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Uniqueness and continuous dependence in the linear elastodynamic exterior and half-space problems

By G. P. GALDI, R. J. KNOPS and S. RIONERO
Istituto di Matematica, Università di Napoli, Napoli, Italy and Department of Mathematics, Heriot-Watt University, Edinburgh

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1. *Introduction*

A method based upon the Lagrange identity has been used by Brun [**2**] in the linear theories of thermoelasticity and viscoelasticity to establish uniqueness of the solution to the initial boundary value problem on bounded three-dimensional regions. A major feature of Brun's analysis is that it does not require any sign-definiteness assumptions on, for instance, the elasticities. The technique was extended by Knops and Payne [**14**] to derive certain continuous dependence results in linear elastodynamics, again for a bounded region. These authors had earlier recovered Brun's uniqueness result for linear elasticity [**11**] and derived other continuous dependence results based upon logarithmic convexity arguments [**12, 14**] (see also [**13**] for a similar treatment of thermoelasticity). Levine [**18**] later treated an abstract version of the Brun approach and applied it to a family of abstract linear operator equations. Among his results is a simplified proof that equipartition of the kinetic and potential energies occurs. Other applications of the Lagrange identity in proofs of uniqueness for bounded regions include those by Naghdi and Trapp [**19**] for a Cosserat surface, and by Green [**9**] for a theory of linear thermoelasticity that allows second sound.

In this paper, we extend the Lagrange identity method to obtain results for uniqueness and continuous dependence on the initial data in the initial boundary value problem of linear elasticity on unbounded three-dimensional regions. Specifically, we treat exterior regions and the half-space although our results may be easily extended to regions that are contained in the half-space. Many authors have previously considered exterior regions under a variety of conditions on the asymptotic behaviour of the solution, and under the assumption that the elasticities are strongly elliptic or positive-definite. (See, e.g. Carbonaro and Russo [**3**], Galdi and Rionero [**6**], Jones [**9, 10**], Knops and Straughan [**16**], Kupradze [**17**], and Wheeler [**20**].) However, in extending the technique of logarithmic convexity, Galdi and Rionero [**7**] were the first to prove that uniqueness and continuous dependence results need not depend upon sign-definite elasticities, thus properly generalizing to exterior regions the earlier conclusions due to Brun and Knops and Payne. Of course, Galdi and Rionero impose certain asymptotic behaviour, but this is less restrictive than that required for straightforward adaptation of the methods for bounded domains. In any case, some such restriction is to be expected from the nature of the problem, since it is well-known that there is non-uniqueness in the presence of harmonic disturbances (cf. Kupradze *et al.* [**17**], chapter VIII).

In [**7**], Galdi and Rionero combined their method of weighted energy (see also [**4, 5**]) with that of logarithmic convexity to derive their results. Here, we also use weighted

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energy but in conjunction with the Lagrange identity. One advantage of our approach is that no appeal need be made to conservation of energy and hence it may be readily adapted to deal with a weak solution whose definition can be formulated without this property (cf. Levine[18], Brockway[1]). Furthermore, our approach delivers somewhat different continuous dependence estimates which perhaps are sharper than previous authors', although it must be noted that Galdi and Rionero obtain continuous dependence (in the sense of Hölder) on data other than the initial set of concern here. Like Galdi and Rionero, we do not require the elasticities to be sign-definite, and in the case of exterior regions, we impose essentially the same asymptotic behaviour. (For a discussion of the corresponding nonlinear elastic problem, see Knops and Payne[15].)

We are, however, also able to treat the half-space and similar problems by the present method and, moreover, under similar mild restrictions. As far as the present authors are aware, there are few, if any, previous results of this kind available in the literature despite the obvious practical significance of the problem.

The main argument of the paper is described in Section 3 where the exterior problem is examined in detail. Modifications essential for the half-space or for regions contained in the half-space are discussed in Section 4. In Section 2, we set down the basic equations and collect together the basic hypotheses required for the exterior problem. As already noted, it is possible to include in our treatment suitably defined weak solutions allowing, in particular, an extension of our conclusions to composite bodies, but since this generalization is somewhat standard, we prefer instead to emphasize the technique itself by admitting only classical solutions. These, of course, are always assumed to exist.

2. *Notation, basic equations and hypotheses*

Let Ω be a three-dimensional unbounded region with smooth boundary $\partial\Omega$. We suppose either that $\partial\Omega$ is closed and compact, in which case Ω forms an exterior region, or that $\partial\Omega$ is non-compact, in which case for ease of presentation we further suppose that $\partial\Omega$ is planar and then Ω is the half-space. In what follows, Ω could be taken to lie in n-dimensional space ($n \geqslant 2$). However, we work specifically with $n = 3$.

The region Ω is occupied by a linear non-homogeneous anisotropic body which is set in motion by initial data. The body-force is held zero throughout, and the surface $\partial\Omega$ is subject to zero displacement. Other types of boundary conditions can be included (for example, those in which the traction on $\partial\Omega$ is specified to be zero), but again because the treatment is fairly routine we omit explicit reference to them. The asymptotic behaviour at large spatial distances is also prescribed and the precise conditions to be satisfied are given in the general formulation of the problem which now follows.

The equations of motion governing the cartesian components of displacement $u_i(x,t)$, are

$$(c_{ijkl}\,u_{k,l})_{,j} = \rho(x)\,u_{i,tt} \quad (x,t) \in \Omega \times (0, 2T], \tag{2·1}$$

in which t denotes the time variable, $x = (x_1, x_2, x_3)$ denotes the position vector, $\rho(x)$ is the non-homogeneous density, assumed positive, and $c_{ijkl}(x)$ are the non-homogeneous elasticities assumed to be continuously differentiable and to possess the symmetry

$$c_{ijkl} = c_{klij}. \tag{2·2}$$

By $(0, T]$ is meant the half-open time interval of length T. For uniqueness, T may be taken as infinite but for the continuous dependence results, T is to be supposed finite.

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Throughout, the usual summation and comma notation is adopted, with latin suffixes ranging over 1, 2, 3.

In addition to (2·1), the displacement also satisfies the prescribed initial conditions

$$u_i(x,0) = f_i(x), \quad u_{i,t}(x,0) = h_i(x), \quad x \in \Omega, \tag{2·3}$$

in which the functions f and h are further assumed to satisfy the boundedness condition

$$\int_\Omega (f_i f_i + h_i h_i)\, dx \leqslant M^2 \tag{2·4}$$

for some positive constant M. The prescribed boundary conditions are given by

$$u_i(x,t) = 0, \quad x \in \partial\Omega \times (0,T]. \tag{2·5}$$

As already noted, other standard boundary conditions can be easily included in the discussion but are omitted for simplicity.

The formulation of the initial boundary value problem is completed by specifying the asymptotic behaviour of the solution at large spatial distances. We set down these conditions only for the *exterior problem*, and postpone until Section 4 the corresponding condition for the half-space. Thus, we require

$$|u_i|,\ |c_{ijkl} u_{k,l}| = o(e^{\alpha R}) \quad \text{as} \quad R = |x| \to \infty, \tag{2·6}$$

where α is a positive constant. (An alternative prescription to (2·6) is

$$\lim_{R\to\infty} \int_0^{2T} \int_{\partial\Omega(\infty)} e^{-\alpha^2 R^2} (u_i u_i + c_{ijkl} u_{k,l} c_{ijpq} u_{p,q})\, dx\, d\eta = 0, \tag{2·7}$$

where $\partial\Omega(\infty)$ denotes the sphere at infinity.) We also impose the condition

$$\lim_{\alpha\to 0} \alpha^2 S(T) = 0, \tag{2·8}$$

in which $S(t)$ is defined by

$$S(t) = \int_0^{2t} \int_\Omega (e^{-\alpha R}/\rho)\, c_{ijkl} u_{k,l} c_{ijpq} u_{p,q}\, dx\, d\eta. \tag{2·9}$$

Conditions guaranteeing (2·8) have been described by Galdi and Rionero [7] and include, for example,

$$|c_{ijkl} u_{k,l} c_{ijpq} u_{p,q}| = O(R^{-\frac{1}{2}-\epsilon}), \quad \epsilon > 0, \quad \text{as } R \to \infty. \tag{2·10}$$

As mentioned previously, we suppose that a twice continuously differentiable displacement exists satisfying conditions (2·1)–(2·9), although it should be noted that the argument of the next section could employ a weighted cut-off function in the manner of Galdi and Rionero [7]. Then, the asymptotic behaviour just imposed could be replaced by the requirement that the displacement is confined to appropriate function classes, e.g. $L^2(\Omega \times (0,T])$. Indeed, we must impose this condition in the derivation of continuous dependence, but not of uniqueness. Thus, we restrict the displacement to the class of functions satisfying the bound

$$\int_0^T \int_\Omega \rho u_i(x,\eta)\, u_i(x,\eta)\, dx\, d\eta \leqslant M_1^2, \tag{2·11}$$

where M_1 is a positive constant.

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3. *Exterior regions*

In this section, we establish the uniqueness of the solution to the initial boundary value problem given in Section 2 in the case when Ω is an exterior region. We also derive an estimate from which we obtain the continuous dependence of the solution on the initial data. Thus, let us introduce the weight function

$$g(x) = e^{-\alpha R}, \quad R = |x|, \tag{3·1}$$

where α is a positive constant, and let $w_i(x,t)$, $v_i(x,t)$ be solutions to (2·1). We then have the following identity

$$\int_0^t \int_\Omega \rho(x)\, g(x)\, [w_{i,\eta}(x,\eta)\, v_{i,\eta}(x,\eta) - w_{i,\eta}(x,\eta)\, v_{i,\eta}(x,\eta)]\, dx\, d\eta = 0 \tag{3·2}$$

which, after an integration with respect to time, gives the basic weighted Lagrange identity:

$$\begin{aligned}\int_\Omega \rho(x)\, g(x)\, [w_i(x,t)\, v_{i,t}(x,t) - w_{i,t}(x,t)\, v_i(x,t)]\, dx \\ = \int_0^t \int_\Omega \rho(x)\, g(x)\, [w_i(x,\eta)\, v_{i,\eta\eta}(x,\eta) - w_{i,\eta\eta}(x,\eta)\, v_i(x,\eta)]\, dx\, d\eta \\ + \int_\Omega \rho(x)\, g(x)\, [w_i(x,0)\, v_{i,t}(x,0) - w_{i,t}(x,0)\, v_i(x,0)]\, dx. \end{aligned} \tag{3·3}$$

The identity (3·3) may be used in several different ways, in analogy with the treatment by Brun [2] for a bounded region. In particular, an energy conservation law may be established. However, for present purposes, we proceed by setting

$$w_i(x,\tau) = u_i(x,\tau), \quad v_i(x,\tau) = u_i(x,2t-\tau), \quad 0 \leqslant \tau < 2t, \tag{3·4}$$

so that (3·3) becomes

$$\begin{aligned}2\int_\Omega \rho g u_i(t)\, u_{i,t}(t)\, dx = \int_0^t \int_\Omega \rho g [u_{i,\eta\eta}(\eta)\, u_i(2t-\eta) - u_i(\eta)\, u_{i,\eta\eta}(2t-\eta)]\, dx\, d\eta \\ + \int_\Omega \rho g [f_i\, u_{i,t}(2t) - h_i\, u_i(2t)]\, dx, \end{aligned} \tag{3·5}$$

where we have omitted explicit dependence of functions on their spatial argument, and also employed the initial conditions (2·3). We next eliminate the inertial terms on the right of (3·5) by means of (2·1) and then integrate spatially by parts the resulting expression to obtain

$$\begin{aligned}2\int_\Omega \rho g u_i(t)\, u_{i,t}(t)\, dx = \int_0^t \int_\Omega g_{,j} [u_i(\eta)\, c_{ijkl}\, u_{k,l}(2t-\eta) - u_i(2t-\eta)\, c_{ijkl}\, u_{k,l}(\eta)]\, dx\, d\eta \\ + \int_\Omega \rho g [f_i\, u_{i,t}(2t) - h_i\, u_i(2t)]\, dx, \end{aligned} \tag{3·6}$$

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where we have appealed to (2·6) or (2·7). We next apply the arithmetic-geometric mean inequality to the first term on the right of (3·6) to get

$$\begin{aligned}
&\int_0^t \int_\Omega g_{,j}\, u_i(\eta)\, c_{ijkl}\, u_{k,l}(2t-\eta)\, dx\, d\eta \\
&\quad \leqslant \int_0^t \int_\Omega |g_{,j}\, u_i(\eta)\, c_{ijkl}\, u_{k,l}(2t-\eta)|\, dx\, d\eta \\
&\quad \leqslant \tfrac{1}{2}\alpha^2 \int_0^t \int_\Omega (g/\rho)\, c_{ijkl}\, u_{k,l}(2t-\eta)\, c_{ijpq}\, u_{p,q}(2t-\eta)\, dx\, d\eta \\
&\qquad + \frac{1}{2}\int_0^t \int_\Omega \rho g u_i(\eta)\, u_i(\eta)\, dx\, d\eta \\
&\quad = \tfrac{1}{2}\alpha^2 \int_t^{2t} \int_\Omega (g/\rho)\, c_{ijkl}\, u_{k,l}(\eta)\, c_{ijpq}\, u_{p,q}(\eta)\, dx\, d\eta + \frac{1}{2}\int_0^t \int_\Omega \rho g u_i(\eta)\, u_i(\eta)\, dx\, d\eta. \qquad (3\cdot7)
\end{aligned}$$

By similarly treating the second term on the right of (3·6) and combining the resulting estimate with (3·7), we may deduce from (3·6) that

$$\begin{aligned}
2\int_\Omega \rho g u_i(t)\, u_{i,t}(t)\, dx \leqslant \tfrac{1}{2}\alpha^2 S(t) + \frac{1}{2}\int_0^{2t} \int_\Omega \rho g u_i(\eta)\, u_i(\eta)\, dx\, d\eta \\
+ \int_\Omega \rho g [f_i\, u_{i,t}(2t) - h_i\, u_i(2t)]\, dx, \qquad (3\cdot8)
\end{aligned}$$

where $S(t)$ is given by (2·9)

Now we introduce the functions $F(t)$ and $G(t)$ respectively defined by

$$F(t) = \int_\Omega \rho g u_i(t)\, u_i(t)\, dx \qquad (3\cdot9)$$

and

$$G(t) = \int_0^{2t} F(\eta)\, d\eta. \qquad (3\cdot10)$$

It follows immediately that

$$\frac{d}{dt} G(t) = 2F(2t). \qquad (3\cdot11)$$

We next integrate (3·8) over $(0, 2\tau)$, where $t < \tau < T/4$. This yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{d\tau} G(\tau) &\leqslant \tfrac{1}{2}\alpha^2 \int_0^{2\tau} S(\eta)\, d\eta + \frac{1}{2}\int_0^{2\tau} \left(\int_0^{2t} \int_\Omega \rho g u_i(\eta)\, u_i(\eta)\, dx\, d\eta \right) dt \\
&\quad + \frac{1}{2}\int_\Omega \rho g f_i\, u_i(4\tau)\, dx + \frac{1}{2}\int_\Omega \rho g f_i f_i\, dx - \int_0^{2\tau} \int_\Omega \rho g h_i\, u_i(2\eta)\, dx\, d\eta \qquad (3\cdot12) \\
&\leqslant \alpha^2 \tau S(2\tau) + \tau G(\tau) + \frac{1}{2}\int_\Omega \rho g f_i f_i\, dx + \frac{1}{2}\int_\Omega \rho g f_i\, u_i(4\tau)\, dx \\
&\quad + \left(\tfrac{1}{2} T \int_\Omega \rho g h_i h_i\, dx \right)^{\frac{1}{2}} \left(\int_0^{2\tau} \int_\Omega \rho g u_i(2\eta)\, u_i(2\eta)\, dx\, d\eta \right)^{\frac{1}{2}}. \qquad (3\cdot13)
\end{aligned}$$

Now we appeal to condition (2·11), and then rewrite (3·13) as

$$\frac{d}{d\tau}(e^{-2\tau} G(\tau)) \leqslant 2\alpha^2 e^{-2\tau} \tau S(\tau) + e^{-2\tau} \int_\Omega \rho g f_i\, u_i(4\tau)\, dx + c e^{-2\tau} \qquad (3\cdot14)$$

where

$$c = \int_\Omega \rho g f_i f_i\, dx + M_1 \left(2T \int_\Omega \rho g h_i h_i\, dx \right)^{\frac{1}{2}}. \qquad (3\cdot15)$$

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An integration of (3·14) over $(0, s)$, where $\tau < s < T/4$, then gives the inequality:

$$G(s) \leqslant \alpha^2 S(T)\, T^2 e^{2T} + e^{2T} \int_0^s \int_\Omega \rho g f_i\, u_i(4\tau)\, dx\, d\tau + cTe^{2T}$$

$$\leqslant \alpha^2 S(T)\, T^2 e^{2T} + e^{2T} \left(T \int_\Omega \rho g f_i f_i\, dx \right)^{\frac{1}{2}} M_1 + cTe^{2T}.$$

Finally, we let $\alpha \to 0$ and use (2·8) to obtain the estimate

$$\int_0^t \int_\Omega \rho u_i(\eta)\, u_i(\eta)\, dx \leqslant e^{2T} T \left[\int_\Omega \rho f_i f_i\, dx + T^{-\frac{1}{2}} M_1 \left\{ \left(\int_\Omega \rho f_i f_i\, dx \right)^{\frac{1}{2}} \right.\right.$$

$$\left.\left. + T^{\frac{1}{2}} \left(2 \int_\Omega \rho h_i h_i\, dx \right)^{\frac{1}{2}} \right\} \right], \quad t \in (0, T], \tag{3·16}$$

which establishes the continuous dependence of the displacement upon the initial data in the class of functions given by (2·11).

Uniqueness of the solution follows at once from (3·16). However, upon repeating the above calculation with $f_i = h_i = 0$, it is clear that assumption (2·11) becomes superfluous for uniqueness.

4. *The half-space and similar regions*

We now turn our attention to unbounded regions that are contained in the half-space and for ease of presentation discuss the explicit example of the half-space for which the boundary $\partial\Omega$ is planar. However, it is clear from the treatment that the restriction to plane boundaries can be relaxed and any non-compact boundary may be considered that is sufficiently smooth to permit application of the divergence theorem. We briefly comment further at the end of the section on this generalization.

Thus let us consider the half-space and adapt the argument of the previous section in order to derive an estimate analogous to (3·16). This requires modification of some of the postulates introduced in Section 2, and hence we start by stating these. We choose our cartesian axes (x, y, z) such that the half-space is the region $z \geqslant 0$ and $\partial\Omega$ is the plane $z = 0$. The formulation of the initial boundary value problem continues to be given by (2·1)–(2·5), but the asymptotic behaviour (2·6) is now replaced by

$$|u_i|, |c_{ijkl} u_{k,l}| = o(e^{\gamma r^\beta} e^{\delta z}) \quad \text{as} \quad z, r = (x^2 + y^2)^{\frac{1}{2}} \to \infty \tag{4·1}$$

for positive constants β, γ, δ. More generally, we could require

$$\lim_{R \to \infty} \int_0^{2T} \int_{\partial\Omega^+(\infty)} e^{-\gamma r^\beta} e^{-\delta z} [u_i\, u_i + c_{ijkl} u_{k,l}\, c_{ijpq} u_{p,q}]\, dx\, d\eta = 0,$$

where $\partial\Omega^+(\infty)$ is the hemi-sphere at infinity.

Instead of (2·8), we suppose that

$$\lim_{\beta, \delta \to 0} [\delta^2 R(T) + \gamma^2 \beta^2 P(T)] = 0, \tag{4·2}$$

where

$$R(t) = \int_0^{2t} \int_\Omega e^{-\gamma r^\beta} e^{-\delta z} \rho^{-1} c_{ijkl} u_{k,l}(\eta)\, c_{ijpq} u_{p,q}(\eta)\, dx\, d\eta \tag{4·3}$$

and

$$P(t) = \int_0^{2t} \int_\Omega e^{-\gamma r^\beta} e^{-\delta z} r^{2(\beta-1)} \rho^{-1} c_{ijkl} u_{k,l}(\eta)\, c_{ijpq} u_{p,q}(\eta)\, dx\, d\eta. \tag{4·4}$$

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In the Appendix, we derive sufficient conditions for (4·2) to hold. These include as a special case the asymptotic condition (cf. (A 15))

$$|c_{ijkl}u_{k,l}(x,t)| \leqslant Kp(z), \quad z > z_1, \quad r > r_1, \tag{4·5}$$

where K is constant, z_1, r_1 are sufficiently large, and $p(z) \in L(0,\infty)$.

A particular example of the last condition is supplied by (cf. (A. 14))

$$|c_{ijkl}u_{k,l}(x,t)| \leqslant Kz^{-\frac{1}{2}}, \quad z > z_1, \quad r > r_1. \tag{4·6}$$

It should be observed that both these conditions do not involve any decay restrictions in the radial direction in contrast with the corresponding behaviour (2·10) for the exterior problem.

We now turn to the proof of uniqueness and continuous dependence of the solution on the initial data which will be established in the class of functions satisfying the boundedness condition

$$\int_0^T \int_\Omega \rho u_i(\eta)\, u_i(\eta)\, dx\, d\eta \leqslant M_2^2, \tag{4·7}$$

where M_2 is a positive constant. We introduce the weight function

$$g(r,z) = e^{-\gamma r^\beta} e^{-\delta z} \tag{4·8}$$

for positive constants β, γ and δ, and substitute into the Lagrange identity (3·5). As before, the inertial terms may be eliminated by means of (2·1) and then a spatial integration by parts combined with (4·1) leads to an equation of similar form to (3·6) but with g given by (4·8). We now examine separately the space-time integrals appearing in the latter equation. Thus, we have

$$\begin{aligned}
&\int_0^t \int_\Omega g_{,j} c_{ijkl} u_{k,l}(2t-\eta)\, u_i(\eta)\, dx\, d\eta \\
&\quad \leqslant \int_0^t \int_\Omega |g_{,3} c_{i3kl} u_{k,l}(2t-\eta)\, u_i(\eta)|\, dx\, d\eta \\
&\qquad + \int_0^t \int_\Omega |g_{,\nu} c_{i\nu kl} u_{k,l}(2t-\eta)\, u_i(\eta)|\, dx\, d\eta \quad (\nu = 1,2) \\
&\quad \leqslant \frac{1}{2}\int_0^t \int_\Omega [\delta^2 + \gamma^2\beta^2 r^{2(\beta-1)}]\, (g/\rho)\, c_{ijkl} u_{k,l}(2t-\eta)\, c_{ijpq} u_{p,q}(2t-\eta)\, dx\, d\eta \\
&\qquad + \frac{1}{2}\int_0^t \int_\Omega g\rho\, u_i(\eta)\, u_i(\eta)\, dx\, d\eta \\
&\quad = \frac{1}{2}\int_t^{2t} \int_\Omega [\delta^2 + \gamma^2\beta^2 r^{2(\beta-1)}]\, (g/\rho)\, c_{ijkl} u_{k,l}(\eta)\, c_{ijpq} u_{p,q}(\eta)\, dx\, d\eta \\
&\qquad + \frac{1}{2}\int_0^t \int_\Omega g\rho\, u_i(\eta)\, u_i(\eta)\, dx\, d\eta,
\end{aligned} \tag{4·9}$$

where the arithmetic-geometric mean inequality is used. Similarly, we have

$$\begin{aligned}
&-\int_0^t \int_\Omega g_{,j} c_{ijkl} u_{k,l}(\eta)\, u_i(2t-\eta)\, dx\, d\eta \\
&\qquad \leqslant \frac{1}{2}\int_0^t \int_\Omega [\delta^2 + \gamma^2\beta^2 r^{2(\beta-1)}]\, (g/\rho)\, c_{ijkl} u_{k,l}(\eta)\, c_{ijpq} u_{p,q}(\eta)\, dx\, d\eta \\
&\qquad\qquad + \frac{1}{2}\int_t^{2t} \int_\Omega \rho g u_i(\eta)\, u_i(\eta)\, dx\, d\eta.
\end{aligned} \tag{4·10}$$

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On recalling the definitions (4·3) and (4·4), we see it therefore follows that

$$2\int_\Omega \rho g u_i(t)\, u_{i,t}(t)\, dx \leqslant \tfrac{1}{2}[\delta^2 R(t) + \gamma^2\beta^2 P(t)] + \int_0^{2t}\int_\Omega \rho g u_i(\eta)\, u_i(\eta)\, dx\, d\eta + \int_\Omega \rho g[f_i\, u_{i,t}(2t) - h_i\, u_i(2t)]\, dx, \quad (4\cdot11)$$

which is the equation analogous to (3·8). It may therefore be discussed in exactly the same way as (3·8) and hence we arrive at uniqueness and continuous dependence of the solution upon the initial data in the class of functions given by (4·7).

Extension to regions with non-compact boundary other than the half-space. The discussion of this section can be applied to regions contained in some half-space, provided the boundary is such that application of the divergence theorem remains valid. The argument leading to continuous data dependence, and hence uniqueness, is unaltered and the assumptions (4·1), (4·2) regarding asymptotic behaviour are the same. Indeed, the condition (4·2) governs behaviour far from the boundary $\partial\Omega$ and thus, in particular (4·6) or (4·5) are sufficient whenever the region Ω is a half-neighbourhood of infinity containing a plane.

Appendix

In this Appendix we give a sufficient condition in order for assumption (4·2) to hold. In fact, we assume that the stress possesses the asymptotic behaviour

$$|c_{ijpq}\, u_{p,q}(x,t)| \leqslant K z^k r^m, \quad z > z_1, \quad r > r_1, \quad t \in (0,T], \quad (A\,1)$$

where z_1, r_1 are sufficiently large and positive, K is a positive constant and k, m are constant. We shall show that (A 1) implies (4·2), on taking for example, $k < -\frac{1}{2}, m = 0$. However, we first consider the behaviour of $R(t)$ defined in (4·3), and for this require only that $k < \frac{1}{2}$.

On inserting (A 1) into (4·3) we obtain

$$\delta^2 R(t) \leqslant K_1 \delta^2 \int_0^\infty e^{-\delta z} z^{2k}\, dz \int_0^\infty e^{-\gamma r^\beta} r^{2m+1} dr + \delta^2 M_3^2, \quad (A\,2)$$

where $K_1(T)$ is a constant and

$$M_3^2(t) \equiv \int_0^{2t}\int_{\Omega_1} e^{-\delta z} e^{-\gamma r^\beta} c_{ijkl}\, u_{k,l}\, c_{ijpq}\, u_{p,q}\, dx\, d\eta < \infty, \quad (A\,3)$$

with
$$\Omega_1 = \{x \in \mathbb{R}^3 : 0 \leqslant r \leqslant r_1,\ 0 \leqslant z \leqslant z_1\}. \quad (A\,4)$$

Let
$$K_2 = K_1 \int_0^\infty e^{-s} s^{2k}\, ds.$$

Then (A 2) becomes

$$\delta^2 R(t) \leqslant K_2\, \delta^{(1-2k)} \int_0^\infty e^{-\gamma r^\beta} r^{2m+1}\, dr + \delta^2 M_3^2. \quad (A\,5)$$

We next employ the inequality

$$\exp(-\gamma r^\beta)\, r^n \leqslant \exp(-n/\beta)\, (n/\gamma\beta)^{n/\beta}, \quad (A\,6)$$

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to deduce from (A 5) that

$$\delta^2 R(t) \leqslant K_2 \delta^{1-2k}\left\{e^{-(2m+3)/\beta}[(2m+3)/\gamma\beta]^{(2m+3)/\beta}\int_1^\infty r^{-2}\,dr + \int_0^1 e^{-\alpha r^\beta} r^{2m+1}\,dr\right\} + \delta^2 M_3^2. \quad \text{(A 7)}$$

Put
$$k < \tfrac{1}{2}, \quad \delta^{1-2k} = [(2m+3)/\gamma\beta]^{-(2m+3)/\beta}, \quad \text{(A 8)}$$

so that $\delta \to 0$ as $\beta \to 0$. Inequality (A 7) then immediately gives

$$\lim_{\delta\to 0,\ \beta\to 0} R(t) = 0. \quad \text{(A 9)}$$

We next consider conditions ensuring the asymptotic behaviour

$$\lim_{\beta\to 0} P(t) = 0, \quad \text{(A 10)}$$

where $P(t)$ is defined in (4·4). In this way we obtain sufficient conditions for (4·2). We again assume the estimate (A 1), but it will turn out that the choice of k and m is more restricted. From the argument leading to (A 5), it is clear that we have the estimate

$$\beta^2 P(t) \leqslant K_2 \delta^{-(1+2k)} \beta^2 \int_0^\infty e^{-\gamma r^\beta} r^{2m+2\beta-1}\,dr + \beta^2 M_3^2,$$

which on using (A 6) together with (A 8) gives, for $\epsilon > 0$,

$$\beta^2 P(t) \leqslant K_2 \beta^2 [(2m+3)/\gamma\beta]^{(2m+3)(1+2k)/\beta(1-2k)} \times \left\{[(2m+2\beta+\epsilon)/\gamma\beta]^{(2m+2\beta+\epsilon)/\beta} e^{-(2m+2\beta+\epsilon)/\beta}\int_1^\infty r^{-(1+\epsilon)}\,dr + [(2m+2\beta-\epsilon)/\gamma\beta]^{(2m+2\beta-\epsilon)/\beta} e^{-(2m+2\beta-\epsilon)/\beta}\int_0^1 r^{\epsilon-1}\,dr\right\} + \beta^2 M_3^2. \quad \text{(A 11)}$$

The limit (A 10) then follows on taking

$$2(3-\epsilon)k < -(4m+3+\epsilon), \quad \epsilon > 0. \quad \text{(A 12)}$$

We note that a particular choice of k, m satisfying (A 12), and also (A 8), as $\epsilon \to 0$ is

$$k < -\tfrac{1}{2}, \quad m = 0, \quad \text{(A 13)}$$

when the corresponding asymptotic behaviour (A 1) is given by

$$|c_{ijkl} u_{k,l}(x,t)| \leqslant K z^{-\frac{1}{2}}, \quad z > z_1, \quad r > r_1, \quad \text{(A 14)}$$

Inspection of the derivation of (A 14) shows that it may be slightly generalized to the case

$$|c_{ijkl} u_{k,l}(x,t)| \leqslant K p(z), \quad z > z_1, \quad r > r_1, \quad \text{(A 15)}$$

provided the function $p(z)$ is of class $L(0,\infty)$.

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