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ON CLASSES OF LIFE DISTRIBUTIONS BASED ON THE MEAN TIME TO FAILURE FUNCTION

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Abstract

The performance and effectiveness of an age replacement policy can be assessed by its mean time to failure (MTTF) function. We develop shock model theory in different scenarios for classes of life distributions based on the MTTF function where the probabilities $\bar{P}_k$ of surviving the first k shocks are assumed to have discrete DMTTF, IMTTF and IDMTTF properties. The cumulative damage model of A-Hameed and Proschan [1] is studied in this context and analogous results are established. Weak convergence and moment convergence issues within the IDMTTF class of life distributions are explored. The preservation of the IDMTTF property under some basic reliability operations is also investigated. Finally we show that the intersection of IDMRL and IDMTTF classes contains the BFR family and establish results outlining the positions of various non-monotonic ageing classes in the hierarchy.

Keywords: Age replacement policy; IDMTTF class; pure-birth process; cumulative damage model; weak convergence

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Secondary 60K10

1. Introduction

In-service failures of a device or system or a component thereof are not only very expensive but can also pose serious threats. To prevent such unexpected failures, a typical practice is to take recourse to maintenance policies. One of the most common and popular maintenance policies is an age replacement policy where a component is replaced by a new one on failure or at a pre-specified time t after installation, whichever is earlier. Let X denote the lifetime of a new component having *cumulative distribution function* (CDF) $F(x)$ and *survival function* $\bar{F}(x) := 1 - F(x)$. Let $X_{[t]}$ denote the time to the first in-service failure of a component under the said age replacement policy. Then the survival function of $X_{[t]}$ can be represented as

$$S_t(x) = \sum_{n=0}^{\infty} [\bar{F}(t)]^n \bar{F}(x - nt) I_{[nt,\,(n+1)t]}(x), \quad x > 0,$$

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where I denotes an indicator function. In this context, Barlow and Proschan [7] introduced the *mean time to failure* (MTTF) function which is given by

$$M_F(t) := \mathbb{E}(X_{[t]}) = \frac{\int_0^t \bar{F}(x)\,\mathrm{d}x}{F(t)}, \quad F(t) > 0.$$

Since then, this function has proved to be a very useful characteristic to investigate the expected life cycles of a system and for the assessment of performance and effectiveness of age replacement policies. Due to its manifold potential applications in reliability analysis, the MTTF function has received significant attention in the literature; see Klefsjö [32], Li and Xu [39], Kayid *et al.* [28], Izadi *et al.* [26] and others.

Throughout this paper, the words 'non-decreasing' ('non-increasing') and 'increasing' ('decreasing') are used interchangeably. It is to be assumed that $F(t) > 0$ for $t > 0$ and $F(0) = 0$ unless otherwise mentioned. The expression $A \overset{\text{sgn}}{=} B$ indicates that both A and B have the same sign. Contrary to popular usage, we denote the set of non-negative integers by $\mathbb{N}$.

Definition 1. A life distribution function F is called *decreasing mean time to failure* (DMTTF) (*increasing mean time to failure* (IMTTF)) if $M_F(t)$ is decreasing (increasing) in $t \in (0, \infty)$.

The above definition of DMTTF (IMTTF) indicates a positive (negative) ageing pattern. Barlow and Proschan [7] showed that the DMTTF class contains the *increasing failure rate* (IFR) class and is contained in the *new better than used in expectation* (NBUE) class. Thereafter Klefsjö [32] established that the DMTTF class also contains all *increasing failure rate average* (IFRA) distributions. The following hierarchy shows the interrelationships between the DMTTF family and some well-known classes of life distributions:

$$\text{IFRA} \Longrightarrow \text{DMTTF} \Longrightarrow \text{NBUE} \quad \text{and} \quad \text{DFRA} \Longrightarrow \text{IMTTF} \Longrightarrow \text{NWUE}.$$

For definitions and further properties of IFR, IFRA, NBUE and other well-known monotonic ageing classes (and their duals), the interested reader is referred to Lai and Xie [38].

Knopik [35] established some preservation results for the DMTTF class under certain reliability operations. Subsequently, the closure of DMTTF distributions under weak convergence and convolutions was also investigated in Knopik [36]. In a different context, Li and Xu [39] introduced a new class of life distributions, namely, *new better than renewal used in the reversed hazard rate* order (NBRUrh) (*new worse than renewal used in the reversed hazard rate* order (NWRUrh)) which is equivalent to the DMTTF (IMTTF) class. Now we will highlight this fact briefly. Suppose X and Y are two non-negative random variables with CDFs F and G respectively. Then X is said to be smaller than Y in reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $G(x)/F(x)$ increases in x for which this ratio is well-defined (see Shaked and Shanthikumar [48]). If μ is the finite mean of X, then the survival function of the equilibrium distribution of X, associated with the random variable $\tilde{X}$, is given by

$$\bar{F}_{\tilde{X}}(t) = \frac{1}{\mu}\int_t^\infty \bar{F}(x)\,\mathrm{d}x, \quad t \geq 0.$$

Thus X is said to be DMTTF or NBRUrh (IMTTF or NWRUrh) if $\tilde{X} \leq_{rh} X$ ($\tilde{X} \geq_{rh} X$). Li and Xu [39] proved some important properties of these classes and also considered the problem of testing exponentiality against the DMTTF property using a U-statistic approach. Asha and

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Nair [6] introduced a new stochastic ordering in terms of the MTTF function to compare life distributions and connected it with some known ageing orderings. Some preservation properties of the MTTF order and a test for exponentiality against DMTTF alternatives were considered by Kayid *et al.* [28]. Kattumannil and Anisha [27] developed a test of exponentiality against DMTTF alternatives using a U-statistic methodology while Sudheesh *et al.* [50] studied some properties of the MTTF function in discrete time. Recently Bhattacharyya *et al.* [13] have proposed a new test for the DMTTF property exploiting L-statistic theory.

It is worth noting that the ageing patterns exhibited in all the classes discussed above happen to be monotonic. However, in many practical situations the ageing profile turns out to be non-monotonic, typically characterized by a 'burn-in phase', a 'useful life phase' and finally a 'wear-out phase'. To model such scenarios, there are several non-monotonic ageing classes of distributions in the literature, for example *bathtub failure rate* (BFR) (Glaser [23]), *increasing initially then decreasing mean residual life* (IDMRL) (Guess *et al.* [24]), NBUE-NWUE (Klefsjö [34]), *new worse then better than used in expectation* (NWBUE) (Mitra and Basu [41]), *increasing initially then decreasing residual life in the Laplace order* (IDRL_{Lt}) (Belzunce *et al.* [11]), etc. In the same spirit, Izadi *et al.* [26] introduced the following two important non-monotonic ageing classes of distributions based on the MTTF function.

Definition 2. A life distribution F is said to be an *increasing then decreasing mean time to failure* (IDMTTF) distribution if there exists a change point $\tau \geq 0$ such that $M_F(t)$ is increasing on $(0, \tau]$ and decreasing on (τ, ∞).

In the above definition Izadi *et al.* [26] considered the domain of the MTTF function as $[0, \infty)$, but we have excluded the point 0 from the domain since $M_F(t)$ is not defined at $t = 0$. For $\tau = 0$, the IDMTTF class reduces to DMTTF. We also include all IMTTF distributions in the IDMTTF class by adopting the convention that these distributions have 'a change point at infinity'. It is worth mentioning that the IDMTTF class can also be considered as the 'NWBRUrh class' following the concepts introduced by Li and Xu [39]. The dual class DIMTTF of life distributions is defined by simply changing the order of monotonicity in the above definition.

Izadi *et al.* [26] studied the relationship between the expected cost rate function and the MTTF function and investigated how τ approximates the optimal replacement time which minimizes the expected cost rate function. They also characterized IDMTTF and DIMTTF distributions via scaled total time on test transform and proved the following chain of implications:

$$\text{BFR} \Longrightarrow \text{IDMTTF} \Longrightarrow \text{NWBUE} \quad \text{and} \quad \text{UBFR} \Longrightarrow \text{DIMTTF} \Longrightarrow \text{NBWUE}.$$

Izadi and Fathimanesh [25] also provided a large-sample test of exponentiality against a trend change in the MTTF function. Recently Khan *et al.* [30] have developed a general methodology for estimating the change point of IDMTTF distributions.

Highlighting the main contributions of our work, we organize the rest of the paper as follows. Results in the spirit of Esary *et al.* [18] and A-Hameed and Proschan [2] are proved in Section 2 in the context of classes of life distributions based on the MTTF function; more precisely, we develop shock model theory in different scenarios where the probabilities $\bar{P}_k$ of surviving the first k shocks are assumed to have discrete DMTTF, IMTTF, IDMTTF and DIMTTF properties. Different properties of the discrete failure distribution $\bar{P}_k$ are shown to be reflected in corresponding properties of the continuous life distribution function $H(t)$ of a system. In the context of monotonic ageing classes, we have considered that the shocks arrive

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according to a non-stationary pure-birth process, while for non-monotonic ageing classes we have considered that the shocks arrive according to a homogeneous Poisson process. We also study the cumulative damage model of A-Hameed and Proschan [1] and establish analogous results. In Section 3 we prove closure under weak convergence within the IDMTTF family and investigate related issues. We also explore the connection between weak convergence and the convergence of moments within the IDMTTF family. In Section 4 we study the preservation of the IDMTTF property under some basic reliability operations such as formation of series, parallel, k-out-of-n, coherent structures and mixtures. Using characterization via scaled total time on test transform, we derive a result to show that the BFR property necessarily implies the IDMRL property. Further, we have established some results to place in clear perspective the position of each of the well-known non-monotonic ageing classes such as BFR, IDMRL, IDMTTF and NWBUE. In Section 5 we summarize the main contributions of our work along with a couple of comments. Finally, we briefly explore possible avenues of future work in this direction.

2. Shock model theory

Suppose that a system is subjected to a sequence of shocks which arrive randomly in time according to a counting process $\{N(t)\colon t \geq 0\}$ and cause damage to the system. Assume that the system under consideration is not subject to constant wear and tear and only the shocks are responsible for its failure. Let $\bar{P}_k$ be the probability that the system survives the first k shocks, $k = 0, 1, 2, \ldots$, and suppose T denotes the time to failure. We assume that the $\bar{P}_k$ satisfy the following natural condition:

$$1 = \bar{P}_0 \geq \bar{P}_1 \geq \bar{P}_2 \geq \cdots .$$

The probability of the system surviving beyond time t is given by the survival function

$$\bar{H}(t) := \mathbb{P}(T > t) = \sum_{k=0}^{\infty} \mathbb{P}(N(t) = k)\bar{P}_k. \tag{1}$$

At this juncture it is pertinent to mention that shock model theory plays a vital role in diverse areas (see Nakagawa [44], Finkelstein and Cha [21] and the references therein). Before proceeding further, a motivating example would be appropriate. Consider an insurance company where claims arrive randomly in time. Claims can be interpreted as 'shocks' and the magnitude X_i of the ith claim can be interpreted as the damage caused by the ith shock. When the total claim exceeds a threshold Y (insurance company's capital), the company goes bankrupt. This idea is applicable in various scenarios such as risk analysis, inventory control, dam theory, biometry, etc., by simply considering appropriate analogues. For instance, 'shock', 'system failure' and 'survival until time t' can be substituted, respectively, by 'claim', 'bankruptcy' and 'all claims met during $[0, t]$' in risk analysis or by 'demand', 'shortage' and 'no shortage during $[0, t]$' in inventory control; for more details, see A-Hameed and Proschan [2]. Using the prior knowledge of the fact that the survival function $\bar{H}(t)$ belongs to a certain class of life distributions, it is feasible to obtain sharp bounds on the survival function and other parameters of the distribution. Such knowledge produces proper maintenance strategies in reliability contexts, higher accuracy in estimation results and functional financial policies in risk analysis. In the huge body of literature concerning shock model theory, the works of Shanthikumar

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and Sumita [49], Anderson [4], Yamada [52], Kochar [37], Pérez-Ocón and Gámiz-Pérez [47], Pellerey [46], Ebrahimi [17] and others are significant.

An underlying fundamental question which is inextricably linked to reliability shock models is as follows: If the sequence $\{\bar{P}_k\}_{k=0}^{\infty}$ possesses a certain discrete ageing property (such as IFR, IFRA, NBU, NBUE and so on), does $\bar{H}(t)$ inherit the corresponding continuous version of the said ageing property under the transformation (1) when $N(t)$ is a homogeneous Poisson process (HPP), non-homogeneous Poisson process (NHPP), stationary or non-stationary pure-birth process.

For the IFR, IFRA, NBU, NBUE and *decreasing mean residual life* (DMRL) classes, Esary *et al.* [18] provided an affirmative answer to the above question when $N(t)$ is an HPP. A-Hameed and Proschan [1] showed that under some assumptions on the intensity function $\Lambda(t)$, the same preservation result holds for the above classes when $N(t)$ is an NHPP. Later they also considered the same problem for the above classes when shocks arrive according to stationary and non-stationary pure-birth processes and proved analogous results; see A-Hameed and Proschan [2]. Stronger results in a similar vein are available for the NBUE, *harmonic new better than used in expectation* (HNBUE) and $\mathcal{L}$ classes in Block and Savits [15], Klefsjö [31] and Klefsjö [33] respectively. Thereafter, analogous results for the *new better than used in failure rate* (NBUFR) and *new better than average failure rate* (NBAFR) classes of life distributions were derived by Abouammoh *et al.* [3] and Fagiuoli and Pellerey [19]. Shock models leading to bivariate IFR and DMRL distributions are discussed in Ghosh and Ebrahimi [22]. In the context of non-monotonic ageing phenomena, Mitra and Basu [43] were the first to establish such results for the NWBUE and BFR classes. Anis [5] established the analogous result in the context of IDMRL distributions. We now proceed to establish the corresponding results in the context of classes of life distributions based on the MTTF function. We first recall the notions of discrete versions of DMTTF and IMTTF properties introduced by Li and Xu [39].

Definition 3. (*Li and Xu* [39].) A distribution $P_k = 1 - \bar{P}_k$, $k \in \mathbb{N}$, is said to be D-DMTTF (D-IMTTF) if $P_k{}^{-1} \sum_{i=0}^{k-1} \bar{P}_i$ is decreasing (increasing) in k. D-DMTTF (D-IMTTF) means discrete DMTTF (IMTTF).

In the same spirit, we now introduce the notions of discrete versions of IDMTTF and DIMTTF properties.

Definition 4. A distribution $P_k = 1 - \bar{P}_k$, $k \in \mathbb{N}$, is said to be D-IDMTTF(k_0) (D-DIMTTF(k_0)) if there exists $k_0 \geq 1$ such that

$$\frac{1}{P_k} \sum_{i=0}^{k-1} \bar{P}_i = \begin{cases} \text{is increasing (decreasing) in } k \text{ for } k < k_0, \\ \text{is decreasing (increasing) in } k \text{ for } k \geq k_0, \end{cases}$$

where D-IDMTTF (D-DIMTTF) denotes discrete IDMTTF (DIMTTF).

We are now in a position to develop shock model theory for pure-birth processes.

2.1. Pure-birth shock models

Now consider a pure-birth shock model where a system is subjected to shocks governed by a birth process with intensities λ_k, $k = 0, 1, 2, \ldots$. Let V_{k+1} denote the interarrival time between the kth and $(k+1)$th shocks. Assume the V_k are independent with V_{k+1} being exponentially

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distributed with mean $1/\lambda_k$ for $k = 0, 1, 2, \ldots$. Then the survival function $\bar{H}(t)$ of the system can be written as

$$\bar{H}(t) = \sum_{k=0}^{\infty} Z_k(t)\bar{P}_k \tag{2}$$

with

$$Z_k(t) := \mathbb{P}(N(t) = k),$$

where $N(t)$ is the pure-birth process defined above. We assume that the intensities $\{\lambda_k\}_{k=0}^{\infty}$ are such that the probability of infinitely many shocks in $(0, t]$ is 0 (i.e. $\sum_{k=0}^{\infty} Z_k(t) = 1$), which is equivalent to the condition $\sum_{k=0}^{\infty} \lambda_k^{-1} = \infty$; see Feller [20, page 452]. To develop sufficient conditions for the preservation of DMTTF (IMTTF) properties, we will use the following lemmas.

Lemma 1. (A-Hameed and Proschan [2].) *Let $Z_k(t)$ be as defined in (2). Then*

$$\int_0^t Z_k(u)\,\mathrm{d}u = \frac{1}{\lambda_k} \sum_{j=k+1}^{\infty} Z_j(t).$$

Lemma 2. *If for any $c \in \mathbb{R}$, $\int_0^t \bar{H}(u)\,\mathrm{d}u - cH(t)$ has at most one change of sign from $+$ to $-$, then $H(t)$ is DMTTF.*

Proof. Suppose $0 < x_1 < x_2$. We want to show that $M_H(x_1) \geq M_H(x_2)$. Note that for any $c \in \mathbb{R}$, $\int_0^t \bar{H}(u)\,\mathrm{d}u - cH(t)$ has at most one change of sign from $+$ to $-$. So this happens for c_1 and c_2 in particular. By choice of c_i, $i = 1, 2$, we have

$$\int_0^t \bar{H}(u)\,\mathrm{d}u \begin{cases} \geq c_i H(t) & \text{for } t \leq x_i, \\ \leq c_i H(t) & \text{for } t > x_i. \end{cases}$$

If possible, let $c_1 = M_H(x_1) < M_H(x_2) = c_2$. Take $x_1 < t < x_2$. Then

$$\int_0^t \bar{H}(u)\,\mathrm{d}u \leq c_1 H(t) < c_2 H(t) \leq \int_0^t \bar{H}(u)\,\mathrm{d}u,$$

a contradiction. Thus $M_H(x_1) \geq M_H(x_2)$ and hence $H(t)$ is DMTTF. □

In the next theorem we show how the D-DMTTF (D-IMTTF) property is inherited by the survival function $\bar{H}(t)$ defined in (1).

Theorem 1. *Suppose that $\bar{H}(t)$ is defined as in (2). Then $H(t)$ is DMTTF (IMTTF) if*

$$\frac{1}{P_j} \sum_{k=0}^{j-1} \frac{\bar{P}_k}{\lambda_k}$$

is decreasing (increasing) in j, where $j \in \mathbb{N}$.

Proof. Note that

$$\bar{H}(t) = \sum_{k=0}^{\infty} Z_k(t)\bar{P}_k \quad \text{and} \quad H(t) = \sum_{k=0}^{\infty} Z_k(t)P_k = \sum_{k=1}^{\infty} Z_k(t)P_k \quad \text{as } P_0 = 0.$$

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Now for any $c \in \mathbb{R}$ we have

$$\begin{aligned}\int_0^t \bar{H}(u)\,\mathrm{d}u - cH(t) &= \int_0^t \sum_{k=0}^{\infty} Z_k(u)\bar{P}_k\,\mathrm{d}u - c\sum_{k=0}^{\infty} Z_k(t)P_k \\ &= \sum_{k=0}^{\infty} \bar{P}_k \int_0^t Z_k(u)\,\mathrm{d}u - c\sum_{k=0}^{\infty} Z_k(t)P_k \\ &= \sum_{k=0}^{\infty} \bar{P}_k \frac{1}{\lambda_k} \sum_{j=k+1}^{\infty} Z_j(t) - c\sum_{k=0}^{\infty} Z_k(t)P_k \quad \text{(by Lemma1)} \\ &= \sum_{j=1}^{\infty} Z_j(t) \sum_{k=0}^{j-1} \frac{\bar{P}_k}{\lambda_k} - c\sum_{j=1}^{\infty} Z_j(t)P_j \\ &= \sum_{j=1}^{\infty} Z_j(t)\left[\sum_{k=0}^{j-1} \frac{\bar{P}_k}{\lambda_k} - cP_j\right].\end{aligned}$$

By hypothesis,

$$\sum_{k=0}^{j-1} \frac{\bar{P}_k}{\lambda_k} - cP_j$$

has at most one change of sign, and if one change of sign occurs, it is from $+$ to $-$. Hence, by the variation diminishing property, it follows that the same conclusion can be drawn for $\int_0^t \bar{H}(u)\,\mathrm{d}u - cH(t)$. Now an application of Lemma 2 yields that $H(t)$ is DMTTF. The proof for the dual class is analogous. □

A pure-birth process reduces to an HPP if all the λ_k are equal to λ. Then Theorem 16 of Li and Xu [39] is obtained as an easy corollary of Theorem 1.

Corollary 1. *When $N(t)$ is an HPP, the survival function $\bar{H}(t)$ possesses the DMTTF (IMTTF) property if the sequence $\{P_k\}_{k=0}^{\infty}$ belongs to the D-DMTTF (D-IMTTF) class.*

Now we consider the non-stationary pure-birth shock model of A-Hameed and Proschan [2]: *shocks occur according to a Markov process; given that k shocks have arrived in* $(0, t]$, *the probability of a shock occurring in* $(t, t + \Delta t]$ *is* $\lambda_k\lambda(t)\Delta t + \mathrm{o}(\Delta t)$, *while the probability of more than one shock occurring in* $(t, t + \Delta t]$ *is* $\mathrm{o}(\Delta t)$. The probability of one shock arriving in a time interval of length Δt is allowed to depend on the time point t and also on the number k of shocks that have already arrived, and this dependence is conveyed via the function $\lambda(t)$ and a sequence of constants λ_k respectively. Here $\mathrm{o}(h)$ stands for any function f satisfying

$$\lim_{h\to 0} \frac{f(h)}{h} = 0.$$

Using Remark 1.1 of A-Hameed and Proschan [2], the survival function $\bar{H}^*(t)$ of the system for this shock model can be written as

$$\bar{H}^*(t) = \sum_{k=0}^{\infty} Z_k(\Lambda(t))\bar{P}_k, \tag{3}$$

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where $Z_k(t)$ is as defined in (2) and $\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u$. Before proving the preservation result in this more general set-up, we first establish the following theorem.

Theorem 2. *Let F be DMTTF (IMTTF) and $g\colon [0,\infty) \to [0,\infty)$ be a differentiable convex (concave) function such that $g(0) = 0$. Then $R(t) = F(g(t))$ is DMTTF (IMTTF).*

Proof. Assume that F is DMTTF with density f. Now

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\int_0^t \bar{R}(u)\,\mathrm{d}u}{R(t)}\right] &\overset{\text{sgn}}{=} R(t)\bar{R}(t) - \frac{\mathrm{d}R}{\mathrm{d}t}\int_0^t \bar{R}(u)\,\mathrm{d}u \\ &= F(g(t))\bar{F}(g(t)) - f(g(t))g'(t)\int_0^t \bar{F}(g(u))\,\mathrm{d}u \\ &\le F(g(t))\bar{F}(g(t)) - f(g(t))\int_0^t \bar{F}(g(u))g'(u)\,\mathrm{d}u\end{aligned}$$

since $g'(x)$, the derivative of $g(x)$ with respect to x, is increasing. Putting $y = g(u)$ in the integral, we get

$$F(g(t))\bar{F}(g(t)) - f(g(t))\int_0^{g(t)} \bar{F}(y)\,\mathrm{d}y \le 0,$$

since F is DMTTF. Hence R is DMTTF. We can draw the same conclusion using limiting arguments even if F does not have a density. An analogous result can be obtained for the IMTTF class of distributions by simply reversing the inequalities. □

We are now in a position to prove the following theorem.

Theorem 3. *Suppose $\lambda(t)$ is an increasing (decreasing) function and*

$$\frac{1}{P_j}\sum_{k=0}^{j-1}\frac{\bar{P}_k}{\lambda_k}$$

is decreasing (increasing) in j, where $j \in \mathbb{N}$. Then $\bar{H}^*(t)$ is DMTTF (IMTTF).

Proof. Note that

$$\bar{H}^*(t) = \bar{H}(\Lambda(t)) = \sum_{k=0}^{\infty} Z_k(\Lambda(t))\bar{P}_k.$$

One can easily show that $\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u$ is a convex function since $\lambda(u)$ is increasing. Now applying Theorem 2, it is enough to show that $H(t)$ is DMTTF, which follows from Theorem 1. Similarly, we can establish the same preservation results for the dual class. □

A non-stationary pure-birth process reduces to an NHPP if all the λ_k are equal to λ. Equation (3) now yields, as a special case, the following corollary, which is just Theorem 17 of Li and Xu [39].

Corollary 2. *When $N(t)$ is an NHPP, the survival function $\bar{H}(t)$ possesses the DMTTF (IMTTF) property if the sequence $\{P_k\}_{k=0}^{\infty}$ belongs to the D-DMTTF (D-IMTTF) class.*

Now we will investigate the same preservation for the non-monotonic classes IDMTTF and DIMTTF in the case of Poisson flow of shocks, i.e. $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$. Before proving the main result, we will, for completeness, state a result which is proved in Belzunce *et al.* [11] using total positivity theory.

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Lemma 3. (Belzunce et al. [11].) *Suppose that* $\{\alpha_k/\beta_k\}_{k=0}^{\infty}$ *is a real sequence with the property that*

$$\frac{\alpha_k}{\beta_k}\begin{cases}\text{is increasing (decreasing) in } k \text{ for } k < k_0,\\ \text{is decreasing (increasing) in } k \text{ for } k \geq k_0,\end{cases}$$

for some positive integer k_0 *and also assume that the power series defined by*

$$\Phi(s) := \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} s^k, \quad s \geq 0 \quad \text{and} \quad \psi(s) := \sum_{k=0}^{\infty} \frac{\beta_k}{k!} s^k, \quad s \geq 0$$

converge absolutely for all $s \geq 0$. *Then there exists* $s_0 \in \mathbb{R}$ *such that*

$$\frac{\phi(s)}{\psi(s)}\begin{cases}\text{is increasing (decreasing) in } s \text{ for } s < s_0,\\ \text{is decreasing (increasing) in } s \text{ for } s \geq s_0.\end{cases}$$

Theorem 4. *Let* $\{\bar{P}_k\}_{k=0}^{\infty}$ *be a sequence of D-IDMTTF* (i_0) *(D-DIMTTF* (i_0)*). Then* $\bar{H}(t)$ *is IDMTTF (DIMTTF) if N(t) is an HPP.*

Proof. For a homogeneous Poisson shock model,

$$\bar{H}(t) = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda t} \frac{(\lambda t)^k}{k!} \bar{P}_k.$$

Now

$$\begin{aligned}\int_0^t \bar{H}(u)\,\mathrm{d}u &= \int_0^t \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda u} \frac{(\lambda u)^k}{k!} \bar{P}_k\,\mathrm{d}u\\ &= \sum_{k=0}^{\infty} \frac{\bar{P}_k}{\lambda} \int_0^t \frac{\lambda^{k+1}}{\Gamma(k+1)} \mathrm{e}^{-\lambda u} u^k\,\mathrm{d}u\\ &= \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} \frac{\bar{P}_k}{\lambda} \frac{(\lambda t)^i}{i!} \mathrm{e}^{-\lambda t}\\ &= \frac{\mathrm{e}^{-\lambda t}}{\lambda} \sum_{i=1}^{\infty} \frac{(\lambda t)^i}{i!} \left(\sum_{k=0}^{i-1} \bar{P}_k\right),\end{aligned}$$

and

$$M_H(t) = \frac{\dfrac{\mathrm{e}^{-\lambda t}}{\lambda} \displaystyle\sum_{i=1}^{\infty} \frac{(\lambda t)^i}{i!} \left(\sum_{k=0}^{i-1} \bar{P}_k\right)}{\displaystyle\sum_{k=0}^{\infty} \mathrm{e}^{-\lambda t} \frac{(\lambda t)^k}{k!} P_k} = \frac{1}{\lambda} \frac{\displaystyle\sum_{k=1}^{\infty} \frac{(\lambda t)^k}{k!} \sum_{j=0}^{k-1} \bar{P}_j}{\displaystyle\sum_{k=1}^{\infty} \frac{(\lambda t)^k}{k!} P_k}.$$

We want to show that the function $M_H(t)$ is increasing in $(0, t_0)$ and decreasing in $[t_0, \infty)$ for some $t_0 \geq 0$. Putting $s = \lambda t$, since $\lambda > 0$ and $t \geq 0$ it is enough to show that for some $s_0 \geq 0$

$$\frac{\displaystyle\sum_{k=1}^{\infty} \frac{s^k}{k!} \left(\sum_{j=0}^{k-1} \bar{P}_j\right)}{\displaystyle\sum_{k=1}^{\infty} \frac{s^k}{k!} P_k} \begin{cases}\text{is increasing in } s \text{ for } s < s_0,\\ \text{is decreasing in } s \text{ for } s \geq s_0.\end{cases} \tag{4}$$

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Writing $i-1$ in place of k in Lemma 3, if

$$\left\{\frac{\alpha_{i-1}}{\beta_{i-1}},\, i=1,2,\ldots\right\}$$

is increasing in i for $i<i_0$ and decreasing in i for $i\geq i_0$, then there exists some $s_0\geq 0$ such that

$$\frac{\displaystyle\sum_{i-1=0}^{\infty}\frac{s^{i-1}}{(i-1)!}\alpha_{i-1}}{\displaystyle\sum_{i-1=0}^{\infty}\frac{s^{i-1}}{(i-1)!}\beta_{i-1}}\begin{cases}\text{is increasing in } s \text{ for } s<s_0,\\ \text{is decreasing in } s \text{ for } s\geq s_0.\end{cases}\tag{5}$$

Now the expression before the parenthesis in (4) can be written as

$$\frac{\displaystyle\sum_{i=1}^{\infty}\frac{s^{i-1}}{(i-1)!}\left(\frac{1}{i}\sum_{j=0}^{i-1}\bar{P}_j\right)}{\displaystyle\sum_{i=1}^{\infty}\frac{s^{i-1}}{(i-1)!}\left(\frac{P_i}{i}\right)}=\frac{\displaystyle\sum_{i-1=0}^{\infty}\frac{s^{i-1}}{(i-1)!}\alpha_{i-1}}{\displaystyle\sum_{i-1=0}^{\infty}\frac{s^{i-1}}{(i-1)!}\beta_{i-1}},$$

which is the same as the expression before the parenthesis in (5), where

$$\alpha_{i-1}=\frac{1}{i}\sum_{j=0}^{i-1}\bar{P}_j$$

and $\beta_{i-1}=P_i/i$. Note that

$$\frac{\alpha_{i-1}}{\beta_{i-1}}=\frac{1}{P_i}\sum_{j=0}^{i-1}\bar{P}_j\begin{cases}\text{is increasing in } i \text{ for } i<i_0,\\ \text{is decreasing in } i \text{ for } i\geq i_0,\end{cases}$$

by the D-IDMTTF property which implies $\bar{H}(t)$ is IDMTTF. The proof for the dual class is analogous. □

2.2. The cumulative damage shock model

In this model, each arriving shock causes damage of random magnitude to the system. The system fails when the accumulated damage exceeds a certain threshold. If there is significant individual variation in the ability to withstand shocks, this threshold must, of necessity, be regarded as a random variable Y having a CDF G with $G(0)=0$. Let X_i be the damage caused to the system by the ith shock, $i=1,2,3,\ldots$. We assume that $X_1, X_2, X_3, \ldots$ are independent with CDFs $F_1, F_2, F_3, \ldots$ and are further independent of the critical threshold. Here the F_i and G are lifetime distributions as damage and critical threshold, respectively, are positive. Now the probability of surviving k shocks is

$$\bar{P}_k=\mathbb{P}[X_1+X_2+\cdots+X_k\leq Y],\quad k=1,2,\ldots.$$

Clearly $\bar{P}_0=1$ and for $k=1,2,\ldots,$

$$\bar{P}_k=\int_0^{\infty}F^{[k]}(x)\,\mathrm{d}G(x),\tag{6}$$

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where $F^{[k]}$ is used to denote the convolution defined by $F_1 * F_2 * \cdots * F_k$. Esary *et al.* [18] established that the survival probabilities preserve the discrete NBU property for all choices of life distributions $\{F_j\}_{j=1}^{\infty}$ having $F_j(x)$ decreasing in j for all x if and only if G is NBU. Later A-Hameed and Proschan [2] and Klefsjö [31] considered a more special choice of the distributions $\{F_j\}_{j=1}^{\infty}$ where for all $j = 1, 2, \ldots,$

$$F_j(x) = \int_0^x \frac{b^{a_j} t^{a_j - 1}}{\Gamma(a_j)} \, \mathrm{e}^{-bt} \, \mathrm{d}t, \quad x \geq 0, \ a_j, b > 0. \tag{7}$$

Then (6) can be explicitly written as

$$\bar{P}_k = b \int_0^{\infty} \bar{G}(x) \, \mathrm{e}^{-bx} \frac{(bx)^{A_k - 1}}{\Gamma(A_k)} \, \mathrm{d}x, \tag{8}$$

where $A_k = \sum_{j=1}^{k} a_j$. At this juncture we now prove the appropriate theorem in the context of DMTTF and IMTTF distributions.

Theorem 5. *Let the survival probability $\bar{P}_k$ be as defined in (8) for $k = 1, 2, \ldots$ and $\bar{P}_0 = 1$, where the damage distributions F_k are given by (7). If $A_k = k + 1$ and $\bar{G}$ is DMTTF (IMTTF) then $\bar{P}_k$ is DMTTF (IMTTF).*

Proof. Now

$$\begin{aligned}
\sum_{j=0}^{k-1} \bar{P}_j &= b \sum_{j=0}^{k-1} \int_0^{\infty} \bar{G}(x) \, \mathrm{e}^{-bx} \frac{(bx)^j}{j!} \, \mathrm{d}x \\
&= b \int_0^{\infty} \bar{G}(x) \sum_{j=0}^{k-1} \mathrm{e}^{-bx} \frac{(bx)^j}{j!} \, \mathrm{d}x \\
&= b \int_0^{\infty} \bar{G}(x) \left(b \int_x^{\infty} \mathrm{e}^{-by} \frac{(by)^{k-1}}{(k-1)!} \, \mathrm{d}y \right) \mathrm{d}x \\
&= b \int_0^{\infty} \left(\int_0^x \bar{G}(y) \, \mathrm{d}y \right) \mathrm{e}^{-bx} \frac{(bx)^{k-1}}{(k-1)!} \, \mathrm{d}(bx).
\end{aligned}$$

Thus, for $c \in \mathbb{R}$,

$$\sum_{j=0}^{k-1} \bar{P}_j - cP_k = b \int_0^{\infty} \left[\int_0^x \bar{G}(y) \, \mathrm{d}y - \frac{c}{b} G(x) \right] \mathrm{e}^{-bx} \frac{(bx)^{k-1}}{(k-1)!} \, \mathrm{d}(bx).$$

Note that $\int_0^x \bar{G}(y) \, \mathrm{d}y - (c/b)G(x)$ has at most one change of sign in $x \geq 0$, and if one change of sign occurs, it is from $+$ to $-$. Again we know that the kernel $\mathrm{e}^{-bx}(bx)^{k-1}/(k-1)!$ is TP_{∞} in $x \geq 0$. Now using the variation diminishing property of totally positive functions, the same conclusion can be argued for $\sum_{j=0}^{k} \bar{P}_j - cP_k$. This implies that ${P_k}^{-1} \sum_{j=0}^{k-1} \bar{P}_j$ is decreasing in k. Hence $\bar{P}_k$ is DMTTF. The proof in the context of the dual class is similar. □

3. Weak convergence of IDMTTF distributions

Suppose $\mathcal{C}$ denotes a specific class of life distributions. Let $F_n, \ n = 1, 2, \ldots$ be a sequence of life distributions belonging to $\mathcal{C}$ and $F_n \to F$ in law. If $F \in \mathcal{C}$, we say that $\mathcal{C}$ is closed under

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the formation of weak limits. This has been established in the context of monotonic ageing classes such as IFR (Basu and Simons [10]), HNBUE (Basu and Bhattacharjee [9]) etc. Similarly, in the case of non-monotonic ageing classes, weak convergence of the NWBUE, BFR and IDMRL families have been studied (see Mitra and Basu [41,42] and Anis [5]). We investigate the above question for the IDMTTF class. Suppose

$$x^*(F) = \inf\{t > 0 \colon F(t) > 0\} \geq 0$$

denotes the lower end point of the support of a distribution $F \in L^1$ on $[0, \infty)$. We start by proving an important theorem which will be used to prove the main theorem of this section.

Theorem 6. *Suppose F_n, $n = 1, 2, \ldots$ is a sequence of life distributions. Assume that $F_n \longrightarrow F$ in law, where F is a continuous distribution function. Then $M_{F_n}(x) \to M_F(x)$ at every point $x > x^*(F)$ as $n \to \infty$.*

Proof. Choose any $x > x^*(F)$. Then there exists $n_0(x)$ such that $n \geq n_0(x)$ implies $F_n(x) > 0$. Now the corresponding MTTF function given by

$$M_{F_n}(x) = \frac{1}{F_n(x)} \int_0^x \bar{F}_n(u)\, \mathrm{d}u, \quad n \geq n_0(x)$$

exists and is finite for every such x, since $F_n(x)$ is positive and $\bar{F}_n(u) I_{(0,\, x)}(u)$ is bounded by an integrable function $g(u)$ which is given by

$$g(u) = \begin{cases} 1 & \text{when } u \in (0, x), \\ 0 & \text{otherwise,} \end{cases}$$

where I denotes an indicator function. Now, by Lebesgue's dominated convergence theorem, we get

$$\lim_{n \to \infty} \int_0^x \bar{F}_n(u)\, \mathrm{d}u = \int_0^x \bar{F}(u)\, \mathrm{d}u,$$

so that

$$M_{F_n}(x) = \frac{1}{F_n(x)} \int_0^x \bar{F}_n(u)\, \mathrm{d}u \to \frac{1}{F(x)} \int_0^x \bar{F}(u)\, \mathrm{d}u = M_F(x) \quad \text{as } n \to \infty.$$

So $M_{F_n}(x)$ converges to $M_F(x)$ at every point $x > x^*(F)$ as $n \to \infty$. □

Next we apply this theorem to prove closure under weak convergence for IDMTTF distributions.

Theorem 7. *Let F_n, $n = 1, 2, \ldots$ be a sequence of IDMTTF life distributions with change point t_{0n}. Suppose $F_n \longrightarrow F$ in law, where F is a continuous distribution function. Then F is IDMTTF.*

Proof. For an IDMTTF distribution G, we define

$$T_G = \{t_0 \colon M_G(t) \text{ is increasing in } (0, t_0) \text{ and decreasing in } [t_0, \infty)\}.$$

Let $t_{0n} \in T_{F_n}$, $n = 1, 2, \ldots$ and consider the following two cases.

Case I: $\{t_{0n}\}_{n=1}^{\infty}$ is bounded. Since $\{t_{0n}\}$ is bounded, applying the Bolzano–Weierstrass theorem we get a subsequence $\{t_{0n_k}\}$ of $\{t_{0n}\}$ converging to some finite limit α as $k \to \infty$. So for a given $\epsilon > 0$ there exists $k_0 \geq 1$ such that

$$\alpha - \epsilon < t_{0n_k} < \alpha + \epsilon \quad \text{for all } k \geq k_0.$$

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Choose any $x, y > 0$ such that $x < y < \alpha - \epsilon$. Since $M_{F_{n_k}}(t)$ is increasing in $t \in (0, t_{0n_k})$, we have $M_{F_{n_k}}(x) \le M_{F_{n_k}}(y)$. Taking limits of both sides as $k \to \infty$ and using Theorem 6, we get $M_F(x) \le M_F(y)$. Hence $M_F(t)$ is increasing in $(0, \alpha - \epsilon)$. An analogous argument shows that $M_F(t)$ is decreasing in $(\alpha + \epsilon, \infty)$. Consequently, $\alpha \in T_F$, since $\epsilon > 0$ is arbitrary. Thus F is IDMTTF with α as one of its change points.

Case II: $\{t_{0n}\}_{n=1}^{\infty}$ *is unbounded.* Since $\{t_{0n}\}$ is unbounded, there exists a subsequence $\{t_{0n_k}\}$ such that $t_{0n_k} \to \infty$ as $k \to \infty$. Choose any $x, y > 0$ such that $x < y < \infty$. Then there exists $k_0 \ge 1$, sufficiently large, such that

$$x \le y < t_{0n_k} \quad \text{for all } k \ge k_0.$$

Now $M_{F_{n_k}}(t)$ is increasing in $t \in (0, t_{0n_k})$ implies $M_{F_{n_k}}(x) \le M_{F_{n_k}}(y)$. Thus $M_F(x) \le M_F(y)$ on taking limits as $k \to \infty$ and using Theorem 6 once again. Hence $M_F(t)$ is IMTTF, i.e. IDMTTF with change point at infinity. □

As a by-product of the proof of Theorem 7, we get the following theorem.

Theorem 8. *Let F_n, $n = 1, 2, \ldots$ be a sequence of IDMTTF distributions with unique change points t_{0n}. Suppose $F_n \longrightarrow F$ in law, where F is a continuous distribution function having unique change point t_0. Then* $\lim_{n\to\infty} t_{0n} = t_0 (\le \infty)$.

Proof. If $\{t_{0n}\}$ is bounded above then the proof follows trivially as in Case I of the proof of Theorem 7 as $\alpha = t_0$, since F has a unique change point. From Case II in the proof of Theorem 7, we get that if $\{t_{0n}\}$ is unbounded then F is necessarily an IMTTF distribution and hence it has a unique change point at infinity. Now suppose that $\{t_{0n}\}$ does not converge to infinity. Then there exists $B > 0$ such that $t_{0n} \le B$ infinitely often, so that one can choose a subsequence $\{t_{0n_k}\}$ converging to a finite limit β, say. But Case I in the proof of Theorem 7 and uniqueness of the change point of F shows that $\infty = \beta < \infty$, a contradiction. So in this case $t_{0n} \to \infty$ as $n \to \infty$, thus establishing our claim. □

Theorem 9. *Suppose F_n, $n = 1, 2, \ldots$ is a sequence of IDMTTF life distributions with mean μ_n and change point t_{0n}. Suppose that both the sequences $\{\mu_n\}_{n=1}^{\infty}$ and $\{t_{0n}\}_{n=1}^{\infty}$ are bounded and $F_n \to F$ in law where F is a continuous distribution function. Then, for every $r > 0$,*

$$\lim_{n\to\infty} \int_0^{\infty} x^r \, \mathrm{d}F_n(x) = \int_0^{\infty} x^r \, \mathrm{d}F(x).$$

Proof. From Theorem 5 of Izadi *et al.* [26], we get that for all $n = 1, 2, \ldots, F_n$ is NWBUE with change point τ_{0n} (in the NWBUE sense), say, and $\tau_{0n} < t_{0n}$ since F_n is IDMTTF with change point t_{0n} (in the IDMTTF sense). By Theorem 7, F is IDMTTF and hence F is NWBUE. Now, applying Theorem 4.1 of Mitra and Basu [41], we get the desired result. □

Theorem 10. *Let F be IDMTTF with finite change point t_0 and finite mean μ. Then F is uniquely determined by its moment sequence.*

Proof. Direct application of Theorem 5 of Izadi *et al.* [26] shows that F is also NWBUE with change point x_0 (in the NWBUE sense), say, and $x_0 < t_0$. Hence using Corollary 3.5 (iv) of Mitra and Basu [41], we get that for all $r \ge 1$,

$$\mathbb{E}_F(X^r) := \int_0^{\infty} x^r \, \mathrm{d}F(x) \le \mu^r \Gamma(r+1) \, \mathrm{e}^{(x_0/\mu)},$$

from which it follows that the power series $\sum_{r=0}^{\infty} (u^r/r!)\mathbb{E}_F(X^r)$ has a non-null radius of convergence. The theorem now follows from a result of Loève [40, page 217]. □

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Theorem 9 implies that under some conditions, weak convergence of a sequence of IDMTTF distributions establishes convergence of moments of the sequence of distributions to the corresponding moments of the limiting distribution. The following theorem explores the converse of Theorem 9. □

Theorem 11. *Let* $F_n,\ n = 1, 2, \ldots$ *be a sequence of IDMTTF life distributions with change point* $t_{0n} < \infty$ *for all* $n \geq 1$ *and suppose that* F *is IDMTTF with finite change point* t_0 *such that, for all integers* $r \geq 1$,

$$\lim_{n\to\infty} \int_0^\infty x^r \, \mathrm{d}F_n(x) = \int_0^\infty x^r \, \mathrm{d}F(x). \tag{9}$$

Then $F_n \to F$ *in law.*

Proof. By virtue of Theorem 10 and (9), the limiting distribution of every weakly convergent subsequence of $\{F_n\}_{n=1}^\infty$ happens to be necessarily F, which completes the proof. □

Example 1. Let

$$\bar{F}(x) = \frac{1}{(x^2+1)}, \quad x > 0$$

be a survival function of a life distribution. Here the MTTF function is of the form $(1 + 1/x^2)\tan^{-1} x$, which is strictly decreasing in (0,2) and is increasing in $[2, \infty)$. Hence F is DIMTTF with finite change point 2. The mean of this distribution is $\pi/2 < \infty$, and for $r > 1$, $\mathbb{E}_F(X^r) = \infty$. Hence this distribution has a finite mean and finite change point but does not possess finite moments of any order greater than one. Hence, in the context of DIMTTF distributions, versions of Theorems 9, 10 and 11 are not meaningful.

4. Basic closure properties and interrelationships

The closure properties of interest concerning ageing classes mainly involve the following.

(a) Formation of coherent systems consisting of independent components.

(b) Mixtures of distributions.

(c) Addition of life lengths (convolution of distributions).

The reliability of a system of components depends on reliability of each component and also on the structure function of the system. In a coherent system, each working component is beneficial to the system. A simple example of a coherent system is a k-out-of-n system, which is a very popular structure of redundancy in fault-tolerant systems. Series and parallel systems arise as special cases of k-out-of-n systems. For more details, one may refer to Lai and Xie [38], Barlow and Proschan [8] and the references therein. In the context of the above-mentioned reliability operations, Park [45] summarized the preservation properties of several well-known classes of life distributions (see Table 10.1 in Park [45], which was itself an update of the original table given by Barlow and Proschan [8]). The major parts of this section are devoted to investigating the preservation of the IDMTTF property under some of these reliability operations, as well as to place in proper perspective the position of the IDMTTF class in the hierarchy of non-monotonic ageing families.

We now present the following result, which deals with the formation of series structure where the components are all IDMTTF having a common absolutely continuous distribution.

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Theorem 12. *Consider a series system with i.i.d. component lifetimes having a common absolutely continuous distribution function F which is IDMTTF with change point t_0. Then the MTTF function of the system is increasing in $(0, t_0]$ and decreasing in (t', ∞) for some $t_0 \le t' (\le \infty)$.*

Proof. Let $F(x)$ and $f(x)$, respectively, denote the CDF and PDF of F. Then the survival function and PDF of the lifetime of the system are $\bar{F}^n(x)$ and $nf(x)\bar{F}^{n-1}(x)$ respectively. Now the MTTF function of the system is given by

$$\frac{\int_0^t \bar{F}^n(x)\,\mathrm{d}x}{1-\bar{F}^n(t)}.$$

To prove the theorem it is enough to show that

$$(1-\bar{F}^n(t))\bar{F}^n(t) - nf(t)\bar{F}^{n-1}(t)\int_0^t \bar{F}^n(x)\,\mathrm{d}x \begin{cases} \ge 0 & \text{for } t \le t_0, \\ \le 0 & \text{for } t > t'. \end{cases}$$

Since $F(t)$ is IDMTTF with change point t_0, we have

$$F(t)\bar{F}(t) - f(t)\int_0^t \bar{F}(x)\,\mathrm{d}x \begin{cases} \ge 0 & \text{for } t \le t_0, \\ \le 0 & \text{for } t > t_0. \end{cases} \tag{10}$$

Now integrating the expression before the parenthesis in (10) yields

$$\int_0^u \left[F(t)\bar{F}(t) - f(t)\int_0^t \bar{F}(x)\,\mathrm{d}x\right]\mathrm{d}t = [1+\bar{F}(u)]\int_0^u \bar{F}(x)\,\mathrm{d}x - 2\int_0^u \bar{F}^2(x)\,\mathrm{d}x.$$

Similarly integrating $(n-1)$ times, we get

$$[1+\bar{F}(u)+\bar{F}^2(u)+\cdots+\bar{F}^{n-1}(u)]\int_0^u \bar{F}(x)\,\mathrm{d}x - n\int_0^u \bar{F}^n(x)\,\mathrm{d}x. \tag{11}$$

Now from (10) and (11) we can say that $\exists\ t' \ge t_0$ such that

$$[1+\bar{F}(t)+\bar{F}^2(t)+\cdots+\bar{F}^{n-1}(t)]\int_0^t \bar{F}(x)\,\mathrm{d}x \begin{cases} \ge n\displaystyle\int_0^t \bar{F}^n(x)\,\mathrm{d}x & \text{for } t < t', \\ \le n\displaystyle\int_0^t \bar{F}^n(x)\,\mathrm{d}x & \text{for } t \ge t'. \end{cases} \tag{12}$$

Note that here $t' = \infty$ is also a possibility. An application of (12) yields

$$(1-\bar{F}^n(t))\bar{F}^n(t) - nf(t)\bar{F}^{n-1}(t)\int_0^t \bar{F}^n(x)\,\mathrm{d}x \begin{cases} \ge \dfrac{(1-\bar{F}^n(t))\bar{F}^{n-1}(t)}{F(t)}W(t) & \text{for } t < t', \\ \le \dfrac{(1-\bar{F}^n(t))\bar{F}^{n-1}(t)}{F(t)}W(t) & \text{for } t \ge t', \end{cases}$$

where

$$W(t) := F(t)\bar{F}(t) - f(t)\int_0^t \bar{F}(x)\,\mathrm{d}x,$$

which proves the theorem since F is IDMTTF with change point t_0. □

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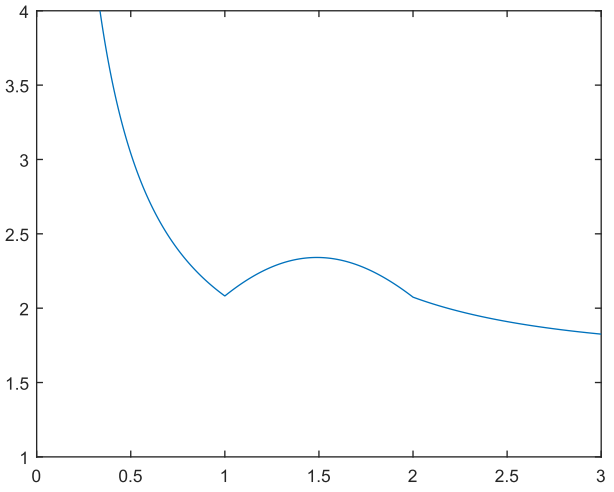


FIGURE 1. MTTF function of G.

The above theorem demonstrates that the MTTF function of a series system having i.i.d. IDMTTF components is initially increasing in $(0, t_0]$ but remains indeterminate in (t_0, t'). However, the question of closure under the formation of series structure still remains open for IDMTTF distributions. The following example shows that the IDMTTF property is not preserved under the formation of parallel systems.

Example 2. Consider the life distribution given by the following survival function:

$$\bar{F}_1(x) = \begin{cases} \mathrm{e}^{-x} & 0 < x \le 1, \\ \dfrac{16}{15e}\left(1 - \dfrac{x^3}{16}\right) & 1 \le x < 2, \\ \dfrac{8}{15}\,\mathrm{e}^{-x+1} & x \ge 2. \end{cases} \tag{13}$$

Izadi *et al.* [26] showed that F_1 is IDMTTF with change point $\tau = 1.7389$. Now we examine the behaviour of a system consisting of two such components connected in parallel. Then the CDF of the parallel system is given by $G(x) = (F_1(x))^2$ and the corresponding MTTF function is given by

$$M_G(x) = \frac{1}{(F_1(x))^2} \int_0^x (1 - (F_1(u))^2)\, \mathrm{d}u.$$

Note that $M_G(0.5) = 3.0415$, $M_G(1) = 2.0820$, $M_G(1.5) = 2.3409$ and $M_G(2) = 2.0741$, which reveal clearly that G does not possess the IDMTTF property. The MTTF function of G is displayed in Figure 1.

Remark 1. Example 2 also shows that IDMTTF property is not preserved under the formation of k-out-of-n systems as well as coherent systems.

The next example illustrates that the mixture of two IDMTTF distributions need not be IDMTTF.

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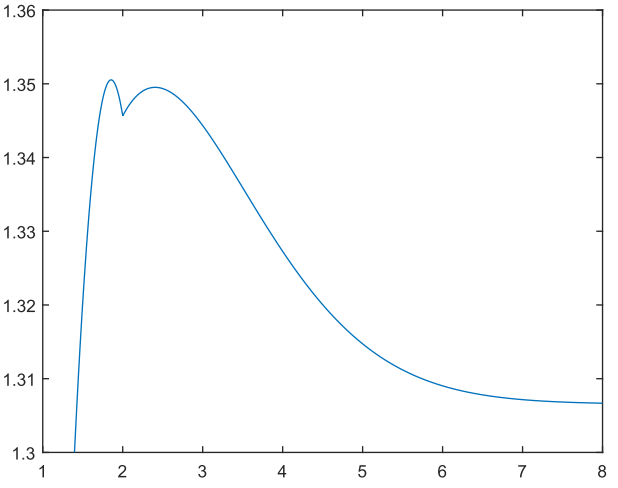


FIGURE 2. MTTF function of H.

Example 3. Consider the life distribution given by the survival function

$$\bar{H}(x) = \frac{1}{2}\{\bar{F}_1(x) + \bar{F}_2(x)\}.$$

This is a mixture of the survival functions $\bar{F}_1(x)$ defined in (13) and $\bar{F}_2(x)$ defined by

$$\bar{F}_2(x) = \begin{cases} \dfrac{1}{(1+x)} & 0 < x \le 1, \\ \dfrac{1}{2}\,\mathrm{e}^{(x-1)/2} & 1 \le x < 2, \\ \dfrac{1}{2}\,\mathrm{e}^{-x^2/8} & x \ge 2. \end{cases}$$

Mitra and Basu [42] showed that F_2 is a BFR distribution (see Definition 5 below) and every $x \in [1, 2]$ is a change point of F_2. Again Izadi *et al.* [26] showed that the BFR class is contained in the IDMTTF class of life distributions. Consequently F_2 is a member of the IDMTTF class having change point $\tau = 2.52914$. As already mentioned in Example 2, $\bar{F}_1(x)$ is IDMTTF with change point $\tau = 1.7389$. Now the MTTF function corresponding to H is given by

$$M_H(x) = \frac{1}{\bar{H}(x)} \int_0^x \bar{H}(u)\,\mathrm{d}u.$$

Here the values $M_H(1.85) = 1.3505$, $M_H(2) = 1.3456$, $M_H(2.5) = 1.3494$ and $M_H(7.5) = 1.3068$ clearly indicate that H is not IDMTTF. The MTTF function of H is plotted in Figure 2.

At this juncture we present an interesting observation in the following theorem, though it is somewhat outside the main focus of this section.

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Theorem 13. *If F is an IMTTF distribution, then the reversed hazard rate of the equilibrium distribution of F is decreasing.*

Proof. The equilibrium distribution of F is given by

$$F_{(1)}(t) = \frac{1}{\mu}\int_0^t \bar{F}(u)\,\mathrm{d}u.$$

Then its reversed hazard rate $r_{F_{(1)}}(t)$ equals

$$\frac{\bar{F}(t)}{\int_0^t \bar{F}(u)\,\mathrm{d}u}.$$

Now the MTTF function of F can be written as

$$M_F(t) = \frac{\int_0^t \bar{F}(u)\,\mathrm{d}u}{F(t)} = \frac{\bar{F}(t)/F(t)}{r_{F_{(1)}}(t)}.$$

The theorem follows easily using the fact that $\bar{F}(t)/F(t)$ is decreasing and F is IMTTF. □

Now we shall attempt to study the interrelationships between various notions of non-monotonic ageing, namely the BFR class of Glaser [23], the IDMRL class of Guess *et al.* [24], the NWBUE class of Mitra and Basu [41] and the IDMTTF class of Izadi *et al.* [26]. Before proceeding further, we recapitulate the definitions of the above-mentioned non-monotonic ageing classes.

Definition 5. A life distribution F is called a bathtub failure rate (BFR) (upside-down bathtub failure rate (UBFR)) distribution if there exists a point $t_0 \geq 0$ such that $-\ln \bar{F}(t)$ is concave (convex) on $[0, t_0)$ and convex (concave) on $[t_0, \infty)$. The point t_0 is referred to as a change point (or turning point) of the distribution in the BFR (UBFR) sense.

Definition 6. A life distribution F with a finite first moment is called an increasing then decreasing mean residual life (IDMRL) (decreasing then increasing mean residual life (DIMRL)) distribution if there exists a point $\tau \geq 0$ such that $e_F(t)$ is increasing (decreasing) on $[0, \tau)$ and decreasing (increasing) on $[\tau, \infty)$, where

$$e_F(t) := \mathbb{E}(X - t \mid X > t) = \begin{cases} \displaystyle\int_t^\infty \bar{F}(u)\,\mathrm{d}u/\bar{F}(t) & \text{if } \bar{F}(t) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The point τ is referred to as a change point (or turning point) of the distribution in the IDMRL (DIMRL) sense.

Definition 7. A life distribution F with a finite first moment is called *new worse then better than used in expectation* (NWBUE) (*new better then worse than used in expectation* (NBWUE)) if there exists a point $x_0 \geq 0$ such that $e_F(t) \geq e_F(0)$ for $[0, x_0)$ and $e_F(t) \leq e_F(0)$ for $[x_0, \infty)$, where $e_F(t)$ is defined in (14). The point x_0 is referred to as a *change point (or turning point)* of the distribution in the NWBUE (NBWUE) sense.

Mitra and Basu [41] proved that the NWBUE family of life distributions is strictly larger than both IDMRL and BFR classes. An analogous chain of implications holds for the dual classes, i.e. {UBFR} ⊂ {NBWUE} and {DIMRL} ⊂ {NBWUE}. Izadi *et al.* [26] showed that {BFR} ⊂ {IDMTTF} ⊂ {NWBUE} and {UBFR} ⊂ {DIMTTF} ⊂ {NBWUE}. We now explore

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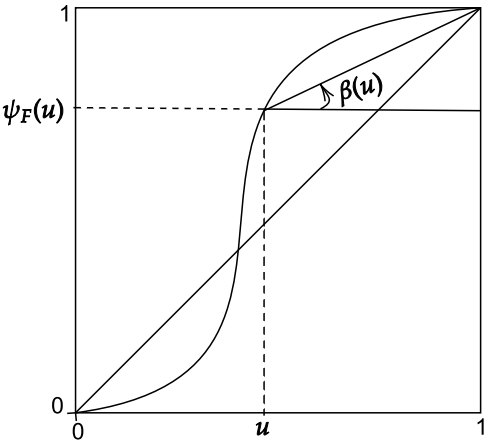


FIGURE 3. Scaled TTT transform of a distribution which is BFR, IDMTTF and IDMRL.

the relation between the BFR and IDMRL classes of distributions using the total time on test (TTT) transform as a tool.

The scaled TTT transform of a life distribution F is defined by

$$\psi_F(u) = \frac{\int_0^{F^{-1}(u)} \bar{F}(x)\,\mathrm{d}x}{\mu}, \quad u \in [0, 1],$$

where $F^{-1}(u) = \inf\{x\colon F(x) \geq u\}$. The TTT transform plays an important role in characterizing ageing classes of life distributions. Our goal is to show that the BFR class is contained in the IDMRL family of life distributions. To this end, we first recall the following two characterization theorems of Deshpande and Suresh [16].

Theorem 14. (Deshpande and Suresh [16].) *A life distribution F is BFR if and only if there exists $u_0 \in [0, 1]$ such that $\psi_F(u)$ is convex on $[0, u_0)$ and concave on $[u_0, 1]$.*

Theorem 15. (Deshpande and Suresh [16].) *A life distribution F is IDMRL if and only if there exists $\tilde{u}_0 \in [0, 1)$ such that $(1 - \psi_F(u))/(1 - u)$ is increasing on $[0, \tilde{u}_0)$ and decreasing on $[\tilde{u}_0, 1)$.*

Figure 3 shows the plot of the scaled TTT transform of a distribution which is both IDMTTF and BFR. Geometrically, from Theorem 15, a distribution is IDMRL if and only if the angle $\beta(u)$ is increasing and then decreasing on $(0, 1)$. We next prove a key lemma which will be useful in establishing the main result.

Lemma 4. *Let $x^*, x^{**} \in [0, 1)$ and $f\colon [x^*, x^{**}] \to \mathbb{R}^+$ be a convex function on $[x^*, x^{**}]$. Suppose $S_f = \{a\colon f(x) - 1 + a - ax = 0$ has at least one root in $[x^*, x^{**}]\}$. Now define*

$$\tilde{a} = \max S_f,$$

$$\tilde{x}_1 = \min\{x\colon x \text{ is the root of } f(x) - 1 + \tilde{a} - \tilde{a}x = 0 \text{ in } [x^*, x^{**}]\},$$

$$\tilde{x}_2 = \max\{x\colon x \text{ is the root of } f(x) - 1 + \tilde{a} - \tilde{a}x = 0 \text{ in } [x^*, x^{**}]\}.$$

(i) *If $\tilde{x}_2 = x^{**}$, then $(1 - f(x))/(1 - x)$ is increasing on $[x^*, x^{**})$.*

(ii) *If $\tilde{x}_1 = x^*$, then $(1 - f(x))/(1 - x)$ is decreasing on $(x^*, x^{**}]$.*

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(iii) *If* $x^* < \tilde{x}_1 \leq \tilde{x}_2 < x^{**}$, *then* $(1-f(x))/(1-x)$ *is increasing on* $[x^*, \tilde{x})$ *and decreasing on* $[\tilde{x}, x^{**}]$ *where* $\tilde{x}$ *is any value in* $[\tilde{x}_1, \tilde{x}_2]$.

Proof. It is easy to see that S_f is a non-empty set. We now prove this lemma by contradiction.

(i) Suppose that $(1-f(x))/(1-x)$ is not increasing on $[x^*, x^{**})$. Then there exists x_1, x_2 such that $x^* \leq x_1 < x_2 < x^{**}$ and

$$a_1 = \frac{1-f(x_1)}{1-x_1} > \frac{1-f(x_2)}{1-x_2} = a_2.$$

Now convexity of f implies that

$$\frac{a_2(1-x_2)-(1-f(\tilde{x}_2))}{\tilde{x}_2-x_2} = \frac{f(\tilde{x}_2)-f(x_2)}{\tilde{x}_2-x_2} \geq \frac{f(\tilde{x}_2)-f(x_1)}{\tilde{x}_2-x_1} > \frac{a_2(1-x_1)-(1-f(\tilde{x}_2))}{\tilde{x}_2-x_1}.$$

Some lengthy algebraic calculations now yield $(1-f(\tilde{x}_2))/(1-\tilde{x}_2) < a_2$, i.e. $\tilde{a} < a_2$. This contradicts the definition of $\tilde{a}$.

(ii) Suppose that $(1-f(x))/(1-x)$ is not decreasing on $(x^*, x^{**}]$. Then there exists x_1, x_2 such that $x^* < x_1 < x_2 \leq x^{**}$ and

$$a_1 = \frac{1-f(x_1)}{1-x_1} < \frac{1-f(x_2)}{1-x_2} = a_2.$$

Now convexity of f yields that

$$\frac{a_1(1-x_1)-(1-f(\tilde{x}_1))}{\tilde{x}_1-x_1} = \frac{f(\tilde{x}_1)-f(x_1)}{\tilde{x}_1-x_1} \leq \frac{f(\tilde{x}_1)-f(x_2)}{\tilde{x}_1-x_2} < \frac{a_1(1-x_2)-(1-f(\tilde{x}_1))}{\tilde{x}_1-x_2}.$$

Algebraic manipulations now yield $(1-f(\tilde{x}_1))/(1-\tilde{x}_1) < a_1$, i.e. $\tilde{a} < a_1$. Once again we arrive at a contradiction to the definition of $\tilde{a}$.

(iii) The result follows by applying parts (i) and (ii) to $[x^*, \tilde{x}_1]$ and $[\tilde{x}_1, x^{**}]$, respectively, and also applying parts (i) and (ii) to $[x^*, \tilde{x}_2]$ and $[\tilde{x}_2, x^{**}]$, respectively. □

Theorem 16. *If a life distribution* F *is BFR with change point* t_0, *then* F *is IDMRL with a change point* $\tau \leq t_0$.

Proof. As F is BFR, Theorem 14 guarantees the existence of $u_0 \in [0, 1]$ such that $\psi_F(u)$ is convex on $[0, u_0)$ and concave on $[u_0, 1]$.

Case I: $u_0 = 0$. In this case $\psi_F(u)$ is concave on $[0, 1]$, i.e. F is IFR and hence F is DMRL. So F is IDMRL with change point $\tau = 0$.

Case II: $u_0 = 1$. In this case $\psi_F(u)$ is convex on [0,1), i.e. F is DFR and hence F is IMRL. So F is IDMRL with a change point at infinity.

Case III: $0 < u_0 < 1$. Lemma 2.1 of Suresh [51] shows that the mean residual life (MRL) function $e_F(t)$ is decreasing for $t \geq F^{-1}(u_0)$. Now it is easy to show that $(1-\psi_F(u))/(1-u)$ is decreasing on $[u_0, 1)$ using an argument similar to the one in the proof of Theorem 2.5 of Klefsjö [32]. Note that using a slight modification of the arguments in the proof of Theorem 3.2 in Deshpande and Suresh [16], we conclude that $\psi_F(u)$ is a convex, positive function on $[0, u_0]$. Now in Lemma 4, we take $f = \psi_F$, $x^* = 0$ and $x^{**} = u_0$.

Subcase i: $0 < \tilde{x}_1 \leq \tilde{x}_2 < u_0$. By (iii) in Lemma 4, there exists $\tilde{u} \in [x^*, x^{**}] = [0, u_0]$ such that $(1-\psi_F(u))/(1-u)$ is increasing on $[0, \tilde{u})$ and decreasing on $[\tilde{u}, u_0]$. Now applying Theorem 15 we get that F is IDMTTF with change point $\tau = F^{-1}(\tilde{u})$.

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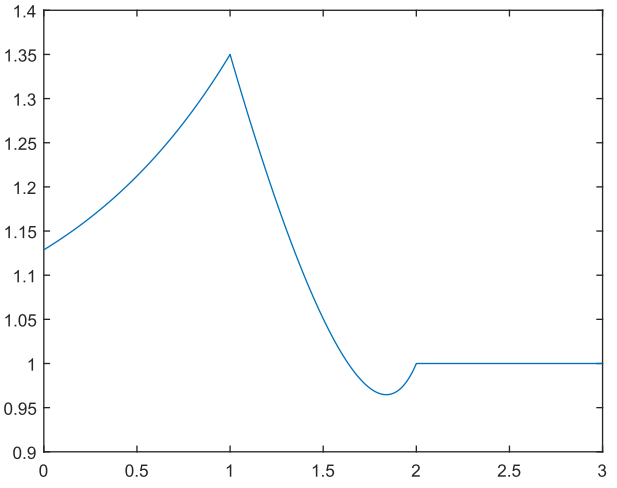


FIGURE 4. MRL function of F_1.

Subcase ii: $\tilde{x}_1 = 0 = x^*$. By (ii) in Lemma 4, we get that $(1 - \psi_F(u))/(1 - u)$ is decreasing in $(0, u_0]$. Hence $(1 - \psi_F(u))/(1 - u)$ is decreasing in [0,1). Now Theorem 2.5 of Klefsjö [32] shows that F DMRL, i.e. F is IDMRL with change point $\tau = 0$.

Subcase iii: $\tilde{x}_2 = x^{**} = u_0$. By (i) in Lemma 4, we get that $(1 - \psi_F(u))/(1 - u)$ is increasing in $[0, u_0)$. Hence applying Theorem 15 we get that F is IDMRL with change point $\tau = F^{-1}(u_0)$.

In all the above cases $\tilde{u} \le u_0$, i.e. $\tau \le t_0$, which completes the proof. □

Thus the IDMTTF and IDMRL classes both contain the BFR family of life distributions and are contained in the NWBUE class. A natural question arises: Does any one of the IDMTTF and IDMRL properties imply the other? The following two examples answer this question in the negative.

Example 4. Consider the IDMTTF distribution given by the survival function $\bar{F}_1$ in (13). Denoting the MRL function of F_1 by $e_{F_1}(x)$, we have $e_{F_1}(0.5) = 1.2123$, $e_{F_1}(1) = 1.3500$, $e_{F_1}(1.8) = 0.9662$ and $e_{F_1}(2.5) = 1$, which clearly demonstrate that F_1 does not possess the IDMRL property. The MRL function of F_1 is plotted in Figure 4.

Example 5. Consider the life distribution given by the following survival function:

$$\bar{G}(x) = \begin{cases} \dfrac{1}{(1+x)^2} & 0 < x \le 1, \\ \dfrac{\mathrm{e}^{(1-x)/2x}}{4x^2} & 1 \le x < 2, \\ \dfrac{18 - x}{2^8\, \mathrm{e}^{1/4}} & 2 \le x < 10, \\ \dfrac{(6 - x/5)^4}{2^{13}\, \mathrm{e}^{1/4}} & 10 \le x < 25, \\ \dfrac{\mathrm{e}^{3/4}}{2^{13}}\, \mathrm{e}^{x-25-\mathrm{e}^{x-25}} & x \ge 25. \end{cases}$$

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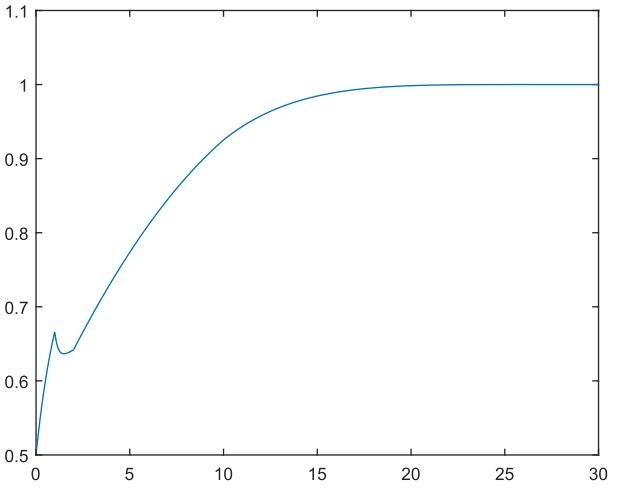


FIGURE 5. MTTF function of G.

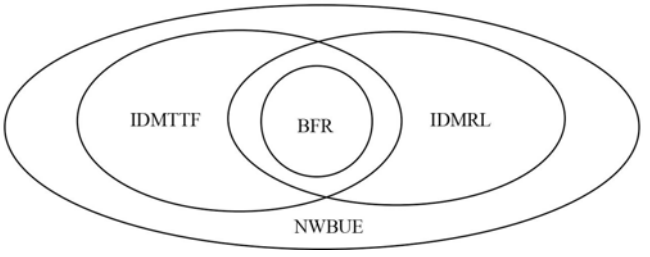


FIGURE 6. The relationships between certain non-monotonic classes of life distributions.

Anis [5] showed that G is IDMRL with change point 2. Now the MTTF function of G is denoted by $M_G(x)$. Note that $M_G(0.9) = 0.6552$, $M_G(1.5) = 0.6366$, $M_G(2) = 0.6418$ and $M_G(9) = 0.9015$, which clearly indicate that G does not have the IDMTTF property. Figure 5 shows the MTTF function of G.

Combining our findings together with the results of Izadi *et al.* [26] and Mitra and Basu [41], we provide in Figure 6 diagrammatic representation of interrelationships between the BFR, IDMRL, IDMTTF and NWBUE classes of life distributions.

Using analogous arguments, similar diagrammatic representation of interrelationships between the UBFR, DIMRL, DIMTTF and NBWUE classes of life distributions can be obtained.

5. Discussion

The first part of this work focuses on developing some shock model theory for ageing classes (both monotonic and non-monotonic) based on the MTTF function. However, a lot of work still remains to be done, for example a general counting process of shock arrivals for the monotonic cases and pure-birth shock models for the non-monotonic scenario remain to be considered.

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More general shock models, where failure occurs due to shocks as well as continuous wear and tear, provide scope for further research.

The authors feel that the weak convergence issues have been adequately dealt with in Section 3. Regarding Section 4, it is pertinent to mention that the question of closure under convolutions still remains open for IDMTTF distributions.

At this juncture, some comments would be in order on the subject of interrelationships between the various classes of life distributions discussed. Suppose {BFRs}, {IDMRLs} and {IDMTTFs} denote the respective families of non-monotonic ageing classes in the strict sense while {BFR} is used to denote the family of bathtub distributions as defined in Lai and Xie [38] (see Definition 3.3, page 74), i.e. $\{\text{BFR}\} = \{\text{BFRs}\} \cup \{\text{IFR}\} \cup \{\text{DFR}\}$. In this context, it is worth noting that the same spirit has been reflected in Definition 1.1 of Guess *et al.* [24] and Definition 1 of Izadi *et al.* [26], which indicate that $\{\text{IDMRL}\} = \{\text{IDMRLs}\} \cup \{\text{DMRL}\} \cup \{\text{IMRL}\}$ and $\{\text{IDMTTF}\} = \{\text{IDMTTFs}\} \cup \{\text{DMTTF}\} \cup \{\text{IMTTF}\}$ respectively. An example in Suresh [51] actually shows that {BFRs} may not imply {IDMRLs}. But our Theorem 16 shows that the {IDMRL} class contains {BFR}. Even if one chooses to define the {BFR} class excluding the DFR distributions (and naturally the {IDMRL} class excluding all IMRL distributions), our theorem still remains valid.

For future work it is well worth investigating the existence of any one-to-one correspondence between a CDF and its MTTF function. It would also be interesting to determine sharp bounds on the survival function $\bar{F}(t)$ for a given t, assuming that the life distribution is a member of some class based on the MTTF function, perhaps with certain additional specifications (see e.g. Khan and Mitra [29] and the references therein). It would also be interesting to develop nonparametric tests for comparing two MTTF functions, perhaps utilizing the intersection-union principle (see e.g. Bhattacharyya *et al.* [14] and Berger *et al.* [12]).

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References

[1] A-Hameed, M. and Proschan, F. (1973). Nonstationary shock models. *Stoch. Process. Appl.* **1**, 383–404.

[2] A-Hameed, M. and Proschan, F. (1975). Shock models with underlying birth process. *J. Appl. Prob.* **12**, 18–28.

[3] Abouammoh, A., Hindi, M. and Ahmed, A. (1988). Shock models with NBUFR and NBAFR survivals. *Trabajos de Estadistica* **3**, 97–113.

[4] Anderson, K. K. (1987). Limit theorems for general shock models with infinite mean intershock times. *J. Appl. Prob.* **24**, 449–456.

[5] Anis, M. (2012). On some properties of the IDMRL class of life distributions. *J. Statist. Planning Infer.* **142**, 3047–3055.

[6] Asha, G. and Nair, U. (2010). Reliability properties of mean time to failure in age replacement models. *Internat. J. Reliab. Quality Safety Eng.* **17**, 15–26.

[7] Barlow, R. E. and Proschan, F. (1965). *Mathematical Theory of Reliability*. Wiley, New York.

[8] Barlow, R. E. and Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing*. To Begin With, Silver Spring, MD.

[9] Basu, S. K. and Bhattacharjee, M. C. (1984). On weak convergence within the HNBUE family of life distributions. *J. Appl. Prob.* **21**, 654–660.

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[10] BASU, S. K. AND SIMONS, G. (1982). Moment spaces for IFR distributions, applications and related material. In *Contributions to Statistics: Essays in Honour of Norman L. Johnson*, ed. P. K. Sen. North-Holland, Amsterdam.
[11] BELZUNCE, F., ORTEGA, E.-M. AND RUIZ, J. M. (2007). On non-monotonic ageing properties from the Laplace transform, with actuarial applications. *Insurance Math. Econom.* **40**, 1–14.
[12] BERGER, R. L., BOOS, D. D. AND GUESS, F. M. (1988). Tests and confidence sets for comparing two mean residual life functions. *Biometrics* **44**, 103–115.
[13] BHATTACHARYYA, D., KHAN, R. A. AND MITRA, M. (2020). A test of exponentiality against DMTTF alternatives via L-statistics. *Statist. Prob. Lett.* **165**, 1–9.
[14] BHATTACHARYYA, D., KHAN, R. A. AND MITRA, M. (2020). A nonparametric test for comparison of mean past lives. *Statist. Prob. Lett.* **161**. Available at https://doi.org/10.1016/j.spl.2020.108722.
[15] BLOCK, H. W. AND SAVITS, T. H. (1978). Shock models with NBUE survival. *J. Appl. Prob.* **15**, 621–628.
[16] DESHPANDE, J. V. AND SURESH, R. P. (1990). Non-monotonic ageing. *Scand. J. Statist.* **17**, 257–262.
[17] EBRAHIMI, N. (1999). Stochastic properties of a cumulative damage threshold crossing model. *J. Appl. Prob.* **36**, 720–732.
[18] ESARY, J., MARSHALL, A. AND PROSCHAN, F. (1973). Shock models and wear processes. *Ann. Prob.* **1**, 627–649.
[19] FAGIUOLI, E. AND PELLEREY, F. (1994). Preservation of certain classes of life distributions under Poisson shock models. *J. Appl. Prob.* **31**, 458–465.
[20] FELLER, W. (1968). *An Introduction to Probability Theory and its Applications*, Vol. I. John Wiley, New York.
[21] FINKELSTEIN, M. AND CHA, J. H. (2013). Stochastic modeling for reliability. In *Shocks, Burn-in and Heterogeneous Populations* (Springer Series in Reliability Engineering). Springer, London.
[22] GHOSH, M. AND EBRAHIMI, N. (1982). Shock models leading to increasing failure rate and decreasing mean residual life survival. *J. Appl. Prob.* **19**, 158–166.
[23] GLASER, R. E. (1980). Bathtub and related failure rate characterizations. *J. Amer. Statist. Assoc.* **75**, 667–672.
[24] GUESS, F., HOLLANDER, M. AND PROSCHAN, F. (1986). Testing exponentiality versus a trend change in mean residual life. *Ann. Statist.* **14**, 1388–1398.
[25] IZADI, M. AND FATHIMANESH, S. (2019). Testing exponentiality against a trend change in mean time to failure in age replacement. *Commun. Statist. Theory Methods.* Available at https://doi.org/10.1080/03610926.2019.1702693.
[26] IZADI, M., SHARAFI, M. AND KHALEDI, B.-E. (2018). New nonparametric classes of distributions in terms of mean time to failure in age replacement. *J. Appl. Prob.* **55**, 1238–1248.
[27] KATTUMANNIL, S. K. AND ANISHA, P. (2019). A simple non-parametric test for decreasing mean time to failure. *Statist. Papers* **60**, 73–87.
[28] KAYID, M., AHMAD, I., IZADKHAH, S. AND ABOUAMMOH, A. (2013). Further results involving the mean time to failure order, and the decreasing mean time to failure class. *IEEE Trans. Reliab.* **62**, 670–678.
[29] KHAN, R. A. AND MITRA, M. (2019). Sharp bounds for survival probability when ageing is not monotone. *Prob. Eng. Inf. Sci.* **33**, 205–219.
[30] KHAN, R. A., BHATTACHARYYA, D. AND MITRA, M. (2020). A change point estimation problem related to age replacement policies. *Operat. Res. Lett.* **48**, 105–108.
[31] KLEFSJÖ, B. (1981). HNBUE survival under some shock models. *Scand. J. Statist.* **8**, 39–47.
[32] KLEFSJÖ, B. (1982). On aging properties and total time on test transforms. *Scand. J. Statist.* **9**, 37–41.
[33] KLEFSJÖ, B. (1983). A useful ageing property based on the Laplace transform. *J. Appl. Prob.* **20**, 615–626.
[34] KLEFSJÖ, B. (1989). Testing against a change in the NBUE property. *Microelectron. Reliab.* **29**, 559–570.
[35] KNOPIK, L. (2005). Some results on the ageing class. *Control Cybernet.* **34**, 1175–1180.
[36] KNOPIK, L. (2006). Characterization of a class of lifetime distributions. *Control Cybernet.* **35**, 407–414.
[37] KOCHAR, S. C. (1990). On preservation of some partial orderings under shock models. *Adv. Appl. Prob.* **22**, 508–509.
[38] LAI, C. D. AND XIE, M. (2006). *Stochastic Ageing and Dependence for Reliability*. Springer, New York.
[39] LI, X. AND XU, M. (2008). Reversed hazard rate order of equilibrium distributions and a related aging notion. *Statist. Papers* **49**, 749–767.
[40] LOÈVE, M. (1963). *Probability Theory*, 3rd edn. Van Nostrand, New York.
[41] MITRA, M. AND BASU, S. K. (1994). On a nonparametric family of life distributions and its dual. *J. Statist. Planning Infer.* **39**, 385–397.
[42] MITRA, M. AND BASU, S. K. (1996). On some properties of the bathtub failure rate family of life distributions. *Microelectron. Reliab.* **36**, 679–684.
[43] MITRA, M. AND BASU, S. K. (1996). Shock models leading to non-monotonic ageing classes of life distributions. *J. Statist. Planning Infer.* **55**, 131–138.
[44] NAKAGAWA, T. (2007). *Shock and Damage Models in Reliability Theory*. Springer, London.

https://doi.org/10.1017/jpr.2020.91 Published online by Cambridge University Press

[45] PARK, D. H. (2003). Class of NBU-t_0 life distribution. In *Handbook of Reliability Engineering*, ed. H. Pham, pp. 181–197. Springer, London.
[46] PELLEREY, F. (1993). Partial orderings under cumulative damage shock models. *Adv. Appl. Prob.* **25**, 939–946.
[47] PÉREZ-OCÓN, R. AND GÁMIZ-PÉREZ, M. L. (1995). On the HNBUE property in a class of correlated cumulative shock models. *Adv. Appl. Prob.* **27**, 1186–1188.
[48] SHAKED, M. AND SHANTHIKUMAR, J. G. (2007). *Stochastic Orders*. Springer, NewYork.
[49] SHANTHIKUMAR, J. G. AND SUMITA, U. (1983). General shock models associated with correlated renewal sequences. *J. Appl. Prob.* **20**, 600–614.
[50] SUDHEESH, K. K., ASHA, G. AND KRISHNA, K. M. J. (2019). On the mean time to failure of an age-replacement model in discrete time. *Commun. Statist. Theory Meth.* Available at https://doi.org/10.1080/03610926.2019.1672742.
[51] SURESH, R. (2003). On the inter-relationships between some classes of life distributions. *Calcutta Statist. Assoc. Bull.* **54**, 261–268.
[52] YAMADA, K. (1989). Limit theorems for jump shock models. *J. Appl. Prob.* **26**, 793–806.

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