}_t), \tag{4}$$

where $w_t^{\bullet} = \Phi(1)\varepsilon_t$ and $\widetilde{w}_t = \sum_{i=0}^{\infty} \widetilde{\Phi}_i \varepsilon_{t-i}$ with $\widetilde{\Phi}_i = \sum_{j=i+1}^{\infty} \Phi_j$. The representation in (4) is motivated by the Beveridge–Nelson decomposition $\Phi(z) = \Phi(1) - (1-z)\widetilde{\Phi}(z)$. As demonstrated in Phillips and Solo (1992), $(w_t)$ behaves like $(w_t^{\bullet})$ in many contexts. We have, in particular, that $\sum|\widetilde{\Phi}_i| < \infty$ under our assumption $\sum i|\Phi_i| < \infty$. Therefore, $(\widetilde{w}_t)$ is well defined as a stationary time series.

For the statistical analysis of our model, we need to invoke an invariance principle. Denote by $[z]$ the largest integer that does not exceed $z$ and let $W$ be

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an $(m+1)$-dimensional vector Brownian motion with variance $\Omega = \Phi(1)\Psi\Phi(1)'$. If we let

$$W_n^{\bullet}(r) = \frac{1}{\sqrt{n}} \sum_{t=1}^{[nr]} w_t^{\bullet} = \Phi(1) \frac{1}{\sqrt{n}} \sum_{t=1}^{[nr]} \varepsilon_t,$$

then it follows immediately from the classical Donsker's theorem that $W_n^{\bullet} \to_d W$ because $\mathbf{E}|\varepsilon_t|^2 < \infty$. Moreover, if we define

$$W_n^{\circ}(r) = \frac{1}{\sqrt{n}} \sum_{t=1}^{[nr]} w_t,$$

then it follows from (4) that

$$\sup_{0 \le r \le 1} |W_n^{\circ}(r) - W_n^{\bullet}(r)| \le 2 \max_{0 \le t \le n} \left| \frac{\widetilde{w}_t}{\sqrt{n}} \right| = o_p(1).$$

Therefore, we also have $W_n^{\circ} \to_d W$, as shown by Phillips and Solo (1992).

However, the invariance principle is not enough for our purpose. It is also necessary to have stronger approximations that involve explicit convergence rates. They are given subsequently as a lemma.

LEMMA 1. *Let Assumption* 1 *hold. Then*

(a) $\sup_{0 \le r \le 1} |W_n^{\circ}(r) - W_n^{\bullet}(r)| = O_p(n^{-a})$, *and*
(b) *there exists* $W_n$ *such that* $W_n^{\bullet} =_d W_n$ *and* $\sup_{0 \le r \le 1} |W_n(r) - W(r)| = O_p(n^{-a})$ *for large* $n$, *where* $a = (p-2)/2p$.

Naturally, the convergence rate is faster if $(\varepsilon_t)$ has higher moments. It approaches $\sqrt{n}$ as $p \to \infty$.

For the development of our theory, we let

$$W_n^{\bullet} = W_n \tag{5}$$

throughout the paper. Our main purpose is to obtain the distributional results for the series estimator of $\alpha$ proposed in the paper. The convention introduced in (5) is therefore innocuous as long as our main results are concerned. Yet, it allows us to avoid repetitious embeddings of $W_n^{\bullet}$ into the probability space where $W_n$ and $W$ are defined and in consequence greatly simplifies the exposition of the paper. However, we emphasize here that all the subsequent results other than the main distribution theory should be interpreted with caution, because they generally do not hold without the convention (5).

As mentioned earlier, we let $\alpha$ in (2) be a smooth function. For the smoothness of $\alpha$, we explicitly require the following assumption.

Assumption 2. We assume that $\alpha$ is $q$-times differentiable with bounded derivative on [0,1] for some $q > 1$.

Assumption 3. $q > (7p-2)/2(p-2)$.

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If $p \geq 4$ as given in Assumption 1, it suffices to have that $\alpha$ is seven times continuously differentiable with bounded derivative. We may allow for less smooth function $\alpha$, under the existence of higher moments of $(\varepsilon_t)$. It is required that $\alpha$ is at least four times differentiable.

For the estimation of $\alpha$, we consider the estimand given by

$$\Pi(\alpha) = (\alpha(r_1)', \ldots, \alpha(r_d)')',$$

where we denote by $r_i$'s the numbers in the unit interval $[0,1]$. Note that the functional $\Pi : C[0,1]^m \to \mathbf{R}^{md}$ is linear and bounded. It is indeed easy to see that we have $\Pi(c_1 f_1 + c_2 f_2) = c_1 \Pi(f_1) + c_2 \Pi(f_2)$ for $c_1, c_2 \in \mathbf{R}$ and $f_1, f_2 \in C[0,1]^m$. Moreover, $\|\Pi(f)\| \leq (md)^{1/2} \|f\|$ for $f \in C[0,1]^m$.

If $\alpha$ is sufficiently smooth, we may approximate it componentwise by a series of polynomial and/or trigonometric functions on $[0,1]$. In particular, we consider in the paper the trigonometric pairs

$$(\cos \lambda_i r, \sin \lambda_i r)_{i \geq 1} \quad \text{with} \quad \lambda_i = 2\pi i \tag{6}$$

and the constant function 1 for the series functions $(\varphi_i)$. Obviously, our asymptotic analysis is not affected by the addition of a finite number of series functions. Therefore, we may also allow for the Fourier flexible form (FFF) functions, which include the linear and quadratic functions, $r$ and $r^2$, in addition to the trigonometric functions in (6). The following lemma shows that $\alpha$ can be arbitrarily well approximated by $\alpha_\kappa$, a finite sum of the series functions $(\varphi_i)_{i=1}^{\kappa}$. It also provides the rate of convergence of $\alpha_\kappa$ to $\alpha$.

LEMMA 2. *Suppose that Assumption* 2 *holds. Then for each* $\kappa \geq 1$, *there exists* $\alpha_\kappa$ *such that* $\kappa^b \|\alpha_\kappa - \alpha\| \to 0$ *as* $\kappa \to \infty$, *where* $b = q - \delta$ *for* $\delta > 0$ *arbitrarily small*.

Because of the boundedness of $\Pi$ noted previously, it follows that $\kappa^b \|\Pi(\alpha_\kappa) - \Pi(\alpha)\| \to 0$ as $\kappa \to \infty$. We may consequently approximate $\Pi(\alpha)$ by $\Pi(\alpha_\kappa)$ for large $\kappa$.

We write $\alpha_\kappa$ more explicitly as

$$\alpha_\kappa = \sum_{i=1}^{\kappa} \beta_{\kappa i} \varphi_i, \tag{7}$$

where $\beta_{\kappa i} \in \mathbf{R}^m$ for $i = 1, \ldots, \kappa$. Furthermore, let $\beta_\kappa = (\beta_{\kappa 1}', \ldots, \beta_{\kappa\kappa}')' \in \mathbf{R}^{m\kappa}$ and $f_\kappa = (\varphi_1, \ldots, \varphi_\kappa)'$, so that $\alpha_\kappa = (f_\kappa' \otimes I_m)\beta_\kappa$. Then we have

$$\Pi(\alpha_\kappa) = (\alpha_\kappa(r_1)', \ldots, \alpha_\kappa(r_d)')' = T_\kappa \beta_\kappa, \tag{8}$$

where $T_\kappa$ is an $md \times m\kappa$ matrix given by $T_\kappa = F_\kappa \otimes I_m$ with $F_\kappa = (f_\kappa(r_1), \ldots, f_\kappa(r_d))'$ for $\kappa \geq 1$.

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It is well known for the trigonometric series functions in (6) that

$$\frac{2}{\kappa}\sum_{i=1}^{\kappa}\varphi_i(r)^2 \to 1 \quad \text{and} \quad \frac{1}{\kappa}\sum_{i=1}^{\kappa}\varphi_i(r_1)\varphi_i(r_2) \to 0 \quad \text{for } r_1 \neq r_2$$

as $\kappa \to \infty$. We therefore have $2T_\kappa T_\kappa'/\kappa \to I_{md}$ as $\kappa \to \infty$. Because $T_\kappa T_\kappa' > 0$ for all $\kappa \geq d$, it follows that

$$c_1 \leq \lambda_{\min}\left(\frac{T_\kappa T_\kappa'}{\kappa}\right) \leq \lambda_{\max}\left(\frac{T_\kappa T_\kappa'}{\kappa}\right) \leq c_2 \tag{9}$$

for all $\kappa \geq d$, where $c_1, c_2 > 0$ are some constants. Here and elsewhere in the paper, we denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively, the minimum and maximum eigenvalues of a matrix $A$.

We can generalize (1) to models containing a constant term and time trend, which are given by

$$y_t = \pi + \alpha_t' x_t + u_t, \tag{10}$$

$$y_t = \pi + \rho t + \alpha_t' x_t + u_t, \tag{11}$$

and $\Delta x_t = \mu + v_t$. Such models may often be more useful for the practical applications. It is also possible to consider models with higher order polynomial time trends or models with time varying coefficients for the deterministic trends. We will, however, mainly concentrate on model (1) in the paper to ease the exposition. The statistical theories for all the extended models are similar to those for the simple model (1).

## 3. BASIC STATISTICAL THEORY

As shown previously, $\alpha$ and $\Pi(\alpha)$ can be approximated by $\alpha_\kappa$ and $\Pi(\alpha_\kappa)$ for large $\kappa$, respectively, in $C[0,1]^m$ and $\mathbf{R}^{md}$. We may, in turn, estimate $\alpha_\kappa$ and $\Pi(\alpha_\kappa)$ using

$$\hat{\alpha}_{n\kappa} = \sum_{i=1}^{\kappa}\hat{\beta}_{n\kappa i}\varphi_i \tag{12}$$

and

$$\Pi(\hat{\alpha}_{n\kappa}) = (\hat{\alpha}_{n\kappa}(r_1)', \ldots, \hat{\alpha}_{n\kappa}(r_d)')' = T_\kappa \hat{\beta}_{n\kappa} \tag{13}$$

defined accordingly as (7) and (8), where $\hat{\beta}_{n\kappa} = (\hat{\beta}_{n\kappa 1}', \ldots, \hat{\beta}_{n\kappa\kappa}')'$ is an estimate for $\beta_\kappa = (\beta_{\kappa 1}', \ldots, \beta_{\kappa\kappa}')'$. It is required that $\kappa \to \infty$ as $n \to \infty$. Later on, we will define $\kappa$ explicitly as a function of $n$.

For our model, it is natural to consider the ordinary least squares (OLS) estimator $\hat{\beta}_{n\kappa}$ of $\beta_\kappa$ from the regression

$$y_t = x_{\kappa t}'\beta_\kappa + u_{\kappa t}, \tag{14}$$

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where, in notation defined earlier,

$$x_{\kappa t} = f_\kappa\left(\frac{t}{n}\right) \otimes x_t,$$

$$u_{\kappa t} = u_t + (\alpha - \alpha_\kappa)\left(\frac{t}{n}\right)' x_t.$$

We may write explicitly

$$\hat{\beta}_{n\kappa} = (X'_{n\kappa} X_{n\kappa})^{-1} X'_{n\kappa} y_{nn} = \beta_\kappa + (X'_{n\kappa} X_{n\kappa})^{-1} X'_{n\kappa} u_{n\kappa}, \tag{15}$$

where $y_{nn} = (y_1, \ldots, y_n)'$, $X_{n\kappa} = (x_{\kappa 1}, \ldots, x_{\kappa n})'$, and $u_{n\kappa} = (u_{\kappa 1}, \ldots, u_{\kappa n})'$.

It is necessary to introduce some additional notation to develop our subsequent statistical theory. We let

$$\Gamma(k) = \mathbf{E} w_t w'_{t-k} \tag{16}$$

be the autocovariance function of $(w_t)$ in (3) and define

$$\Sigma = \Gamma(0), \qquad \Lambda = \sum_{k=1}^{\infty} \Gamma(k), \qquad \Omega = \sum_{k=-\infty}^{\infty} \Gamma(k). \tag{17}$$

The autocovariance function $\Gamma(k)$ in (16) is absolutely summable under the summability condition on $(\Phi_i)$ in Lemma 1. Note that $\Omega = \Sigma + \Lambda + \Lambda'$. The variance of the limit Brownian motion $W$ in Lemma 1 is given by the long-run variance $\Omega$ of $(w_t)$. We also define

$$\Delta = \Sigma + \Lambda. \tag{18}$$

It will be useful to partition the parameter matrices $\Omega$ and $\Delta$ in (17) and (18) accordingly as $(w_t)$. We denote by $\Omega_{ij}$ and $\Delta_{ij}$ for $i, j = 1,2$ the partitioned submatrices. If the partitioned submatrices are indeed scalars or vectors, then the lowercase letters $\omega_{ij}$ and $\delta_{ij}$ will be used. In particular, we denote by $\omega^2 (\equiv \omega_{11})$ the long-run variance of the regression errors $(u_t)$.

Partition the process $W_n$ introduced in Lemma 1 as

$$W_n = (U_n, V'_n)' \tag{19}$$

conformably with $(w_t)$ and define

$$f_{n\kappa}(r) = f_\kappa\left(\frac{[nr]}{n}\right) \quad \text{and} \quad (\alpha_n - \alpha_{n\kappa})(r) = (\alpha - \alpha_\kappa)\left(\frac{[nr]}{n}\right).$$

Then we have the following lemma.

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LEMMA 3. *Let Assumption* 1 *hold. Then, as* $n \to \infty$, *we have for each* $\kappa$ *that*

(a) $\dfrac{1}{n^2}\sum_{t=1}^{n} x_{\kappa t}x'_{\kappa t} = \int_0^1 f_{n\kappa}f'_{n\kappa} \otimes V_nV'_n + O_p(n^{-a})$

(b) $\dfrac{1}{n}\sum_{t=1}^{n} x_{\kappa t}u_t = \int_0^1 f_{n\kappa} \otimes V_n\,dU_n + \int_0^1 f_{n\kappa} \otimes \delta'_{12} + O_p(n^{-a})$

(c) $\dfrac{1}{n}\sum_{t=1}^{n} x_{\kappa t}(u_{\kappa t} - u_t) = n\int_0^1 (f_{n\kappa} \otimes V_nV'_n)(\alpha_n - \alpha_{n\kappa}) + \|\alpha - \alpha_\kappa\| O_p(n^{1-a})$

*independent of* $\kappa$, *where a is given in Lemma* 1.

Now we define

$$A_{n\kappa} = \int_0^1 f_{n\kappa}f'_{n\kappa} \otimes V_nV'_n,$$

$$B_{n\kappa} = \int_0^1 f_{n\kappa} \otimes V_n\,dU_n + \int_0^1 f_{n\kappa} \otimes \delta'_{12},$$

$$C_{n\kappa} = \int_0^1 (f_{n\kappa} \otimes V_nV'_n)(\alpha_n - \alpha_{n\kappa})$$

and let

$$A_\kappa = \int_0^1 f_\kappa f'_\kappa \otimes VV',$$

$$B_\kappa = \int_0^1 f_\kappa \otimes V\,dU + \int_0^1 f_\kappa \otimes \delta'_{12},$$

$$C_\kappa = \int_0^1 (f_\kappa \otimes VV')(\alpha - \alpha_\kappa),$$

where $W = (U, V')'$ partitioned similarly as $W_n = (U_n, V'_n)'$ in (19). The following lemma gives their asymptotic results.

LEMMA 4. *Let Assumptions* 1 *and* 2 *hold. Then we have as* $n, \kappa \to \infty$

(a) $\|A_\kappa\|, \|A_\kappa^{-1}\| = O_p(1)$, $\|A_{n\kappa} - A_\kappa\| = O_p(n^{-a}\kappa) + O_p(n^{-1}\kappa^2)$,
(b) $\|B_\kappa\| = O_p(\kappa^{1/2})$, $\|B_{n\kappa} - B_\kappa\| = O_p(n^{-a}\kappa^{1/2})$,
(c) $\|C_\kappa\|, \|C_{n\kappa}\| = o_p(\kappa^{1/2-b})$,

*where a and b are defined, respectively, in Lemmas* 1 *and* 2.

For the estimator $\Pi(\hat{\alpha}_{n\kappa})$ in (13) to be consistent, we should let $\kappa$ increase as the sample size $n$ tends to large. This is to eliminate the bias in the limit. However, it should not grow too fast, because it would then make the variance of the estimator explode. More precisely, the following assumption is required.

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Assumption 4. We let $\kappa = cn^r$ with $2/(2q-1) < r < (p-2)/3p$, where $c$ is some constant and $p$ and $q$ are given in Assumptions 1 and 2, respectively.

Recall that $p$ and $q$ are the maximum orders of existing moments of $(\varepsilon_t)$ and derivatives $\alpha$, respectively. The smoother the function $\alpha$ is, the smaller is the required rate of increase in the number of approximating terms. The bias would still disappear. On the other hand, as the underlying time series has higher moments, we are allowed to include more terms for a better approximation without making the sampling variance explode. When $p,q = \infty$, we may choose $\kappa = cn^r$ for any $0 < r < \frac{1}{3}$. We will continue to denote the truncation number by $\kappa$ in the subsequent results, though it is more appropriate to use the notation $\kappa_n$ because it is dependent upon $n$.

Define

$$M_{n\kappa} = T_\kappa (X'_{n\kappa} X_{n\kappa})^{-1} T'_\kappa. \tag{20}$$

Propositions 5 and 6 present the asymptotic results for the sampling error variance and the truncation bias, respectively, in estimating $\Pi(\alpha)$ by $\Pi(\hat{\alpha}_{n\kappa})$.

PROPOSITION 5. *Let Assumptions* 1–4 *hold. Then we have*

(a) $\Pi(\hat{\alpha}_{n\kappa}) = \Pi(\alpha_\kappa) + O_p(n^{-1}\kappa)$
(b) $M_{n\kappa}^{-1/2}(\Pi(\hat{\alpha}_{n\kappa}) - \Pi(\alpha_\kappa)) = (T_\kappa A_\kappa^{-1} T'_\kappa)^{-1/2} T_\kappa A_\kappa^{-1} B_\kappa + o_p(1)$

*as* $n \to \infty$.

PROPOSITION 6. *Let Assumptions* 1–4 *hold. Then we have*

(a) $\Pi(\alpha_\kappa) = \Pi(\alpha) + o(\kappa^{-b})$
(b) $M_{n\kappa}^{-1/2}(\Pi(\alpha_\kappa) - \Pi(\alpha)) = o_p(\kappa^{-1/2})$

*as* $n \to \infty$, *where* $b$ *is given in Lemma* 2.

As is well expected, the estimation error increases proportionately as the greater number of series functions is used to approximate $\alpha$. In contrast, the bias due to the approximation of $\alpha$ with a finite number of series functions decreases as more series functions are included.

Because

$$\|\Pi(\hat{\alpha}_{n\kappa}) - \Pi(\alpha)\| \le \|\Pi(\hat{\alpha}_{n\kappa}) - \Pi(\alpha_\kappa)\| + \|\Pi(\alpha_\kappa) - \Pi(\alpha)\|,$$

the next theorem follows immediately from Propositions 5 and 6.

THEOREM 7. *Let Assumptions* 1–4 *hold. Then we have*

(a) $\Pi(\hat{\alpha}_{n\kappa}) = \Pi(\alpha) + O_p(n^{-1}\kappa)$
(b) $M_{n\kappa}^{-1/2}(\Pi(\hat{\alpha}_{n\kappa}) - \Pi(\alpha)) = (T_\kappa A_\kappa^{-1} T'_\kappa)^{-1/2} T_\kappa A_\kappa^{-1} B_\kappa + o_p(1)$

*as* $n \to \infty$.

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We thus have shown that $\Pi(\hat{\alpha}_{n\kappa})$ is consistent for $\Pi(\alpha)$. The convergence rate is given by $n^{-1}\kappa$. It is slower, by a factor of $\kappa$, than the OLS estimator for the fixed coefficient cointegrating regression, which converges at an $n$-rate. The distribution of $\Pi(\hat{\alpha}_{n\kappa})$ is generally non-Gaussian and biased even asymptotically. However, if we consider the special case that $\omega_{12} = \delta_{12} = 0$, its limiting distribution reduces to rightfully centered normal distribution. Such a case arises in particular when the regressors $(x_t)$ are generated independently of the regression errors $(u_t)$. We call such regressors *strictly exogenous*. In the case of strict exogeneity, the limiting distribution of OLS estimator for the fixed coefficient cointegrating regression is Gaussian, as shown in Park and Phillips (1988). We will show subsequently that the Gaussianity extends to our model with time varying coefficients.

COROLLARY 8. *Let Assumptions* 1–4 *hold. If* $\omega_{12} = \delta_{12} = 0$, *then*

$$M_{n\kappa}^{-1/2}(\Pi(\hat{\alpha}_{n\kappa}) - \Pi(\alpha)) \to_d \mathbf{N}(0, \omega^2 I_{md})$$

*as* $n \to \infty$.

The result in Corollary 8 implies, in particular, that for practical purposes we may regard $\omega^2 M_{n\kappa}$ as the sampling variance of the estimator $\Pi(\hat{\alpha}_{n\kappa})$, which is distributed approximately as normal around the true value $\Pi(\alpha)$.

We may consistently estimate the long-run variance $\omega^2$ of the regression errors $(u_t)$ using the residuals from the truncated regression (14). To see this, we first let $(\hat{u}_{\kappa t})$ be the OLS residuals from regression (14) and consider an estimator of the form

$$\hat{\omega}_{n\kappa}^2 = \frac{1}{n} \sum_{|k|<\ell_n} \nu\left(\frac{k}{\ell_n}\right) \sum_{t=k+1}^{n} \hat{u}_{\kappa t}\, \hat{u}_{\kappa, t-k}, \tag{21}$$

where $\nu$ and $\ell_n$ are, respectively, the lag window and truncation number.

Assumption 5. We assume that

(a) $\nu(\cdot)$ is a bounded even function with $\nu(0) = 1$ such that

$$\int_{-1}^{1} \nu(x)^2\, dx < \infty \quad \text{and} \quad \lim_{x\to 0} \frac{1-\nu(x)}{|x|} = \nu_0 < \infty$$

(b) $\ell_n = cn^{1/3}$ for some constant $c$.

The conditions in Assumption 5(a) are satisfied for many commonly used lag windows including Bartlett and Parzen kernels and the rectangular kernel. In Assumption 5(b), we let the lag truncation number increase along with $n$ but at a controlled rate $O(n^{1/3})$. If $\omega^2$ can be estimated directly from $(u_t)$ by the kernel estimator given in (21), the rate is optimal in the sense that it balances off the

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sampling bias and variance. The condition in Assumption 5(b) is not crucial and can be relaxed. It is introduced to simplify proofs for some of our subsequent results. The reader is referred to Andrews (1991b) for more detailed discussions of the choice of $\nu$ and $\ell_n$.

PROPOSITION 9. *Let Assumptions* 1–5 *hold. Then we have*

$$\hat{\omega}_{n\kappa}^2 = \omega^2 + O_p(n^{-1/3})$$

*for large n.*

We may therefore consistently estimate $\omega^2$ using the fitted errors $(\hat{u}_{\kappa t})$ from the truncated regression (14). Moreover, we have under the conditions given in Corollary 8 that

$$\hat{\omega}_{n\kappa}^{-1} M_{n\kappa}^{-1/2}(\Pi(\hat{\alpha}_{n\kappa}) - \Pi(\alpha)) \to_d \mathbf{N}(0, I_{md}) \tag{22}$$

and consequently,

$$\hat{\omega}_{n\kappa}^{-2} \operatorname{tr} M_{n\kappa}^{-1}(\Pi(\hat{\alpha}_{n\kappa}) - \Pi(\alpha))(\Pi(\hat{\alpha}_{n\kappa}) - \Pi(\alpha))' \to_d \chi_{md}^2, \tag{23}$$

which allows us to perform the usual chi-square test for inference.

It is not difficult to show that the asymptotic normality similar to that in Corollary 8 holds also for the parameter estimates in models (10) and (11). The proofs are entirely analogous to those for model (1), if we consider

$$y_t = \pi + x_{\kappa t}' \beta_\kappa + u_{\kappa t}, \tag{24}$$

$$y_t = \pi + \rho t + x_{\kappa t}' \beta_\kappa + u_{\kappa t} \tag{25}$$

in place of regression (14). For each $\kappa$, the regressions introduced in (24) and (25) can be analyzed as in Park and Phillips (1988). Moreover, the consistency of the long-run variance estimator of the errors given in Proposition 9 for model (1) can readily be extended to regressions (10) and (11).

## 4. GENERALIZATION

The asymptotic normality given in Corollary 8 does not hold for general models with $\omega_{12} \neq 0$ or $\delta_{12} \neq 0$. Just as for the fixed coefficient models, non-Gaussianity implies asymptotic inefficiency of the estimator and invalidity of the standard testing procedures. For efficient estimation and valid inference, here we extend to our model of time varying coefficients the method of canonical cointegrating regressions (CCR), which was developed by Park (1992) for the fixed coefficient cointegrating regressions. The procedure requires a transformation of $(y_t)$ and $(x_t)$ in (1), using the stationary components of the model. As introduced before,

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we denote by $\Omega_{ij}$ (or $\omega_{ij}$) and $\Delta_{ij}$ (or $\delta_{ij}$) for $i,j = 1,2$ the partitioned submatrices of $\Omega$ and $\Delta$ in (17) and (18). We also define

$$\Delta_2 = (\delta_{12}', \Delta_{22}'),$$

which will also be used to define the CCR transformation given subsequently.

We now consider the regression

$$y_t^* = x_{\kappa t}^{*\prime} \beta_\kappa + u_{\kappa t}^* \tag{26}$$

in place of (14), where

$$x_{\kappa t}^* = f_\kappa \left( \frac{t}{n} \right) \otimes x_t^* \tag{27}$$

with

$$x_t^* = x_t - \Delta_2 \Sigma^{-1} w_t$$

and

$$y_t^* = y_t - \left( f_\kappa \left( \frac{t}{n} \right) \otimes \Delta_2 \Sigma^{-1} w_t \right)' \beta_\kappa - (0, \omega_{12} \Omega_{22}^{-1}) w_t. \tag{28}$$

Because the previous transformations introduce only stationary deviations to regression (14), the same relationship is sustained in the transformed regression (26).

The stationary errors $(u_{\kappa t})$ in (14), however, undergo some substantial changes. The transformed errors $(u_{\kappa t}^*)$ in (26) are now given by

$$u_{\kappa t}^* = u_t^* + (\alpha - \alpha_\kappa) \left( \frac{t}{n} \right)' x_t$$

with the CCR error

$$u_t^* = u_t - \omega_{12} \Omega_{22}^{-1} \Delta x_t, \tag{29}$$

which constitutes the main nonperishing part. The long-run variance of the CCR error $(u_t^*)$ is

$$\omega_*^2 = \omega_{11} - \omega_{12} \Omega_{22}^{-1} \omega_{21}, \tag{30}$$

which is the conditional long-run variance of the errors $(u_t)$ given the innovations of the regressors $(v_t), v_t = \Delta x_t$, in the original regression (1).

We define the CCR estimator to be the OLS estimator for the transformed regression model (26). Notice that the CCR error $(u_t^*)$ in (29) has zero long-run correlation with the innovations of the regressors $(v_t^*), v_t^* = \Delta x_t^*$. The OLS procedure in the transformed model thus yields an efficient and optimal estimator in the sense of Phillips (1991) and Saikkonen (1991). If we denote by

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$X^*_{n\kappa} = (x^*_{\kappa 1}, \ldots, x^*_{\kappa n})'$ and $y^*_{nn} = (y^*_1, \ldots, y^*_n)'$, then the CCR estimator $\hat{\beta}^*_{n\kappa}$ of $\beta_\kappa$ is defined by

$$\hat{\beta}^*_{n\kappa} = (X^{*\prime}_{n\kappa} X^*_{n\kappa})^{-1} X^{*\prime}_{n\kappa} y^*_{nn}.$$

Similarly as in (12) and (13), the estimators of $\alpha$ and $\Pi(\alpha)$ are given, respectively, by $\hat{\alpha}^*_{n\kappa}$ and $\Pi(\hat{\alpha}^*_{n\kappa})$, where

$$\hat{\alpha}^*_{n\kappa} = \sum_{i=1}^{\kappa} \hat{\beta}^*_{n\kappa i} \varphi_i$$

and

$$\Pi(\hat{\alpha}^*_{n\kappa}) = (\hat{\alpha}^*_{n\kappa}(r_1)', \ldots, \hat{\alpha}^*_{n\kappa}(r_s)')' = T_\kappa \hat{\beta}^*_{n\kappa}$$

with $T_\kappa$ introduced in (8) and $\hat{\beta}^*_{n\kappa} = (\hat{\beta}^{*\prime}_{n\kappa 1}, \ldots, \hat{\beta}^{*\prime}_{n\kappa\kappa})'$.

The CCR transformation involves unknown parameters, which must of course be estimated for practical applications. We may easily get consistent estimates for those parameters from the first-step OLS regression. As defined before, we let $(\hat{u}_{\kappa t})$ be the OLS residuals from regression (14) and define

$$\hat{w}_{\kappa t} = (\hat{u}_{\kappa t}, v_t')'. \tag{31}$$

We may consistently estimate $\Omega$ and $\Delta$ by

$$\hat{\Omega}_{n\kappa} = \frac{1}{n} \sum_{|k| \le \ell_n} \nu\left(\frac{k}{n}\right) \sum_t \hat{w}_{\kappa t} \hat{w}'_{\kappa, t-k}, \tag{32}$$

$$\hat{\Delta}_{n\kappa} = \frac{1}{n} \sum_{0 \le k \le \ell_n} \nu\left(\frac{k}{n}\right) \sum_t \hat{w}_{\kappa t} \hat{w}'_{\kappa, t-k} \tag{33}$$

using $(\hat{w}_{\kappa t})$, similarly as $\hat{\omega}^2_{n\kappa}$ in (21). The estimates for the unknown parameters used in the CCR transformation can be obtained from (32) and (33) and $\hat{\beta}_{n\kappa}$ in (15).

In what follows, we suppose that all $\hat{\alpha}^*_{n\kappa}$, $\hat{\beta}^*_{n\kappa}$, and $\Pi(\hat{\alpha}^*_{n\kappa})$ are obtained from the regression (26) formulated with the estimated parameters in the CCR transformation. Let

$$M^*_{n\kappa} = T_\kappa (X^{*\prime}_{n\kappa} X^*_{n\kappa})^{-1} T'_\kappa$$

similarly as (20). Also, we define

$$B^*_\kappa = \int_0^1 f_\kappa \otimes V \, dU_*,$$

where $U_* = U - \omega_{12} \Omega_{22}^{-1} V$. Note that $U_*$ is a Brownian motion independent of $V$. It has variance $\omega^2_*$, i.e., the long-run variance of the CCR errors given in (30).

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THEOREM 10. *Let Assumptions* 1–5 *hold. Then Propositions* 5 *and* 6 *and Theorem* 7 *hold with* $\hat{\alpha}_{n\kappa}$, $M_{n\kappa}$, *and* $B_\kappa$ *replaced, respectively, by* $\hat{\alpha}^*_{n\kappa}$, $M^*_{n\kappa}$, *and* $B^*_\kappa$. *Moreover*,

$$M^{*-1/2}_{n\kappa}(\Pi(\hat{\alpha}^*_{n\kappa}) - \Pi(\alpha)) \to_d \mathbf{N}(0, \omega^2_* I_{md})$$

*as* $n \to \infty$.

Theorem 10 shows that all our previous results for regressions with strictly exogenous regressors continue to hold for general models with $\omega_{12} \neq 0$ or $\delta_{12} \neq 0$, if formulated with variables transformed by the CCR method.

A consistent estimator of the long-run variance $\omega^2_*$ of the CCR errors in (30) can readily be obtained from $\hat{\Omega}_{n\kappa}$ in (32). For such a consistent estimator $\hat{\omega}^{*2}_{n\kappa}$, we have

$$\hat{\omega}^{*-1}_{n\kappa} M^{*-1/2}_{n\kappa}(\Pi(\hat{\alpha}^*_{n\kappa}) - \Pi(\alpha)) \to_d \mathbf{N}(0, I_{md})$$

and

$$\hat{\omega}^{*-2}_{n\kappa} \operatorname{tr} M^{*-1}_{n\kappa}(\Pi(\hat{\alpha}^*_{n\kappa}) - \Pi(\alpha))(\Pi(\hat{\alpha}^*_{n\kappa}) - \Pi(\alpha))' \to_d \chi^2_{md}$$

corresponding to (22) and (23) for the strictly exogenous case. Therefore, the usual chi-square tests are, in particular, valid for general models if based on the CCR transformed regressions.

The CCR methodology is applicable also for models (10) and (11) with $\Delta x_t = \mu + v_t$. This is well expected, because the asymptotic normality as in Corollary 8 holds for the parameter estimates in such models. This was noted earlier in the previous section. Of course, we need to redefine $(\hat{w}_{\kappa t})$ in (31) as

$$\hat{w}_{\kappa t} = (\hat{u}_{\kappa t}, \hat{v}'_t)'$$

to estimate the unknown parameters in the CCR transformation, where $(\hat{v}_t)$ are the fitted residuals from the regression of $(\Delta x_t)$ on a constant term. The asymptotic normality of the CCR-based estimator in Theorem 10 holds for models (10) and (11) and also for model (1).

## 5. SPECIFICATION TESTS

In this section, we discuss how the adequacy of our model can be tested. In particular, we consider two classes of specification tests that are believed to be powerful against most relevant alternative models. The first class of tests is introduced to discriminate our model against the cointegrating regression with fixed coefficients. We thus assume that the cointegrating relationship is present in both models and concentrate on testing whether the coefficients are fixed or varying. The second class of tests is used to test our model against the alternative of a spurious regression with time varying coefficients. It is therefore maintained that

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the coefficients are time dependent. We only test whether the regression is meaningfully defined and consistently estimated. We subsequently introduce the specification tests with their statistical theories.

The usual fixed coefficient (FC) cointegrating regression model

$$y_t = \bar{\alpha}' x_t + u_t \tag{34}$$

can be tested against our model (1) with time varying coefficients (TVC), using the variable addition approach by Park (1990) or the residual-based method by Shin (1994). We consider two statistics given by

$$\tau_1 = \frac{RSS_{FC} - RSS_{FC}^s}{\hat{\omega}_{n\kappa}^2}, \tag{35}$$

$$\tau_2 = \frac{\sum_{t=1}^{n} \left( \sum_{i=1}^{t} \hat{u}_i \right)^2}{n^2 \hat{\omega}_{n\kappa}^2}, \tag{36}$$

where $RSS_{FC}$ and $RSS_{FC}^s$ are the sums of squared residuals, respectively, from (34) and the regression augmented with some superfluous regressors and $(\hat{u}_t)$ are the fitted residuals from regression (34). The long-run variance estimate $\hat{\omega}_{n\kappa}^2$ is given in (21).

The statistics introduced in (35) and (36) were originally proposed to test the stationarity, against the nonstationarity, of the regression error $(u_t)$ in (34). Because our TVC model (1) can obviously be viewed as the FC model (34) with nonstationary errors, they are applicable in our context. Note that $\tau_1$ is essentially the Wald statistic to test for the significance of the superfluously added regressors. It is modified in a trivial way to allow for the serial correlation in the errors. The statistic relies on the fact that we must be able to detect the redundancy of the superfluous regressors if the errors are stationary and the regression is correctly specified. This is not so when the errors are nonstationary and the regression is fatally misspecified. The superfluousness of the added regressors is more likely to be undetected in this case. The statistic $\tau_2$ more directly tests the stochastic order of the regression errors. It uses the fact that the partial sum of the errors has different orders, depending upon whether they are stationary or nonstationary.

For the implementation of the test based on $\tau_1$, we need to choose appropriate superfluous regressors. In the paper, we use time polynomial terms $1, t, t^2, \ldots, t^{s-1}$ as superfluous regressors, as suggested by Park (1990). Under the null hypothesis of the FC cointegrating regression model (34), the limiting distributions of $\tau_1$ and $\tau_2$ are as given in Park (1990) and Shin (1994). Strictly speaking, $\tau_1$ and $\tau_2$ are slightly different from their original statistics. The difference is in the estimation of the error long-run variance. Our statistics are defined with the long-run variance estimate of errors based on (1), in contrast to theirs with the one based on (34). However, because of Proposition 9, it is obvious that our modification of the long-run variance estimate does not change their limiting distributions.

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The statistics $\tau_1$ and $\tau_2$ diverge under the TVC model (1). To see how $\tau_1$ and $\tau_2$ behave under the alternative hypothesis, we let $\sigma(r) = (1, r, r^2, \ldots, r^{s-1})'$ and denote by $L^2[0,1]$ the Hilbert space of square integrable functions defined on $[0,1]$.

THEOREM 11. *Let the true model be given by* (1) *and let Assumptions* 1 *and* 2 *hold. Then we have*

$$\frac{1}{n^2}\tau_1 \to_d \frac{1}{\omega^2}\int_0^1 PQ'\left(\int_0^1 QQ'\right)^{-1}\int_0^1 QP,$$

*where P and Q are the residuals from the Hilbert space projections of* $\alpha'V$ *and* $\sigma$ *on V, respectively, in* $L^2[0,1]$. *Moreover*,

$$\frac{1}{n^2}r_2 \to_d \frac{1}{\omega^2}\int_0^1 R^2,$$

*where*

$$R(r) = \int_0^r \alpha'V - \int_0^1 \alpha'VV'\left(\int_0^1 VV'\right)^{-1}\int_0^r V.$$

Theorem 11 shows, in particular, that $\tau_1$ and $\tau_2$ diverge at the rate $n^2$ under the TVC model (1). The specification tests based on the statistics are therefore consistent. It is important to use the error long-run variance estimate $\hat{\omega}^2_{n\kappa}$ in (21), which uses the residuals from regression (14). If it is computed from the FC model (34), the divergence rate would be reduced.

We may also test our specification (1) against the alternative of a spurious regression with nonstationary errors having a unit root, which is specified as

$$y_t = \alpha_t' x_t + e_t \quad \text{with } e_t = \sum_{i=1}^{t} u_i. \tag{37}$$

To test the cointegrating regression (1) against the spurious regression given by (37), we consider the statistic

$$\tau = \frac{RSS_{TVC} - RSS^s_{TVC}}{\hat{\omega}^2_{n\kappa}}, \tag{38}$$

where $RSS_{TVC}$ and $RSS^s_{TVC}$ are the squared sums of residuals from regression (14) and the regression augmented with additional regressors $1, t, t^2, \ldots, t^{s-1}$, defined similarly as for $\tau_1$ in (35). The test relying on a statistic analogous to $\tau_2$ in (36) is also expected to have discriminatory power, but its critical value would be dependent upon the unknown function $\alpha$. As for $\tau_1$ and $\tau_2$, we use the long-run variance estimate $\hat{\omega}^2_{n\kappa}$ given in (21).

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THEOREM 12. *Suppose that Assumptions* 1–5 *hold.*

(a) *If the true model is given by* (1) *and if* $\omega_{12} = \delta_{12} = 0$, *then* $\tau \to_d \chi_s^2$ *as* $n \to \infty$.
(b) *If the true model is given by* (37), *then* $\tau \to_p \infty$ *as* $n \to \infty$.

Similarly as for the spurious regression with fixed coefficients, the statistic $\tau$ in (38) diverges if the true model has nonstationary errors and becomes spurious. The result in Park (1990) for the spurious regression with fixed coefficients thus extends well to the model with time varying coefficients. This implies in particular that the spuriousness, or the absence of cointegration, would not disappear by allowing for the coefficients to vary over time. Consequently, the test that rejects the TVC model (1) for a large value of $\tau$ is consistent.

For more general models with $\omega_{12} \neq 0$ or $\delta_{12} \neq 0$, the model specification test statistics $\tau_1^*$ and $\tau_2^*$ can be defined from the transformed regression

$$y_t^* = \bar{\alpha}' x_t^* + u_t^*$$

correspondingly to $\tau_1$ and $\tau_2$ in (35) and (36), where $(x_t^*)$ and $(y_t^*)$ are the CCR transformations of $(x_t)$ and $(y_t)$ introduced in Section 4. The statistic $\tau^*$ can also be defined from regression (26), in the same way as the statistic $\tau$ is defined in (38) for regression (14). The long-run error variance estimate $\hat{\omega}_{n\kappa}^2$ for $\tau$ should now be replaced for $\tau^*$ by $\hat{\omega}_{n\kappa}^{2*}$, which has been introduced earlier. It is straightforward to show the following corollary.

COROLLARY 13. *If Assumptions* 1 *and* 2 *are satisfied, then the results in Theorem* 11 *continue to hold for* $\tau_1^*$ *and* $\tau_2^*$ *with only replacement of* $\omega^2$ *by* $\omega_*^2$. *Moreover, under Assumptions* 1–5, *the results in Theorem* 12 *hold also for* $\tau^*$.

All our previous discussions on $\tau$, $\tau_1$, and $\tau_2$ therefore extend to $\tau^*$, $\tau_1^*$, and $\tau_2^*$, which are applicable for more general models.

The test statistics $\tau, \tau_1, \tau_2$ and their modifications are introduced to compare (1) with (34) and (37). In particular, they are not applicable to test for models (10) and (11). Their behaviors are affected critically by the presence of the deterministic trends. For the specification tests of models (10) and (11), we need to generalize $\tau_1$, $\tau_2$, and $\tau$. The required generalization, however, is trivial. We may, for instance, define test statistics similarly as $\tau_1$ and $\tau_2$ to test

$$y_t = \pi + \bar{\alpha}' x_t + u_t \quad \text{and} \quad y_t = \pi + \rho t + \bar{\alpha}' x_t + u_t, \tag{39}$$

respectively, against (10) and (11). To construct the $\tau_1$ statistic for the regressions in (39), we may use $t, \ldots, t^s$ or $t^2, \ldots, t^{s+1}$ as superfluous regressors and obtain $RSS_{FC}^s$. The $\tau_2$ statistic can also be defined using the residuals from the regressions in (39). Moreover, the generalization of $\tau$ is possible to test models (10) and (11) against

$$y_t = \pi + \alpha_t' x_t + e_t \quad \text{and} \quad y_t = \pi + \rho t + \alpha_t' x_t + e_t,$$

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where $(e_t)$ is given in (37). For the definition of $\tau$, we may use the same superfluous regressors as for $\tau_1$ and obtain $RSS^s_{TVC}$. We may show that Corollary 13 holds also for the generalized statistics, if they are modified appropriately by the CCR method.

## 6. AN EMPIRICAL APPLICATION

To illustrate the practical relevancy of our model and methodology, we study a simple econometric model set up for the U.S. automobile demand function. The model is specified as in the textbook by Pindyck and Rubinfeld (1991), but we allow for some of the regression coefficients to change over time. Our interpretation of the model is also different in that the regression equation is now regarded as representing a long-run cointegrating relationship.

The actual regression we consider is

$$c_t = \pi + \alpha_{1t} y_t + \alpha_2 r_t + \alpha_3 p_t + u_t,$$

where $(c_t)$, $(y_t)$, $(r_t)$, and $(p_t)$ denote, respectively, log of personal consumption expenditure on new automobiles, log of personal disposable income (both in billions of current dollars), 3-month Treasury bill rate in percentage per year, and log of consumer price index of new cars with the base year 1983. The monthly data were collected from the Citibase financial data base, whose time span is from January 1959 to October 1997. All the series are seasonally adjusted.

The unit root tests for individual series were performed. Both the tests by Phillips (1987) and Dickey and Fuller (1979, 1981) strongly support the presence of unit roots in all four series, if they are undetrended. The evidence in favor of the unit root is rather decisive also for detrended $(y_t)$, $(r_t)$, and $(p_t)$. The test result for detrended $(c_t)$, however, is mixed and inconclusive. The reported results for the Dickey–Fuller test in Table 1 are based on the first-order autoregression augmented with 12 differenced lag terms. The test results were rather insensitive to the number of augmented lagged differences. To estimate the long-run error variance for the Phillips test reported in Table 1, we used the Parzen

**TABLE 1.** Unit root tests

| | Demeaned Series | | Detrended Series | |
|---|---|---|---|---|
| Variables | Phillips | Dickey–Fuller | Phillips | Dickey–Fuller |
| $y_t$ | −0.57 | −0.61 | 0.67 | −0.95 |
| $r_t$ | −2.42 | −1.97 | −2.35 | −1.91 |
| $p_t$ | 0.73 | −0.17 | −2.49 | −2.02 |
| $c_t$ | −1.65 | −1.51 | −4.60 | −1.90 |
| 5% critical values | −2.86 | | −3.41 | |

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window with the lag truncation number chosen according to the data-dependent method by Andrews (1991b).

The regression coefficient $\alpha_{1t}$ of $(y_t)$ was fitted using the FFF series functions, which include constant, $t$, $t^2$, and trigonometric functions with fundamental frequencies. The generalized cross validation (GCV) method by Craven and Wahba (1979) was employed to determine the truncation point. The TVC of income was actually fitted with constant, time, time squared, and 15 trigonometric pairs. To implement the CCR procedure, we assume that $(\Delta r_t)$ have zero mean. However, we allowed for the presence of linear time trends in $(y_t)$ and $(p_t)$; i.e., $(\Delta y_t)$ and $(\Delta p_t)$ were permitted to have nonzero means. For the estimation of long-run parameters used in the CCR transformations, the Parzen window was used with the data-dependent selection of lag truncation number as in the tests for the unit root.

The results for the model specification tests introduced in Section 4 are summarized in Table 2. The statistics $\tau_1^*$ and $\tau_2^*$ are used to test the null hypothesis of the FC model against the model with TVC. With the statistic $\tau^*$, we test the null hypothesis of the cointegrating regression with TVC against the alternative of the noncointegrated model with integrated errors. For $\tau_1^*$ and $\tau^*$, we included four time polynomial terms $t$, $t^2$, $t^3$, and $t^4$ as superfluous regressors. The limiting null distribution of the test statistics is therefore chi-square with four degrees of freedom.

It seems evident that the cointegrated model with TVC is strongly supported by the data. The tests based on $\tau_1^*$ and $\tau_2^*$ unambiguously reject the FC model in favor of the TVC model. Moreover, the value of the $\tau^*$ statistic shows that the cointegrating regression with TVC is much preferred to the noncointegrated spurious regression model with a unit root in the regression errors.

The estimated TVC regression model is

$$\underset{}{c_t} = \underset{(-4.6002)}{-22.2128} + \alpha_t \underset{}{y_t} \underset{(-0.2107)}{- 0.0011} \underset{}{r_t} \underset{(-2.8222)}{- 1.1306} \underset{}{p_t}$$

with the time varying $(\alpha_t)$ plotted in Figure 1. In the parentheses are the CCR adjusted $t$-values for the coefficients of the constant term, $(r_t)$ and $(p_t)$. For the time varying coefficient of $(y_t)$, the CCR adjusted 95% confidence band is plotted in Figure 1. Figure 1 shows that the income coefficient smoothly decreases from 4.91 to 3.65 during the estimation period. Hence, it can be said that 1%

**TABLE 2.** Specification tests

| Test statistics | $\tau_1^*$ | $\tau_2^*$ | $\tau^*$ |
|---|---|---|---|
| Test values | 674.69 | 6.89 | 4.21 |
| 5% critical values | 9.49 | 0.16 | 9.49 |

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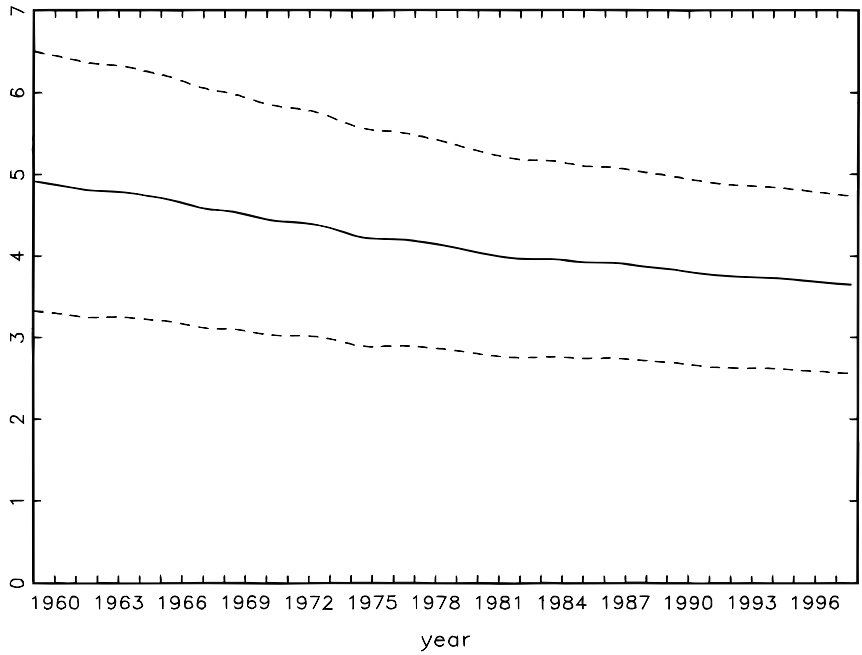



**FIGURE 1.** Income coefficient.

increase in disposable income would have led to 4.91% increase in automobile sales in 1959, with of course the effects of all other variables held constant. In 1997, the income elasticity of automobile demand had become considerably smaller, and 1% increase in disposable income would predict only about 3.65% increase in automobile sales.

For the purpose of comparison, we also fitted the FC model. The fitted regression equation for the FC model is given by

$$c_t = \underset{(-9.3864)}{-1.2083} + \underset{(21.4172)}{0.8272}\, y_t + \underset{(1.8557)}{0.0056}\, r_t - \underset{(-3.2303)}{0.2886}\, p_t.$$

The *t*-values in the parentheses were adjusted again by the CCR procedure. The estimated coefficients for the FC model are rather different from those for the TVC model. The FC model, where the income coefficient is held fixed, suggests that automobile demand is much less elastic with respect to income changes. Moreover, the effect of interest rates on automobile sales is significantly positive in the fitted FC model, in contrast to the fitted TVC model in which it is insignificantly negative. Given our previous results for model specification tests, however, the large *t*-values reported here should be interpreted not as representing the

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significance of the regression coefficients but as evidence for the regression being spurious. It is well known from Phillips (1986) that the *t*-values diverge in a spurious regression as the sample size increases, regardless of the absence of a true relationship.

## 7. CONCLUDING REMARKS

We have considered in the paper the cointegrating regression model with coefficients varying smoothly over time. Such a model seems a very reasonable alternative to the constant coefficient cointegrating model, when the latter fails to fit the actual data. The notion of cointegration is primarily used to describe a long-run relationship, and it is likely that the relationship evolves over time especially when the time span is long. The regression model with time varying coefficients seems thus much more important in our context of cointegrating regression than that of stationary regression, which essentially models a short-run relationship. The empirical relevancy of our model has been successfully demonstrated in the paper through an application to the U.S. automobile demand function.

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# APPENDIX A: MATHEMATICAL PROOFS

**Proof of Lemma 1.** For part (a), note that

$$\sup_{0\le r\le 1} |W_n^{\circ}(r) - W_n^{\bullet}(r)| \le \frac{\widetilde{w}_0}{\sqrt{n}} + \max_{1\le t\le n} \left| \frac{\widetilde{w}_t}{\sqrt{n}} \right|.$$

Because $p > 2$ and $\sum |\tilde{\Phi}_i|^2 < \infty$, we have from the assumption $\mathbf{E}|\varepsilon_t|^p < \infty$ that $\mathbf{E}|\widetilde{w}_t|^p < \infty$. However, it follows from Tchebyshev inequality that

$$\mathbf{P}\{n^{-1/p} \max_{1\le t\le n} |\widetilde{w}_t| > M\} \le \sum_{t=1}^{n} \mathbf{P}\{n^{-1/p}|\widetilde{w}_t| > M\} \le \frac{\mathbf{E}|\widetilde{w}_t|^p}{M^p} \to 0$$

as $M \to \infty$. This proves part (a).

We now show part (b). It follows immediately from Einmahl (1989, Theorem 1) that there exists $W_n$ such that $W_n^{\bullet} =_d W_n$ and

$$\sup_{1\le t\le n} \left| W_n\left(\frac{t}{n}\right) - W\left(\frac{t}{n}\right) \right| = o_p(n^{-a}). \tag{A.1}$$

However, because of the well-known Hölder continuity of $W$ (see, e.g., Karatzas and Shreve 1988, Theorem 9.25, p. 114), we have

$$\left| W\left(\frac{[nr]}{n}\right) - W(r) \right| \le c \left| \frac{[nr]}{n} - r \right|^{1/2-\delta} \tag{A.2}$$

for some constant $c$ and any $\delta > 0$. Because

$$\sup_{0\le r\le 1} \left| \frac{[nr]}{n} - r \right| = O_p(n^{-1})$$

the result stated in part (b) can easily be deduced from (A.1) and (A.2). ■

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**Proof of Lemma 2.** It follows as a special case from more general results in Edmunds and Moscatelli (1977). The reader is also referred to Andrews (1991a) for more readable discussions of their results in the context of nonparametric series estimation of regression function. ■

**Proof of Lemma 3.** Partition $W_n^\circ$ similarly as $W_n$ in (19). Also, define

$$\varphi_{ni}(r) = \varphi_i\left(\frac{[nr]}{n}\right).$$

The result in part (a) is straightforward, because

$$\frac{1}{n^2}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)x_t x_t' = \int_0^1 \varphi_{ni}\varphi_{nj}V_n^\circ V_n^{\circ\prime} + O_p(n^{-1})$$

$$= \int_0^1 \varphi_{ni}\varphi_{nj}V_n V_n' + O_p(n^{-a})$$

uniformly in $i, j \geq 1$. Note that the series functions $(\varphi_i)$ are uniformly bounded.

For part (b), we let $z_t = \sum_1^t w_i$ and show that

$$\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)z_t w_t' = \int_0^1 \varphi_{ni}W_n\,dW_n' + \Delta'\int_0^1 \varphi_{ni} + O_p(n^{-a}) \tag{A.3}$$

uniformly in $i \geq 1$. To prove (A.3), we first write

$$\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)z_t w_t' = \frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)z_{t-1}w_t^{\bullet\prime} + S_{ni} + R_{ni}, \tag{A.4}$$

where

$$S_{ni} = \frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)w_t w_t' + \frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)w_t\widetilde{w}_t' := P_{ni} + Q_{ni}$$

and

$$R_{ni} = \frac{1}{n}\left(\varphi_i(0)z_0\widetilde{w}_0' - \varphi_i(1)z_n\widetilde{w}_n'\right),$$

which follows immediately from (4).

We have

$$\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)z_{t-1}w_t^{\bullet\prime} = \int_0^1 \varphi_{ni}W_n^\circ\,dW_n^{\bullet\prime} + O_p(n^{-1})$$

$$= \int_0^1 \varphi_{ni}W_n\,dW_n + O_p(n^{-a})$$

uniformly in $i \geq 1$. Also, it is obvious that $R_{ni} = O_p(n^{-1/2})$ for all $i \geq 1$. Note that $(\varphi_i)$ are uniformly bounded. To deduce the stated result from (A.4), it therefore suffices to show

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that

$$P_{ni} = P_{ni}^{\circ} + O_p(n^{-1/2}) \quad \text{and} \quad Q_{ni} = Q_{ni}^{\circ} + O_p(n^{-1/2}) \tag{A.5}$$

independent of $i \geq 1$, where

$$P_{ni}^{\circ} = \Sigma\left(\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\right) \quad \text{and} \quad Q_{ni}^{\circ} = \Lambda'\left(\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\right),$$

which we now set out to prove.

Note that

$$\Sigma = \sum_{k=0}^{\infty}\Phi_k\Psi\Phi_k'$$

and

$$\Lambda' = \sum_{k=0}^{\infty}\left(\sum_{j=k+1}^{\infty}\Phi_j\right)\Psi\Phi_k' = \sum_{k=0}^{\infty}\tilde{\Phi}_k\Psi\Phi_k'.$$

We therefore have

$$\begin{aligned} P_{ni} - P_{ni}^{\circ} &= \sum_{k=0}^{\infty}\Phi_k\left(\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)(\varepsilon_{t-k}\varepsilon_{t-k}' - \Psi)\right)\Phi_k' \\ &\quad + \sum_{j\neq k}\Phi_j\left(\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varepsilon_{t-j}\varepsilon_{t-k}'\right)\Phi_k', \\ Q_{ni} - Q_{ni}^{\circ} &= \sum_{k=0}^{\infty}\tilde{\Phi}_k\left(\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)(\varepsilon_{t-k}\varepsilon_{t-k}' - \Psi)\right)\Phi_k' \\ &\quad + \sum_{j\neq k}\tilde{\Phi}_j\left(\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varepsilon_{t-j}\varepsilon_{t-k}'\right)\Phi_k'. \end{aligned}$$

However, it follows from the existence of the fourth moment of $(\varepsilon_t)$ assumed in Assumption 1 and the uniform boundedness of $(\varphi_i)$ that

$$\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)(\varepsilon_{t-k}\varepsilon_{t-k}' - \Psi) = O_p(n^{-1/2}) \tag{A.6}$$

and, for $j \neq k$,

$$\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varepsilon_{t-j}\varepsilon_{t-k}' = O_p(n^{-1/2}) \tag{A.7}$$

uniformly in $i$, $j$, and $k$. We now have (A.5) due to (A.6)–(A.7) and the absolute summability of $(\Phi_i)$ and $(\tilde{\Phi}_i)$.

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For part (c), notice that

$$\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)x_t(u_{\kappa t}-u_t)=\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)x_t x_t'(\alpha-\alpha_\kappa)\left(\frac{t}{n}\right)$$

$$=n\int_0^1(\varphi_{ni}V_n^{\circ}V_n^{\circ\prime})(\alpha_n-\alpha_{n\kappa})+\|\alpha-\alpha_\kappa\|O_p(1)$$

$$=n\int_0^1(\varphi_{ni}V_nV_n')(\alpha_n-\alpha_{n\kappa})+\|\alpha-\alpha_\kappa\|O_p(n^{1-a}),$$

which holds uniformly in $i\geq 1$. ■

**Proof of Lemma 4.** The proof relies on the following lemma.

LEMMA A1. *We have*

(a) $\sup_{\kappa\geq 1}\left\|\int_0^1 f_\kappa f_\kappa'\otimes VV'\right\|=O_p(1)$,

(b) $\sup_{\kappa\geq 1}\left\|\int_0^1 f_\kappa f_\kappa'\otimes VV'\right\|^{-1}=O_p(1)$.

**Proof of Lemma A1.** Note that $\int_0^1 f_\kappa f_\kappa'=I_\kappa$, because the series functions $(\varphi_i)$ in (6) are orthonormal in $L^2[0,1]$. Let $V=(V_1,\ldots,V_m)'$ and define

$$s_{\max}=\sup_{1\leq i\leq m}\sup_{0\leq r\leq 1}|V_i(r)|. \tag{A.8}$$

Then we have

$$\int_0^1 f_\kappa f_\kappa'\otimes(VV'-s_{\max}^2 I_m)\leq 0$$

for all $\kappa\geq 1$. Therefore,

$$\int_0^1 f_\kappa f_\kappa'\otimes VV'\leq s_{\max}^2 I_{m\kappa},$$

and consequently,

$$\lambda_{\max}\left(\int_0^1 f_\kappa f_\kappa'\otimes VV'\right)\leq s_{\max}^2$$

for all $\kappa\geq 1$, which proves the stated result in part (a).

To prove part (b), we first define $V_\varepsilon=(V_1^\varepsilon,\ldots,V_m^3)'$, where

$$V_i^e(r)=V_i(r)1\{|V_i(r)|>\varepsilon\}+\varepsilon 1\{|V_i(r)|\leq\varepsilon\}.$$

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It follows that

$$\int_0^1 f_\kappa f_\kappa' \otimes (V_\varepsilon V_\varepsilon' - \varepsilon^2 I_m) \geq 0$$

for all $\kappa \geq 1$, and therefore,

$$\int_0^1 f_\kappa f_\kappa' \otimes V_\varepsilon V_\varepsilon' \geq \varepsilon^2 I_{m\kappa}$$

for all $\kappa \geq 1$. We thus have

$$\lambda_{\min}\left(\int_0^1 f_\kappa f_\kappa' \otimes V_\varepsilon V_\varepsilon'\right) \geq \varepsilon^2$$

for all $\kappa \geq 1$.

We now define

$$J_\varepsilon = \{r \in [0,1] | \inf_{1 \leq i \leq m} |V_i(r)| \leq \varepsilon\}.$$

Then it follows that

$$\int_0^1 f_\kappa f_\kappa' \otimes V_\varepsilon V_\varepsilon' - \int_0^1 f_\kappa f_\kappa' \otimes VV' \leq 2 \int_{J_\varepsilon} f_\kappa f_\kappa' \otimes \varepsilon^2 I_m.$$

However, because

$$\lim_{\varepsilon \to 0} \frac{\omega_i^2}{2\varepsilon} \int_0^1 1\{|V_i(r)| \leq \varepsilon\}\, dr = L_i(1,0),$$

where $\omega_i^2$ is the variance of $V_i$ and $L_i(1,0)$ is the local time of $V_i$, we have in particular

$$\frac{1}{\varepsilon} \int_0^1 1(J_\varepsilon) = O(1) \quad \text{a.s.}$$

and therefore,

$$\left\| \int_{J_\varepsilon} f_\kappa f_\kappa' \right\| = O(\varepsilon) \quad \text{a.s.}$$

The reader is referred, e.g., to Karatzas and Shreve (1988, p. 203) for the definition of Brownian local time. We now choose $\varepsilon > 0$ sufficiently small to deduce that

$$\lambda_{\min}\left(\int_0^1 f_\kappa f_\kappa' \otimes VV'\right) = \lambda_{\min}\left(\int_0^1 f_\kappa f_\kappa' \otimes V_\varepsilon V_\varepsilon'\right) + O(\varepsilon^3)$$
$$\geq \varepsilon^2(1 + o(1)) > 0 \quad \text{a.s.}$$

for all $\kappa \geq 1$. The proof is therefore complete. ■

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Lemma A1 shows that $\|A_\kappa\|, \|A_\kappa^{-1}\| = O_p(1)$. To show that $\|A_{n\kappa} - A_\kappa\| = O_p(n^{-a}\kappa) + O_p(n^{-1}\kappa^2)$, observe that

$$\sup_{1 \le i \le \kappa} \|\varphi_{ni} - \varphi_i\| = O(n^{-1}\kappa), \tag{A.9}$$

where $(\varphi_{ni})$ are defined in the proof of Lemma 3. It follows from (A.9) and Lemma 1 that

$$\int_0^1 \varphi_{ni}\varphi_{nj}V_nV_n' = \int_0^1 \varphi_i\varphi_j VV' + O_p(n^{-a}) + O_p(n^{-1}\kappa)$$

uniformly in $i, j \ge 1$, and the proof for part (a) is complete.

We now prove the results stated in part (b). It is obvious that $\|B_\kappa\| = O_p(\kappa^{1/2})$. Next, to establish $\|B_{n\kappa} - B_\kappa\| = O_p(n^{-a}\kappa^{1/2})$, it suffices to show that

$$\int_0^1 \varphi_{ni}V_n\,dU_n = \int_0^1 \varphi_i V\,dU + O_p(n^{-a})$$

uniformly in $i \ge 1$. However, this follows directly from

$$\left|\int_0^1 \varphi_{ni}V_n\,dU_n - \int_0^1 \varphi_i V\,dU\right| \le \left|\int_0^1 (\varphi_{ni}V_n - \varphi_i V)\,dU_n\right| + \left|\int_0^1 \varphi_i V\,dU_n - \int_0^1 \varphi_i V\,dU\right|$$

and

$$\int_0^1 \varphi_i V\,dU_n - \int_0^1 \varphi_i V\,dU$$

$$= \varphi_i(1)V(1)(U_n(1) - U(1)) + \int_0^1 \varphi_i(U - U_n)\,dV + \int_0^1 (U - U_n)V\,d\varphi_i$$

along with (A.9) and Lemma 1.

For part (c), we let $s_{\max}$ be defined as in (A.8). Then we have

$$\left\|\int_0^1 \varphi_i VV'(\alpha - \alpha_\kappa)\right\| \le s_{\max}^2\|\alpha - \alpha_\kappa\|$$

uniformly in $i \ge 1$, because $(\varphi_i)$ are uniformly bounded. Moreover, we have

$$\left\|\int_0^1 \varphi_i V_nV_n'(\alpha_{n\kappa} - \alpha_\kappa)\right\| \le (s_{\max} + 1)^2\|\alpha - \alpha_\kappa\|$$

for all large $n$, uniformly in $i \ge 1$. The stated results now follow immediately from the fact that $s_{\max} = O_p(1)$ and Lemma 2. ■

**Proof of Proposition 5.** Define

$$A_{n\kappa}^\circ = \frac{X_{n\kappa}'X_{n\kappa}}{n^2} \quad \text{and} \quad B_{n\kappa}^\circ = \frac{X_{n\kappa}'u_{n\kappa}}{n}$$

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so that we have

$$n(\Pi(\hat{\alpha}_{n\kappa}) - \Pi(\alpha_\kappa)) = nT_\kappa(\hat{\beta}_{n\kappa} - \beta_\kappa) = T_\kappa A_{n\kappa}^{\circ -1} B_{n\kappa}^\circ.$$

For part (a), it suffices to show that

$$T_\kappa A_{n\kappa}^{\circ -1} B_{n\kappa}^\circ = T_\kappa A_\kappa^{-1} B_\kappa + O_p(n^{-a}\kappa^2) + o_p(n\kappa^{1-b}) \tag{A.10}$$

because $n^{-a}\kappa^2, n\kappa^{1-b} = O(\kappa)$. Note that $\kappa = o(n^{2a/3})$ and there exists $b$ such that $\kappa^{-b} = o(n^{-2/(2b-1)})$ under Assumption 4.

To deduce (A.10), note that

$$\|A_{n\kappa}^\circ - A_\kappa\| \le \|A_{n\kappa}^\circ - A_{n\kappa}\| + \|A_{n\kappa} - A_\kappa\| = O_p(n^{-a}\kappa), \tag{A.11}$$

which follows from Lemmas 3 and 4. Moreover, because $\|A_\kappa^{-1}\| = O_p(1)$ and $\|A_{n\kappa} - A_\kappa\| = o_p(1)$, we have $\|A_{n\kappa}^{-1}\| = O_p(1)$. Consequently,

$$\|A_{n\kappa}^{\circ -1} - A_\kappa^{-1}\| \le \|A_{n\kappa}^{\circ -1}\|\|A_\kappa - A_{n\kappa}^\circ\|\|A_\kappa^{-1}\| = O_p(n^{-a}\kappa). \tag{A.12}$$

On the other hand, we have

$$\|B_{n\kappa}^\circ - B_\kappa\| \le \|B_{n\kappa}^\circ - B_{n\kappa}\| + \|B_{n\kappa} - B_\kappa\| = O_p(n^{-a}\kappa^{1/2}) + o_p(n\kappa^{1/2-b}), \tag{A.13}$$

due, again, to Lemmas 3 and 4. Therefore, we have from (A.12) and (A.13)

$$\begin{aligned}
&\|T_\kappa A_{n\kappa}^{\circ -1} B_{n\kappa}^\circ - T_\kappa A_\kappa^{-1} B_\kappa\| \\
&\quad \le \|T_\kappa\|(\|A_{n\kappa}^{\circ -1} - A_\kappa\|\|B_{n\kappa}^\circ\| + \|A_\kappa^{-1}\|\|B_{n\kappa}^\circ - B_\kappa\|) \\
&\quad = O(\kappa^{1/2})(O_p(n^{-a}\kappa)O_p(\kappa^{1/2}) + O_p(1)(O_p(n^{-a}\kappa^{1/2}) + o_p(n\kappa^{1/2-b}))) \\
&\quad = O_p(n^{-a}\kappa^2) + o_p(n\kappa^{1-b}),
\end{aligned}$$

from which (A.10) follows immediately.

For part (b), we first show that

$$\begin{aligned}
(n^2 M_{n\kappa})^{-1/2} &= (T_\kappa A_{n\kappa}^{\circ -1} T_\kappa')^{-1/2} \\
&= (T_\kappa A_\kappa^{-1} T_\kappa')^{-1/2} + O_p(n^{-a}\kappa^{1/2}).
\end{aligned} \tag{A.14}$$

To prove (A.14), note that we have from (9)

$$\lambda_{\min}(T_\kappa A_{n\kappa}^{\circ -1} T_\kappa') \ge \frac{\lambda_{\min}(T_\kappa T_\kappa')}{\lambda_{\min}(A_{n\kappa}^\circ)} \ge \frac{c_1\kappa}{\lambda_{\max}(A_{n\kappa}^\circ)}$$

$$\lambda_{\min}(T_\kappa A_\kappa^{-1} T_\kappa') \ge \frac{\lambda_{\min}(T_\kappa T_\kappa')}{\lambda_{\min}(A_\kappa)} \ge \frac{c_1\kappa}{\lambda_{\max}(A_\kappa)}$$

for some constant $c_1 > 0$, and therefore,

$$(T_\kappa A_{n\kappa}^{\circ -1} T_\kappa')^{-1/2}, (T_\kappa A_\kappa^{-1} T_\kappa')^{-1/2} = O_p(\kappa^{-1/2}). \tag{A.15}$$

Also, we have

$$\|T_\kappa(A_\kappa^{-1} - A_{n\kappa}^{\circ -1})T_\kappa'\| \le \|T_\kappa\|^2\|A_{n\kappa}^{\circ -1} - A_\kappa^{-1}\| = O_p(n^{-a}\kappa^2). \tag{A.16}$$

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It can then be easily deduced from (A.15) and (A.16) that

$$
\begin{aligned}
&(T_\kappa A_{n\kappa}^{\circ -1} T_\kappa')^{-1/2} - (T_\kappa A_\kappa^{-1} T_\kappa')^{-1/2} \\
&\quad = (T_\kappa A_{n\kappa}^{\circ -1} T_\kappa')^{-1/2} (T_\kappa (A_\kappa^{-1} - A_{n\kappa}^{\circ -1}) T_\kappa') \\
&\qquad \times ((T_\kappa A_\kappa^{-1} T_\kappa')^{1/2} + (T_\kappa A_{n\kappa}^{\circ -1} T_\kappa')^{1/2})^{-1} (T_\kappa A_\kappa^{-1} T_\kappa')^{-1/2} \\
&\quad = O_p(n^{-a} \kappa^{1/2}),
\end{aligned}
$$

as was to be shown.

Now the result stated in part (b) follows directly from (A.10) and (A.14) upon noticing that

$$
T_\kappa A_\kappa^{-1} B_\kappa = O_p(\kappa) \quad \text{and} \quad (T_\kappa A_\kappa^{-1} T_\kappa')^{-1/2} = O_p(\kappa^{-1/2})
$$

and that $n^{-a} \kappa^{3/2}, n\kappa^{1/2-b} = o(1)$. This completes the proof. ■

**Proof of Proposition 6.** Part (a) is obvious. For the result in part (b), note that there exists $b$ such that $\kappa^{-b} = o(n^{-1})$ under Assumption 4 and that

$$
M_{n\kappa}^{-1/2} = O_p(n\kappa^{-1/2}),
$$

which follows directly from (A.14) and (A.15). ■

**Proof of Theorem 7.** The stated results are immediate from Propositions 5 and 6. The proofs are therefore omitted. ■

**Proof of Corollary 8.** Under the given condition $\omega_{12} = \delta_{12} = 0$,

$$
B_\kappa = \int_0^1 f_\kappa \otimes V \, dU,
$$

and $V$ is independent of $U$. We may therefore easily deduce, by first conditioning on a realization of $V$ as in Park and Phillips (1988), that

$$
(T_\kappa A_\kappa^{-1} T_\kappa)^{-1/2} T_\kappa A_\kappa^{-1} B_\kappa =_d \mathbf{N}(0, \omega^2 I_{md})
$$

for all $\kappa \geq 1$. The stated result now follows from part (b) of Theorem 7. ■

**Proof of Proposition 9.** Define pseudoestimators

$$
\tilde{\omega}_{n\kappa}^2 = \frac{1}{n} \sum_{|k| \leq \ell_n} \nu\left(\frac{k}{\ell_n}\right) \sum_{t=k+1}^{n} u_{\kappa t} u_{\kappa, t-k},
$$

$$
\tilde{\omega}_n^2 = \frac{1}{n} \sum_{|k| \leq \ell_n} \nu\left(\frac{k}{\ell_n}\right) \sum_{t=k+1}^{n} u_t u_{t-k},
$$

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which are closely related to $\hat{\omega}^2_{n\kappa}$ introduced in (21). To deduce the stated result, we will show that

$$\tilde{\omega}^2_n = \omega^2 + O_p(n^{-1/3}), \tag{A.17}$$

$$\tilde{\omega}^2_{n\kappa} = \tilde{\omega}^2_n + o_p(n^{-1/3}), \tag{A.18}$$

$$\hat{\omega}^2_{n\kappa} = \tilde{\omega}^2_{n\kappa} + o_p(n^{-1/3}) \tag{A.19}$$

under Assumptions 1–5.

To establish (A.17), note that

$$\sum_{k=-\infty}^{\infty} |k||\Gamma(k)| \le |\Sigma|\left(\sum_i \sum_j |i+j||\Phi_i||\Phi_{i+j}| + \sum_i \sum_j |i||\Phi_i||\Phi_{i+j}|\right)$$
$$\le 2|\Sigma|\left(\sum_{i=0}^{\infty} |\Phi_i|\right)\left(\sum_{j=0}^{\infty} j|\Phi_j|\right) < \infty$$

as a result of the summability condition in Assumption 1. The result in (A.17) now follows from Hannan (1970, Theorems 9 and 10, pp. 280–283) under the condition given in Assumption 5(a).

To show (A.18), we write

$$\sum_{t=k+1}^{n} u_{\kappa t} u_{\kappa, t-k} = \sum_{t=k+1}^{n} u_t u_{t-k} + R^1_{n\kappa}, \tag{A.20}$$

where

$$R^1_{n\kappa} = \sum_{t=k+1}^{n} (x_{t-k}u_t)'(\alpha - \alpha_\kappa)\left(\frac{t-k}{n}\right) + \sum_{t=k+1}^{n} (x_t u_{t-k})'(\alpha - \alpha_\kappa)\left(\frac{t}{n}\right)$$
$$+ \sum_{t=k+1}^{n} (\alpha - \alpha_\kappa)\left(\frac{t}{n}\right) x_t x'_{t-k} (\alpha - \alpha_\kappa)\left(\frac{t}{n}\right),$$

and define

$$\delta_\kappa\left(\frac{t}{n}\right) = (\alpha - \alpha_\kappa)\left(\frac{t}{n}\right) - (\alpha - \alpha_\kappa)\left(\frac{t-1}{n}\right)$$

for $\kappa \ge 1$.

We have

$$\sum_{|k|\le \ell_n} \nu\left(\frac{k}{l_n}\right)\left(\sum_{t=1}^{n} (x_t u_t)'(\alpha - \alpha_\kappa)\left(\frac{t}{n}\right) - \sum_{t=k+1}^{n} (x_{t-k}u_t)'(\alpha - \alpha_\kappa)\left(\frac{t-k}{n}\right)\right)$$
$$= \sum_{k=1}^{\ell_n}\left(\sum_{j=k}^{\ell_n} \nu\left(\frac{j}{\ell_n}\right)\right)\left(\sum_{t=k}^{n} (v_{t-k+1}u_t)'\delta_\kappa\left(\frac{t-k+1}{n}\right) - \sum_{t=1}^{n-k} (v_{t+k}u_t)'\delta_\kappa\left(\frac{t+k}{n}\right)\right)$$
$$= o_p(n\kappa^{-b}\ell_n^2)$$

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because, in particular, $\|\delta_\kappa\| = o(\kappa^{-b})$ and

$$\left|\frac{1}{n}\sum_{t=k+1}^{n} v_{t-k}u_t\right| \le \left(\frac{1}{n}\sum_{t=1}^{n}|v_t|^2\right)^{1/2}\left(\frac{1}{n}\sum_{t=1}^{n}u_t^2\right)^{1/2} = O_p(1)$$

uniformly in $k$. Similarly,

$$\sum_{|k|\le\ell_n}\nu\left(\frac{k}{\ell_n}\right)\left(\sum_{t=1}^{n}(x_t u_t)'(\alpha-\alpha_\kappa)\left(\frac{t}{n}\right) - \sum_{t=k+1}^{n}(x_t u_{t-k})'(\alpha-\alpha_\kappa)\left(\frac{t}{n}\right)\right)$$
$$= o_p(n\kappa^{-b}\ell_n^2).$$

Because

$$\left|\sum_{t=1}^{n}(x_t u_t)'(\alpha-\alpha_\kappa)\left(\frac{t}{n}\right)\right| = o_p(n\kappa^{-b})$$

we have

$$\frac{1}{n}\sum_{|k|\le\ell_n}\nu\left(\frac{k}{l_n}\right)\sum_{t=k+1}^{n}(x_{t-k}u_t)'(\alpha-\alpha_\kappa)\left(\frac{t-k}{n}\right) = o_p(\kappa^{-b}\ell_n^2), \tag{A.21}$$

$$\frac{1}{n}\sum_{|k|\le\ell_n}\nu\left(\frac{k}{\ell_n}\right)\sum_{t=k+1}^{n}(x_t u_{t-k})'(\alpha-\alpha_\kappa)\left(\frac{t}{n}\right) = o_p(\kappa^{-b}\ell_n^2) \tag{A.22}$$

uniformly in $k$.

Moreover,

$$\left|\sum_{t=k+1}^{n}(\alpha-\alpha_\kappa)\left(\frac{t}{n}\right)'x_t x_{t-k}'(\alpha-\alpha_\kappa)\left(\frac{t}{n}\right)\right| \le \|\alpha-\alpha_\kappa\|^2\sum_{t=k+1}^{n}|x_t x_{t-k}'|$$
$$\le \|\alpha-\alpha_\kappa\|^2\sum_{t=1}^{n}|x_t|^2$$
$$= o_p(n^2\kappa^{-2b}),$$

and therefore,

$$\frac{1}{n}\sum_{|k|\le\ell_n}\nu\left(\frac{k}{\ell_n}\right)\sum_{t=k+1}^{n}(\alpha-\alpha_\kappa)\left(\frac{t}{n}\right)'x_t x_{t-k}'(\alpha-\alpha_\kappa)\left(\frac{t}{n}\right) = o_p(n\kappa^{-2b}\ell_n) \tag{A.23}$$

for all $k$. Now (A.18) follows directly from (A.20)–(A.23), because $\ell_n = O(n^{1/3})$ and $\kappa^b = o(n^{-1})$.

To deduce (A.19), we write

$$\sum_{t=k+1}^{n}\hat{u}_{\kappa t}\hat{u}_{\kappa,t-k} = \sum_{t=k+1}^{n}u_{\kappa t}u_{\kappa,t-k} + R_{n\kappa}^2, \tag{A.24}$$

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where

$$R_{n\kappa}^2 = -\left(\sum_{t=1}^{n} u_{\kappa t} x_{\kappa t}'\right)\left(\sum_{t=1}^{n} x_{\kappa t} x_{\kappa t}'\right)^{-1}\left(\sum_{t=k+1}^{n} x_{\kappa, t-k} u_{\kappa t}\right)$$
$$- \left(\sum_{t=1}^{n} u_{\kappa t} x_{\kappa t}'\right)\left(\sum_{t=1}^{n} x_{\kappa t} x_{\kappa t}'\right)^{-1}\left(\sum_{t=k+1}^{n} x_{\kappa t} u_{\kappa, t-k}\right)$$
$$+ \left(\sum_{t=1}^{n} u_{\kappa t} x_{\kappa t}'\right)\left(\sum_{t=1}^{n} x_{\kappa t} x_{\kappa t}'\right)^{-1}\left(\sum_{t=k+1}^{n} x_{\kappa t} x_{\kappa, t-k}\right)\left(\sum_{t=1}^{n} x_{\kappa t} x_{\kappa t}'\right)^{-1}\left(\sum_{t=1}^{n} x_{\kappa t} u_{\kappa t}\right).$$

As we have seen earlier,

$$\left\|\sum_{t=1}^{n} x_{\kappa t} u_{\kappa t}\right\| = O_p(n\kappa^{1/2}) \quad \text{and} \quad \left\|\left(\sum_{t=1}^{n} x_{\kappa t} x_{\kappa t}'\right)^{-1}\right\| = O_p(n^{-2}). \tag{A.25}$$

Furthermore, we may easily deduce that

$$\left\|\frac{1}{n}\sum_{|k|\le \ell_n} \nu\left(\frac{k}{\ell_n}\right)\sum_{t=1}^{n} x_{\kappa t} u_{\kappa t}\right\| = O_p(\kappa^{1/2}\ell_n). \tag{A.26}$$

Now define

$$\delta_i\left(\frac{t}{n}\right) = \varphi_i\left(\frac{t}{n}\right) - \varphi_i\left(\frac{t-1}{n}\right).$$

We have

$$\sup_{1\le i\le \kappa}\sup_{1\le t\le n}\left|\delta_i\left(\frac{t}{n}\right)\right| \le \frac{\kappa}{n}$$

for all $\kappa$ and $n$. It follows that

$$\sum_{|k|\le \ell_n} \nu\left(\frac{k}{\ell_n}\right)\left(\sum_{t=1}^{n} \varphi_i\left(\frac{t}{n}\right) x_t u_t - \sum_{t=k+1}^{n} \varphi_i\left(\frac{t-k}{n}\right) x_{t-k} u_t\right)$$
$$= \sum_{k=1}^{\ell_n}\left(\sum_{j=k}^{\ell_n} \nu\left(\frac{j}{\ell_n}\right)\right)\left(\sum_{t=k}^{n} (v_{t-k+1} u_t)'\delta_i\left(\frac{t-k+1}{n}\right) - \sum_{t=1}^{n-k} (v_{t+k} u_t)'\delta_i\left(\frac{t+k}{n}\right)\right)$$
$$= O_p(\kappa \ell_n^2)$$

uniformly in $i \ge 1$. Similarly, we may show that

$$\sum_{|k|\le \ell_n} \nu\left(\frac{k}{\ell_n}\right)\left(\sum_{t=1}^{n} \varphi_i\left(\frac{t}{n}\right) x_t u_t - \sum_{t=k+1}^{n} \varphi_i\left(\frac{t-k}{n}\right) x_{t-k} u_t\right) = O_p(\kappa \ell_n^2)$$

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for all $i \geq 1$. Therefore,

$$\left\| \frac{1}{n} \sum_{|k| \leq \ell_n} \nu\left(\frac{k}{\ell_n}\right) \sum_{t=1}^{n} x_{\kappa t} u_{\kappa t} - \frac{1}{n} \sum_{|k| \leq \ell_n} \nu\left(\frac{k}{\ell_n}\right) \sum_{t=k+1}^{n} x_{\kappa, t-k} u_{\kappa t} \right\| = O_p\left(\frac{\kappa^{3/2} \ell_n^2}{n}\right), \quad \textbf{(A.27)}$$

$$\left\| \frac{1}{n} \sum_{|k| \leq \ell_n} \nu\left(\frac{k}{\ell_n}\right) \sum_{t=1}^{n} x_{\kappa t} u_{\kappa t} - \frac{1}{n} \sum_{|k| \leq \ell_n} \nu\left(\frac{k}{\ell_n}\right) \sum_{t=k+1}^{n} x_{\kappa t} u_{\kappa, t-k} \right\| = O_p\left(\frac{\kappa^{3/2} \ell_n^2}{n}\right). \quad \textbf{(A.28)}$$

Note that $n^{-1}\kappa^{3/2}\ell_n^2 = o(\kappa^{1/2}\ell_n)$.

We now have from (A.25)–(A.28)

$$\left(\sum_{t=1}^{n} u_{\kappa t} x'_{\kappa t}\right)\left(\sum_{t=1}^{n} x_{\kappa t} x'_{\kappa t}\right)^{-1}\left(\frac{1}{n} \sum_{|k| \leq \ell_n} \nu\left(\frac{k}{\ell_n}\right) \sum_{t=k+1}^{n} x_{\kappa, t-k} u_{\kappa t}\right) = O_p\left(\frac{\kappa \ell_n}{n}\right), \quad \textbf{(A.29)}$$

$$\left(\sum_{t=1}^{n} u_{\kappa t} x'_{\kappa t}\right)\left(\sum_{t=1}^{n} x_{\kappa t} x'_{\kappa t}\right)^{-1}\left(\frac{1}{n} \sum_{|k| \leq \ell_n} \nu\left(\frac{k}{\ell_n}\right) \sum_{t=k+1}^{n} x_{\kappa t} u_{\kappa, t-k}\right) = O_p\left(\frac{\kappa \ell_n}{n}\right). \quad \textbf{(A.30)}$$

Moreover,

$$\left| \sum_{t=k+1}^{n} \varphi_i\left(\frac{t}{n}\right) \varphi_j\left(\frac{t-k}{n}\right) x_t x'_{t-k} \right| \leq \sum_{t=1}^{n} |x_t|^2 = O_p(n^2)$$

for all $i, j \geq 1$, and therefore,

$$\left(\sum_{t=1}^{n} u_{\kappa t} x'_{\kappa t}\right)\left(\sum_{t=1}^{n} x_{\kappa t} x'_{\kappa t}\right)^{-1}\left(\frac{1}{n} \sum_{|k| \leq \ell'_n} \nu\left(\frac{k}{\ell_n}\right) \sum_{t=k+1}^{n} x_{\kappa t} x_{\kappa, t-k}\right) \times \left(\sum_{t=1}^{n} x_{\kappa t} x'_{\kappa t}\right)^{-1}\left(\sum_{t=1}^{n} x_{\kappa t} u_{\kappa t}\right) = O_p\left(\frac{\kappa \ell_n}{n}\right). \quad \textbf{(A.31)}$$

We may now easily deduce (A.19) from (A.24) and (A.29)–(A.31), upon noticing that $n^{-1}\kappa\ell_n = o(n^{-1/3})$. The proof is therefore complete. ■

**Proof of Theorem 10.** Denote by $\hat{\Sigma}$, $\hat{\Delta}_2$, $\hat{\Omega}_{22}$, and $\hat{\omega}_{12}$ the estimates of the parameters used in the CCR transformations. For notational brevity, we suppress the subscripts $n$ and $\kappa$ on which they depend. It is obvious from the proof of Proposition 9 that

$$\hat{\Delta}_2 = \Delta_2 + O_p(n^{-1/3}), \qquad \hat{\Omega}_{22} = \Omega_{22} + O_p(n^{-1/3}), \qquad \hat{\omega}_{12} = \omega_{12} + O_p(n^{-1/3}). \quad \textbf{(A.32)}$$

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Define $\hat{x}^*_{\kappa t}$ and $\hat{y}^*_t$ similarly as $x^*_{\kappa t}$ and $y^*_t$ in (27) and (28) with unknown parameters replaced by their estimates introduced earlier. We may write

$$\hat{y}^*_t = \hat{x}^{*\prime}_{\kappa t}\beta_\kappa + \hat{u}^*_{\kappa t}, \tag{A.33}$$

where

$$\hat{u}^*_{\kappa t} = u^*_{\kappa t} - (\hat{\omega}_{12}\hat{\Omega}_{22}^{-1} - \omega_{12}\Omega_{22}^{-1})v_t - \left(f_\kappa\left(\frac{t}{n}\right) \otimes \hat{\Delta}_2\hat{\Sigma}^{-1}\hat{w}_{\kappa t}\right)'(\hat{\beta}_{n\kappa} - \beta_\kappa) \tag{A.34}$$

in notation defined earlier.

Write

$$\begin{aligned}\hat{u}_{\kappa t} &= u_{\kappa t} - \left(\sum_{t=1}^n u_{\kappa t}x'_{\kappa t}\right)\left(\sum_{t=1}^n x_{\kappa t}x'_{\kappa t}\right)^{-1}x_{\kappa t}\\ &= u_t + (\alpha - \alpha_\kappa)\left(\frac{t}{n}\right)'x_t - \left(\sum_{t=1}^n u_{\kappa t}x'_{\kappa t}\right)\left(\sum_{t=1}^n x_{\kappa t}x'_{\kappa t}\right)^{-1}x_{\kappa t}.\end{aligned}$$

Then we have

$$\frac{1}{n}\sum_{t=1}^n \varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)\hat{w}_{\kappa t}\hat{w}'_{\kappa t} = \frac{1}{n}\sum_{t=1}^n \varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)w_t w'_t + O_p(n^{-1}\kappa) \tag{A.35}$$

uniformly in $i, j \geq 1$. It follows immediately from (A.35) that

$$\hat{\Sigma} = \Sigma + O_p(n^{-1/2}) \tag{A.36}$$

because $\sum_t w_t w'_t/n = \Sigma + O_p(n^{-1/2})$ and $n\kappa^{-1} = o(n^{-1/2})$.

Moreover, if we let

$$\begin{aligned}\hat{x}^*_t &= x_t - \hat{\Delta}_2\hat{\Sigma}^{-1}\hat{w}_{\kappa t}\\ &= x^*_t - \hat{\Delta}_2\hat{\Sigma}^{-1}(\hat{w}_{\kappa t} - w_t) - (\hat{\Delta}_2\hat{\Sigma}^{-1} - \Delta_2\Sigma^{-1})w_t,\end{aligned}$$

it can be easily deduced that

$$\frac{1}{n}\sum_{t=1}^n \varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)x_t\hat{w}'_{\kappa t} = O_p(\kappa), \tag{A.37}$$

$$\frac{1}{n}\sum_{t=1}^n \varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)\hat{x}^*_t\hat{w}'_{\kappa t} = O_p(\kappa) \tag{A.38}$$

uniformly in $i, j \geq 1$. Also, we have

$$\frac{1}{n}\sum_{t=1}^n \varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)\hat{x}^*_t w'_{\kappa t} = \frac{1}{n}\sum_{t=1}^n \varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)x^*_t w'_{\kappa t} + O_p(n^{-1/3}). \tag{A.39}$$

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Furthermore,

$$\frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)\hat{x}_t^*w'_{\kappa t} = \frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)x_t^*w'_{\kappa t} + O_p(n^{-1}\kappa)$$
$$= \frac{1}{n}\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)x_t^*w'_t + o_p(n\kappa^{-b}) + O_p(n^{-1}\kappa) \tag{A.40}$$

uniformly in $i, j \geq 1$.

We now have

$$\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)\hat{x}_t^*\hat{x}_t^{*\prime} = \sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)x_t x_t' + R_{n\kappa},$$

where

$$R_{n\kappa} = -\left(\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)x_t\hat{w}'_{\kappa t}\right)\hat{\Sigma}^{-1}\hat{\Delta}_2'$$
$$- \hat{\Delta}_2\hat{\Sigma}^{-1}\left(\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)\hat{w}_{\kappa t}x_t'\right)$$
$$+ \hat{\Delta}_2\hat{\Sigma}^{-1}\left(\sum_{t=1}^{n}\varphi_i\left(\frac{t}{n}\right)\varphi_j\left(\frac{t}{n}\right)\hat{w}_{\kappa t}\hat{w}'_{\kappa t}\right)\hat{\Sigma}^{-1}\hat{\Delta}_2'$$
$$= O_p(n)$$

uniformly in $i, j \geq 1$, due to (A.35) and (A.37). It follows that

$$\left\|\frac{1}{n^2}\sum_{t=1}^{n}\hat{x}_{\kappa t}^*\hat{x}_{\kappa t}^{*\prime} - \frac{1}{n^2}\sum_{t=1}^{n}x_{\kappa t}x'_{\kappa t}\right\| = O_p(n^{-1}\kappa), \tag{A.41}$$

which is of order smaller than $O_p(n^{-a}\kappa)$ in (A.11).

Moreover, we have from (A.32) and (A.38) that

$$\left\|\sum_{t=1}^{n}\hat{x}_{\kappa t}^*v_1'\right\|\|\hat{\Omega}_{22}^{-1}\hat{\omega}_{21} - \Omega_{22}^{-1}\omega_{21}\| = O_p(n\kappa^{1/2})O_p(n^{-1/3}), \tag{A.42}$$

and, because

$$\sum_{t=1}^{n}\hat{x}_{\kappa t}\hat{w}'_{\kappa t} = \sum_{t=1}^{n}\left(f_\kappa\left(\frac{t}{n}\right)f_\kappa\left(\frac{t}{n}\right)'\otimes\hat{x}_t^*\hat{w}'_{\kappa t}\right)(I_\kappa\otimes\hat{\Sigma}^{-1}\hat{\Delta}_2')(\hat{\beta}_{n\kappa} - \beta_\kappa),$$

it follows from (A.40) and our earlier results that

$$\left\|\sum_{t=1}^{n}\hat{x}_{\kappa t}\hat{w}'_{\kappa t}\right\| = O_p(n\kappa^2)O_p(n^{-1}\kappa^{1/2}) = O_p(\kappa^{5/2}). \tag{A.43}$$

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It can also be deduced from (A.39) that

$$\left\| \sum_{t=1}^{n} \hat{x}_{\kappa t}^{*} u_{\kappa t}^{*} - \sum_{t=1}^{n} x_{\kappa t}^{*} u_{\kappa t}^{*} \right\| = O_p(n^{2/3}) \tag{A.44}$$

and that

$$\left\| \sum_{t=1}^{n} x_{\kappa t}^{*} u_{\kappa t}^{*} - \sum_{t=1}^{n} x_{\kappa t}^{*} u_{t} \right\| = o_p(n^2 \kappa^{1/2-b}). \tag{A.45}$$

Therefore, we have from (A.42)–(A.45) that

$$\left\| \frac{1}{n} \sum_{t=1}^{n} \hat{x}_{\kappa t}^{*} \hat{u}_{\kappa t}^{*} - \frac{1}{n} \sum_{t=1}^{n} x_{\kappa t}^{*} u_{t}^{*} \right\| = O_p(n^{-1/3} \kappa^{1/2}) + O_p(n^{-1} \kappa^{5/2}) + o_p(n \kappa^{1/2-b}), \tag{A.46}$$

the order of which is smaller than the one in (A.13).

Finally, we observe that

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^{n} x_{\kappa t}^{*} u_{t}^{*} &= \frac{1}{n} \sum_{t=1}^{n} x_{\kappa t} u_{t}^{*} - \Delta_2 \Sigma^{-1} \frac{1}{n} \sum_{t=1}^{n} w_t u_{t}^{*} \\ &= \int_0^1 f_{n\kappa} \otimes V_n \, dU_n^{*} \\ &\quad + \int_0^1 f_{n\kappa} \otimes \left( \Delta_2 - \Delta_2 \Sigma^{-1} \left( \frac{1}{n} \sum_{t=1}^{n} w_t w_t' \right) \right) \begin{pmatrix} 1 \\ -\Omega_{22}^{-1} \omega_{21} \end{pmatrix} + O_p(n^{-a}) \\ &= \int_0^1 f_{n\kappa} \otimes V_n \, dU_n^{*} + O_p(n^{-a}) \end{aligned} \tag{A.47}$$

because $\sum_t w_t w_t'/n = \Sigma + O_p(n^{-1/2})$. The stated result now follows from (A.41), (A.46), and (A.47), along with our earlier results. ■

**Proof of Theorem 11.** In what follows, we assume w.l.o.g. that the elements in $(s_t)$ are appropriately normalized, namely,

$$s_t = \left( 1, \frac{t}{n}, \ldots, \left( \frac{t}{n} \right)^{s} \right)'.$$

Our subsequent results are invariant with respect to such normalization. Define

$$p_t = \alpha_t' x_t - \left( \sum_{t=1}^{n} \alpha_t' x_t x_t' \right) \left( \sum_{t=1}^{n} x_t x_t' \right)^{-1} x_t + u_t - \left( \sum_{t=1}^{n} u_t x_t' \right) \left( \sum_{t=1}^{n} x_t x_t' \right)^{-1} x_t \tag{A.48}$$

and

$$q_t = s_t - \left( \sum_{t=1}^{n} s_t x_t' \right) \left( \sum_{t=1}^{n} x_t x_t' \right)^{-1} x_t.$$

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Then we may write

$$\frac{1}{n^2}\tau_1 = \hat{\omega}_{n\kappa}^{-2}\left(\frac{1}{n^{3/2}}\sum_{t=1}^{n} p_t q_t'\right)\left(\frac{1}{n}\sum_{t=1}^{n} q_t q_t'\right)^{-1}\left(\frac{1}{n^{3/2}}\sum_{t=1}^{n} q_t p_t\right) \tag{A.49}$$

as is well known.

Let

$$\alpha_n(r) = \alpha\left(\frac{[nr]}{n}\right) \quad \text{and} \quad \sigma_n(r) = \sigma\left(\frac{[nr]}{n}\right).$$

Then it follows immediately from Lemma 1 that

$$\begin{aligned}\frac{1}{n^{3/2}}\sum_{t=1}^{n} p_t s_t' &= \frac{1}{n^{3/2}}\sum_{t=1}^{n} \alpha_t' x_t s_t' \\ &\quad - \frac{1}{n^{3/2}}\left(\sum_{t=1}^{n} \alpha_t x_t x_t'\right)\left(\sum_{t=1}^{n} x_t x_t'\right)^{-1}\left(\sum_{t=1}^{n} x_t s_t'\right) + O_p(n^{-1}) \\ &\to_d \int_0^1 \alpha' V\sigma' - \left(\int_0^1 \alpha' VV'\right)\left(\int_0^1 VV'\right)^{-1}\left(\int_0^1 V\sigma'\right)\end{aligned}$$

and that

$$\frac{1}{n^{3/2}}\sum_{t=1}^{n} s_t x_t' \to_d \int_0^1 \sigma V',$$

which together yield

$$\frac{1}{n}\sum_{t=1}^{n} q_t q_t' \to_d \int_0^1 QQ',$$

$$\frac{1}{n^{3/2}}\sum_{t=1}^{n} q_t p_t \to_d \int_0^1 QP.$$

The stated result for $\tau_1$ now follows from (A.49).

For $\tau_2$, notice that $\hat{u}_t = p_t$ as defined in (A.48) and

$$\frac{1}{n^{3/2}}\sum_{t=1}^{[nr]} p_t \to_d R(r),$$

from which the stated result can be obtained in a straightforward manner. ■

**Proof of Theorem 12.** We let

$$s_{\kappa t} = s_t - \left(\sum_{t=1}^{n} s_t x_{\kappa t}'\right)\left(\sum_{t=1}^{n} x_{\kappa t} x_{\kappa t}'\right)^{-1} x_{\kappa t}.$$

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To prove part (a), suppose that the true model is given by (1). We may then write

$$\tau = \hat{\omega}_{n\kappa}^{-2}\left(\sum_{t=1}^{n} u_{\kappa t} s_{\kappa t}'\right)\left(\sum_{t=1}^{n} s_{\kappa t} s_{\kappa t}'\right)^{-1}\left(\sum_{t=1}^{n} s_{\kappa t} u_{\kappa t}\right). \tag{A.50}$$

Define

$$G_n = \int_0^1 \sigma_n \sigma_n', \qquad H_n = \int_0^1 \sigma_n \, dU_n, \qquad D_{n\kappa} = \int_0^1 f_{n\kappa} \otimes V_n \sigma_n'$$

and let $A_{n\kappa}$ and $B_{n\kappa}$ be given as earlier. It can be shown as in the proof of Lemma 3 that

$$\frac{1}{n}\int_0^1 s_{\kappa t} s_{\kappa t}' = Q_{n\kappa} + O_p(n^{-a}), \tag{A.51}$$

$$\frac{1}{n}\sum_{t=1}^{n} s_{\kappa t} u_{\kappa t} = P_{n\kappa} + O_p(n^{-a}), \tag{A.52}$$

where

$$Q_{n\kappa} = G_n - D_{n\kappa}' A_{n\kappa}^{-1} D_{n\kappa} \quad \text{and} \quad P_{n\kappa} = H_n - D_{n\kappa}' A_{n\kappa}^{-1} B_{n\kappa}$$

in notations defined previously.

Let

$$G = \int_0^1 \sigma\sigma', \qquad H = \int_0^1 \sigma \, dU, \qquad D_\kappa = \int_0^1 f_\kappa \otimes V\sigma'$$

and $A_\kappa$ and $B_\kappa$ be defined as earlier. We may easily deduce that

$$G_n = G + O(n^{-1}), \qquad H_n = H + O_p(n^{-a}) \tag{A.53}$$

and that

$$\|D_{n\kappa} - D_\kappa\| = O_p(n^{-a}\kappa^{1/2}). \tag{A.54}$$

Moreover, it follows that

$$\|D_\kappa\| = O_p(1) \tag{A.55}$$

uniformly in $\kappa \geq 1$. To see this, note that

$$\left\| \int_0^1 f_\kappa \otimes V\sigma' \right\|^2 \leq \sum_{i=1}^{\kappa}\sum_{p=1}^{m}\sum_{q=1}^{s}\left(\int_0^1 \varphi_i V_p \sigma_q\right)^2$$

and that

$$\sum_{i=1}^{\kappa}\left(\int_0^1 \varphi_i V_p \sigma_q\right)^2 \leq \frac{1}{2}\int_0^1 V_p^2 \sigma_q^2,$$

which holds by the Bessel's inequality.

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Furthermore, if we define

$$Q_\kappa = G - D_\kappa' A_\kappa^{-1} D_\kappa \quad \text{and} \quad P_\kappa = H - D_\kappa' A_\kappa^{-1} B_\kappa,$$

then we have from (A.53)–(A.55)

$$\|Q_{n\kappa} - Q_\kappa\| = O_p(n^{-a}\kappa^{1/2}) \quad \text{and} \quad \|P_{n\kappa} - P_\kappa\| = O_p(n^{-a}\kappa). \tag{A.56}$$

It also follows that

$$\|Q_{n\kappa}^{-1} - Q_\kappa^{-1}\| = O_p(1) \tag{A.57}$$

because $\|Q_{n\kappa}^{-1}\|, \|Q_\kappa^{-1}\| = O_p(1)$.

We now have from (A.50)–(A.52), (A.56), and (A.57) that

$$\begin{aligned}\tau &= \hat{\omega}_{n\kappa}^{-2} P_{n\kappa}' Q_{n\kappa}^{-1} P_{n\kappa} + O_p(n^{-a}) \\ &= \omega^{-2} P_\kappa' Q_\kappa^{-1} P_\kappa + O_p(n^{-a}\kappa),\end{aligned}$$

and the stated result follows immediately upon noticing that

$$\omega^{-2} P_\kappa' Q_\kappa^{-1} P_\kappa =_d \chi_s^2$$

for all $\kappa \geq 1$ and that $n^{-a}\kappa = o(1)$.

For part (b), we assume that the true model is given by (37). Let

$$e_{\kappa t} = e_t + (\alpha - \alpha_\kappa)\left(\frac{t}{n}\right)' x_t$$

and write

$$\tau = \hat{\omega}_{n\kappa}^{-2}\left(\sum_{t=1}^{n} e_{\kappa t} s_{\kappa t}'\right)\left(\sum_{t=1}^{n} s_{\kappa t} s_{\kappa t}'\right)^{-1}\left(\sum_{t=1}^{n} s_{\kappa t} e_{\kappa t}\right). \tag{A.58}$$

It can be shown that

$$\begin{aligned} n^{-1}\ell_n^{-1}\hat{\omega}_{n\kappa}^2 &= \frac{1}{n^2\ell_n^{-1}} \sum_{|k|\leq \ell_n} \nu\left(\frac{k}{\ell_n}\right) e_{\kappa t} e_{\kappa, t-k} + o_p(1) \\ &= \left(\frac{1}{\ell_n} \sum_{|k|\leq \ell_n} \nu\left(\frac{k}{\ell_n}\right)\right)\left(\frac{1}{n^2}\sum_{t=1}^{n} e_t^2\right) + o_p(1) \\ &= \left(\int_0^1 \nu\right)\left(\int_0^1 U^2\right) + o_p(1),\end{aligned}$$

and therefore, for any $\delta > 0$,

$$n^{1+\delta}\ell_n \hat{\omega}_{n\kappa}^{-2} \to_p \infty \tag{A.59}$$

as $n \to \infty$.

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Define

$$K_n = \int_0^1 \sigma_n U_n \quad \text{and} \quad K = \int_0^1 \sigma U$$

and note that $K_n = K + O_p(n^{-a})$. Moreover, if we let

$$L_{n\kappa} = \int_0^1 f_{n\kappa} \otimes V_n U_n \quad \text{and} \quad L_\kappa = \int_0^1 f_\kappa \otimes VU,$$

then we may show as previously that

$$\|L_{n\kappa} - L_\kappa\| = O_p(n^{-a}\kappa^{1/2}) \quad \text{and} \quad \|L_\kappa\| = O_p(1)$$

similarly as the results for $\|D_{n\kappa} - D_\kappa\|$ and $\|D_\kappa\|$ in (A.54) and (A.55).

We have

$$\begin{aligned} \frac{1}{n^{3/2}} \sum_{t=1}^{n} s_{\kappa t} e_{\kappa t} &= R_{n\kappa} + O_p(n^{-a}) \\ &= R_\kappa + O_p(n^{-a}\kappa^{1/2}), \end{aligned} \tag{A.60}$$

where $R_{n\kappa} = K_n - D'_{n\kappa} A_{n\kappa}^{-1} L_{n\kappa}$ and $R_\kappa = K - D'_\kappa A_\kappa^{-1} L_\kappa$, because

$$\|R_{n\kappa} - R_\kappa\| = O_p(n^{-a}\kappa^{1/2}).$$

The stated result now follows easily from (A.51), (A.57), and (A.58)–(A.60). ■

**Proof of Corollary 13.** The proofs for the stated results are tedious but straightforward given the results in the proofs of Proposition 9 and Theorems 10–12. The detailed proofs are therefore omitted. ■

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