101.13 The Lester circle once again

After several *Gazette* Notes on the triangle, the Cartesian framework is now an established alternative to areal coordinates.

On this occasion, we will show that the interior point of intersection of two significant chords of the Lester circle lies on the Kiepert hyperbola.

The Cartesian framework

The asymptotes of the circumscribing Kiepert rectangular hyperbola provide our x, y-axes such that the triangle vertices $A\left(1, \frac{1}{a}\right)$, $B\left(1, \frac{1}{b}\right)$, $C\left(1, \frac{1}{c}\right)$, centroid $G\left(g, \frac{1}{g}\right)$ and orthocentre $H\left(h, \frac{1}{h}\right)$ lie on the curve $xy = 1$. Hence $3g = a + b + c$, $\frac{3}{g} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ with $abch = -1$ for rectangularity. The origin $K\,(0,\ 0)$ is the Kiepert centre and g, h and two of the parameters a, b, c may be taken as positive.

Routine coordinate geometry then leads to the following designated points:

Point	x-coordinate	y-coordinate
Kiepert centre K	0	0
Centroid G	g	$\frac{1}{g}$
Orthocentre H	h	$\frac{1}{h}$
Circumcentre O	$\frac{1}{2}(3g - h)$	$\frac{1}{2}\left(\frac{3}{g} - \frac{1}{h}\right)$
Nine-point centre N	$\frac{1}{4}(3g + h)$	$\frac{1}{4}\left(\frac{3}{g} + \frac{1}{h}\right)$
Brocard midpoint W	$\frac{4}{\frac{3}{g} + \frac{1}{h}}$	$\frac{4}{3g + h}$
Euler-Fermat meet M	$\frac{1}{2}(g + h)$	$\frac{1}{2}\left(\frac{1}{g} + \frac{1}{h}\right)$
Lemoine (symmedian) point L	$\frac{2}{\frac{1}{g} + \frac{1}{h}}$	$\frac{2}{g + h}$
First Fermat point P	$\sqrt{gh}$	$\frac{1}{\sqrt{gh}}$
Second Fermat point Q	$-\sqrt{gh}$	$\frac{-1}{\sqrt{gh}}$

Note in passing the collinearity KLM and KWN.

The Lester circle

The *power lemma* may be used to prove that the four points O, N, P, Q are concyclic. Thus we have either $MN.MO = MP.MQ$ or $VO.VP = VN.VQ$, where the interior point V (negative power ([1])) is the OP, NQ meet.

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Now the respective gradients of OP and NQ are given by $-s$ and s, where

$$s = \frac{\dfrac{3}{\sqrt{g}} + \dfrac{1}{\sqrt{h}}}{\left(3\sqrt{g} + \sqrt{h}\right)\sqrt{gh}},$$

whence parallels to the Kiepert asymptotes bisect the angles between the chords OP and NQ. Moreover, V may be shown to lie on the hyperbola with parameter

$$v = \frac{\left(3\sqrt{g} + \sqrt{h}\right)}{\dfrac{3}{\sqrt{g}} + \dfrac{1}{\sqrt{h}}},$$

which result is somewhat less accessible in areal coordinates.

Concluding remarks

Note also that the Brocard midpoint W is the meet of chords OL and NK for the associated circle $ONLK$ ([2]) which has similar angle bisection.

Synthetic proof of Lester's theorem is available in reference [1].

References

1. G. Leversha, *The geometry of the triangle*, UKMT (2013) pp. 37-39, pp. 151-155.
2. J. A. Scott, A nine-point hyperbola, *Math. Gaz.* **89** (March 2005) pp. 93-96.

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101.14 The direction of limiting friction

When a particle is in contact with a surface, the reaction force of the surface on the particle has two components, the *normal reaction* **N** and the *frictional force* **F**. The normal reaction is normal to the surface at the point of contact, whereas the frictional force acts along the tangent plane. There is a constant μ associated with a given particle and surface, called the coefficient of friction. The laws of friction are as follows.

1. *Static friction*

   When there is no relative motion of particle on surface, the friction is just sufficient to prevent sliding, and can assume any magnitude and direction subject to the inequality $F \leqslant \mu N$.

   When the particle is on the point of slipping then $F = \mu N$, and we say that the particle is in limiting equilibrium.

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