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An improved real C^k Koebe principle

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Abstract. We prove an improved C^k Koebe principle for a C^k interval map with non-flat critical points, where $k \geq 3$, that requires no disjointness of the intervals involved.

1. *Introduction*

The real Koebe principle is a major analytic tool in studying iterated interval maps. The most well-known version of this principle provides estimates on the first- or second-order derivatives for diffeomorphisms with negative Schwarzian derivatives; see [**3**]. In recent works [**2, 8**], it is proved that such an estimate remains true for diffeomorphic branches of iterates of a general smooth interval map with non-flat critical points.

In certain circumstances, we need a C^k Koebe principle to obtain estimates on higher-order derivatives. A result in this direction was obtained in [**6**]. However, in order to apply that result, one has to prove an upper bound for the sum of the length of the orbit of an interval, which often requires extra work. In this paper we shall prove an improved version of the C^k Koebe principle for smooth multimodal maps that requires no estimate on the sum of the length.

More precisely, we consider a C^1 map f from the interval $X := [0, 1]$ into itself. Let $\mathrm{Crit}(f)$ denote the set of critical points of f, i.e. the set of points where f' vanishes. We shall always assume that for some $r \geq 2$, f is C^r outside $\mathrm{Crit}(f)$ and for each critical point c, there exist C^r local diffeomorphisms ϕ, ψ defined on a neighborhood of 0 with $\phi(0) = c$, $\psi(0) = f(c)$, and a real number $\ell = \ell_c > 1$ (called the *order* of c) such that $|\psi^{-1} \circ f \circ \phi(x)| = |x|^\ell$ holds when $|x|$ is small. Denote by $\mathcal{F}^r$ the class of C^r maps of $[0, 1]$ with non-flat critical points and all periodic points repelling.

Given a homeomorphism $\phi : I_1 \to I_2$ between compact intervals, let $\alpha_i : I_i \to [0, 1]$ be the orientation-preserving affine homeomorphism ($i = 1, 2$) and let

$$[\phi] = \alpha_2 \circ \phi \circ \alpha_1^{-1} : [0, 1] \to [0, 1].$$

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For a bounded interval J and $\tau > 0$, let $(1+2\tau)J$ denote the concentric open interval of length $(1+2\tau)|J|$. We say that an interval J is τ-*well inside* an interval T if $T \supset (1+2\tau)J$.

The main result of this paper is the following theorem.

THEOREM A. *Let $f \in \mathcal{F}^r$, where $r \geq 3$. For each $\tau > 0$ and each $1 \leq k \leq r$, there exist $K_k = K_k(\tau) > 0$ and $\xi = \xi(f, \tau) > 0$ satisfying the following. Let $T \subset [0, 1]$ be an open interval and let J be a closed subinterval of T such that the following hold:*

- *$f^s : T \to f^s(T)$ is a diffeomorphism;*
- *$|f^s(T)| < \xi$;*
- *$f^s(J)$ is τ-well inside $f^s(T)$.*

Then for $\phi = [f^s|J]$ we have

$$|D^k\phi| \leq K_k.$$

Furthermore, $K_1 \to 1$ and for each $k > 1$, $K_k \to 0$ as $\tau \to \infty$.

A conditional $C^r (r \geq 2)$ Koebe principle was proved in [**6**], where an upper bound on $\sum_{j=0}^{s-1} |f^j(T)|$ was required to obtain the same conclusion of the theorem. It was observed in [**4**] that, for $C^r (r \geq 3)$ maps, we only need an upper bound on $\sum_{j=0}^{s-1} |f^j(J)|^2$. We shall prove here that we always have such an upper bound.

THEOREM B. *Let $f \in \mathcal{F}^3$. For any $\alpha > 0$, there exists a constant $C = C(\alpha) > 0$ satisfying the following. For any interval $T \subset [0, 1]$ and integer $s > 0$, if $f^s : T \to f^s(T)$ is a diffeomorphism, then*

$$\sum_{j=1}^{s} |f^j(T)|^{1+\alpha} \leq C.$$

In [**7**], the author proved this theorem for $C^{2+\eta}$ unimodal maps. The argument given in this paper is significantly simpler.

Proof of Theorem A assuming Theorem B. By [**4**, Theorem 2], for each $\tau > 0$ and each $1 \leq k \leq r$, there exists a constant $\eta = \eta(f, \tau) > 0$ such that, under the further condition $\sum_{j=0}^{s-1} |f^j(J)|^2 \leq \eta$, the conclusion of Theorem A holds. This result was proved by modifying the argument in [**6**]. Although it was also stated for S-unimodal maps in [**4**], the proof works through in the general case by the following modification: in the proof of Claim 2, we replace the Koebe principle for maps with negative Schwarzian derivatives by [**8**, Theorem C].

By Theorem B, there exists a universal constant $C = C(1/2) > 0$ such that $\sum_{j=0}^{s-1} |f^j(T)|^{3/2} \leq C$. Since f has no wandering interval (see [**3**]) and all periodic points are repelling, there exists $\xi > 0$ such that if $|f^s(T)| \leq \xi$, then

$$\max_{j=0}^{s-1} |f^j(T)| \leq (\eta/C)^2.$$

Thus

$$\sum_{j=0}^{s-1} |f^j(J)|^2 \leq \sum_{j=0}^{s-1} |f^j(T)|^2 \leq \sqrt{\max_{j=0}^{s-1} |f^j(T)|} \sum_{j=0}^{s-1} |f^j(T)|^{3/2} \leq \eta.$$

The proof is completed. □

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Remark. Recall that a diffeomorphism $\phi : J' \to J$ between bounded intervals is in the Epstein class $\mathcal{E}_\tau$ if ϕ^{-1} extends to a univalent holomorphic function from $\mathbb{C} \setminus (\mathbb{R} \setminus (1 + 2\tau)J)$ into $\mathbb{C}$. The same outline in fact gives the following stronger estimate on the distortion of $f^s|J$ under the circumstances of Theorem A: given $\varepsilon > 0$, provided that $|f^s(T)|$ is small enough, we can find a map ψ in the Epstein class $\mathcal{E}_{\tau/2}$ such that the C^k distance between $\phi = [f^s|J]$ and ψ is at most ε.

The rest of the paper is devoted to the proof of Theorem B. The real bounds properties developed in [**8**] are the basis for our analysis.

Notation. We say that a constant is universal if it depends only on the number and order of the critical points of f.

2. *A reduction*

For $V \subset [0, 1]$, let

$$\mathrm{D}(V) = \{x \in [0, 1] \mid f^k(x) \in V \text{ for some } k \geq 1\}.$$

The *first entry map* $R_V : \mathrm{D}(V) \to V$ is defined as $x \mapsto f^{k(x)}(x)$, where $k(x)$ is the *entry time* of x into V, i.e. the minimal positive integer such that $f^{k(x)}(x) \in V$. The map $R_V|(\mathrm{D}(V) \cap V)$ is called the *first return map* of I.

An open interval $V \subset [0, 1]$ is called *nice* if $f^n(\partial V) \cap V = \emptyset$ for all $n \geq 0$. A component of $\mathrm{D}(V)$ (respectively $\mathrm{D}(V) \cap V$) is called an *entry domain* (respectively *return domain*) of V. It is well known that the entry time is constant in any component of $\mathrm{D}(V)$.

A sequence of open intervals $\{T_j\}_{j=0}^s$ is called a *chain* if for each $j = 0, 1, \ldots, s-1$, T_j is a component of $f^{-1}(T_{j+1})$. The *intersection multiplicity* of the chain is defined to be the maximal number of the intervals T_j that have a common point.

Given a nice interval V and a chain $\mathbb{T} = \{T_j\}_{j=0}^s$, we say that the pair $(V, \mathbb{T})$ is *admissible* if $f^s : T_0 \to T_s$ is a diffeomorphism and $T_s \subset V$.

A *pull back* of an open interval V is a component of $f^{-n}(V)$ for some $n \geq 0$. Denote by $\rho(V)$ the maximal length of all pull backs of V. Clearly, if U is a pull back of V, then $\rho(U) \leq \rho(V)$.

The following proposition was proved by van Strien and Vargas; see [**8**, Theorem C(1)].

PROPOSITION 1. *Assume $f \in \mathcal{F}^3$. For any $\xi > 0$, there exists $\xi' > 0$ such that if I, V are nice intervals, V is ξ-well inside I, and $x \in I$, $f^k(x) \in V$, then the pull back of V along $\{x, f(x), \ldots, f^k(x)\}$ is ξ'-well inside I.*

To prove Theorem B, it suffices to prove the following Main Lemma.

MAIN LEMMA. *Let $f \in \mathcal{F}^3$. For any $\alpha > 0$, there exists a constant $C = C(\alpha) > 0$ satisfying the following. For any admissible pair $(V, \{T_j\}_{j=0}^s)$, with $|V|$ sufficiently small, there exists $0 \leq s' < s$ such that*

$$\sum_{j=s'+1}^{s} |T_j|^{1+\alpha} \leq C\rho(V)^\alpha,$$

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and one of the following holds:

- $s' = 0$*; or*
- *there is a nice interval* V' *such that* V' *is* ξ*-well inside some pull back of* V *and* $(V', \{T_j\}_{j=0}^{s'})$ *is an admissible pair, where* ξ *is a universal constant.*

Proof of Theorem B. Without loss of generality, we may assume that $f^s(T) \subset V$ for some nice interval V with $|V|$ small. By the Main Lemma, for any $\alpha > 0$, there exist a constant $C = C(\alpha) > 0$ and integers $0 = s_n < \cdots < s_2 < s_1 < s_0 = s$ and nice intervals $V_0 = V, V_1, V_2, \ldots, V_n$, which are pull backs of V, such that:

- $f^{s_i}(T) \subset V_i$ for $0 \le i < n$;
- $\sum_{j=s_i+1}^{s_{i-1}} |f^j(T)|^{1+\alpha} \le C\rho(V_{i-1})^\alpha$ for $1 \le i \le n$;
- V_i is ξ-well inside some pull back of V_{i-1} for $1 \le i < n$.

Let us prove that there exists a constant $\lambda = \lambda(\xi) \in (0, 1)$ such that $\rho(V_i) < \lambda\rho(V_{i-1})$ for $1 \le i < n$. To this end, fix i and let I be a pull back of V_{i-1} with $(1 + 2\xi)V_i \subset I$. Letting U be a pull back of V_i, we shall show that it is well inside a component of $\mathrm{D}(I) \cup I$. If $U \subset I$, then by Proposition 1, U is well inside I. Otherwise, U is contained in a component U' of $\mathrm{D}(I) \setminus I$. Let r be the entry time of U' into I. Then $f^r(U)$ is contained in a pull back of V_i, hence well inside I, by Proposition 1 once again. Since the chain $\{U'_j\}_{j=0}^r$ with $U'_r = I$ and $U'_0 = U'$ has uniformly bounded intersection multiplicity, we obtain that U is well inside U' by the usual Koebe principle.

Therefore

$$\sum_{j=1}^{s} |f^j(T)|^{1+\alpha} = \sum_{i=1}^{n} \sum_{j=s_i+1}^{s_{i-1}} |f^j(T)|^{1+\alpha} \le \sum_{i=1}^{n} C\rho(V_{i-1})^\alpha$$
$$< \sum_{i=1}^{n} C\lambda^{(i-1)\alpha}\rho(V)^\alpha < C',$$

where $C' = C'(\alpha)$ is a constant. □

3. *Pull back by monotone branches*

The goal of this section is to prove the following proposition.

PROPOSITION 2. *There exists* $\tau > 0$*, and for any* $\alpha > 0$ *there exists* $C = C(\alpha) > 0$ *with the following property. Let* I *be a small nice interval and* J *be a return domain of* I *with return time* r *such that* $(1 + 2\tau)J \supset I$*. Let* K *be a subinterval of* J *such that* $f^r : K \to f^r(K)$ *is monotone. If* $f^{qr} : T \to f^{qr}(T)$ *is a diffeomorphism and* $f^{ir}(T) \subset K$ *for all* $0 \le i < q$*, then*

$$\sum_{i=1}^{q} |f^{ir}(T)|^{1+\alpha} \le C|I|^{1+\alpha}.$$

The constant τ is chosen according to the following proposition, which provides a useful estimate in the case that $|J|$ and $|I|$ have almost the same length.

PROPOSITION 3. *There exist universal constants* $\tau > 0$ *and* M *with the following property. Let* I *be a small nice interval and let* J *be a return domain of* I *that has return time* r*. Assume that* $I \subset (1 + 2\tau)J$*. Let* $\mathbb{U} = \{U_j\}_{j=0}^r$ *and* $\{I_j\}_{j=0}^r$ *be the chains with*

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$U_r = (1+2\tau)I \supset I_r = I$ and $U_0 \supset I_0 = J$. Then:

(1) *the intersection multiplicity of $\mathbb{U}$ is not greater than M;*

(2) *for each $x \in U_0$, $|(f^r)'(x)| \le C$;*

(3) *if $c \in U_0$ is a critical point of f^r, then*

$$d(f^r(c), c) \ge \kappa |U_0|;$$

(4) *if $f^r|U_0$ has a critical point, then the Schwarzian derivative $Sf^r(x) < -\kappa/|J|^2$ for all $x \in U_0$, where $\kappa > 0$, $C > 0$ are universal constants.*

This is [**5**, Proposition 5.2], except that we have changed the assumption of the last item from '*$f^r|J$ has a critical point*' to '*$f^r|U_0$ has a critical point*' and changed the notation 'T_j' to 'U_j'. The same proof works: indeed, the assumption that '*$f^r|J$ has a critical point*' was only used to guarantee that $T_0 (= U_0)$ contains a critical point, and the proof showed that $|T_0|/|J|$ is bounded from above.

Recall that the Schwarzian derivative of a C^3 map φ is defined as

$$S\varphi = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2.$$

The following lemma, due to Yoccoz, provides useful estimates in the almost parabolic case.

LEMMA 1. (Yoccoz lemma) *Let $a_0 < a_1 < a_2 < \cdots < a_n < a_{n+1}$, and let $\varphi : [a_0, a_n] \to [a_1, a_{n+1}]$ be a C^3 diffeomorphism with $S\varphi < 0$ such that $\varphi(a_i) = a_{i+1}$ for all $0 \le i \le n$. For any $\sigma > 0$, there exists $C = C(\sigma) > 1$ such that if $|a_1 - a_0| > \sigma |a_0 - a_n|$ and $|a_n - a_{n-1}| > \sigma |a_0 - a_n|$, then for all $1 \le i \le n$,*

$$\frac{1}{C}\frac{1}{\min(i+1, n+1-i)^2} \le \frac{|a_i - a_{i-1}|}{|a_0 - a_n|} \le C\frac{1}{\min(i+1, n+1-i)^2}.$$

Proof. See [**1**, Appendix A] for a proof. □

Proof of Proposition 2. Clearly, we may assume that $q \ge 2$. Note that this assumption implies that $K \cap f^r(K) \ne \emptyset$. The argument will be divided into two cases.

Case 1. Assume that $f^r|K$ has no fixed point. Since $K \cap f^r(K) \ne \emptyset$, this implies that $f^r : K \to f^r(K)$ is monotone increasing. Without loss of generality, let us assume that $f^r(x) > x$ for all $x \in K$.

We only need to consider the case that $f^r : K \to f^r(K)$ is a diffeomorphism. To see this fact, let $c_1 < c_2 < \cdots < c_m$ be the critical points of $f^r|K$, and let $K_1, K_2, \ldots, K_{m+1}$ be the components of $K \setminus \{c_1, c_2, \ldots, c_m\}$, labeled from left to right. Then m is uniformly bounded and $f^r|K_j$ is diffeomorphic for all $1 \le j \le m+1$. Since $f^r|K$ moves every point to its right-hand side, we can find $0 = q_0 \le q_1 \le q_2 \le \cdots \le q_{m+1} = q$ such that $f^{ir}(T) \subset K_{j+1}$ for all $q_j \le i < q_{j+1}$, $j = 0, 1, \ldots, m$. Once we prove the proposition in the diffeomorphic case, we shall obtain upper bounds for $\sum_{i=q_j+1}^{q_{j+1}} |f^{ir}(T)|^{1+\alpha}$ for all $0 \le j \le m$, which clearly implies an upper bound for $\sum_{i=1}^{q} |f^{ir}(T)|^{1+\alpha}$.

Let $V_0 = (a, b)$ be the maximal open subinterval of U_0 that contains K and such that $f^r|V_0$ has no critical point.

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Subcase 1.1. Assume that $V_0 = U_0$. Then $f^r|U_0$ is a monotone increasing diffeomorphism, hence so is $f^r : J \to I$. Since $J \subset I$, f^r has a fixed point p in J. Since f^r has no attracting fixed point, we have $V_0 \subset f^r(V_0)$. Let $\lambda = Df^r(p)$ and let $J_n = (f^r|J)^{-n}(I)$, $n = 0, 1, \ldots$. By [**3**, Theorem IV.B], $\lambda > 1$ is bounded away from 1. Since $f^{nr} : J_n \to I$ is a diffeomorphism that extends to a diffeomorphism onto $U_r = (1 + 2\tau)I$, we have

$$|J_n| \le C\lambda^{-n}|I|,$$

where C is a constant depending on τ. Since $f^{ir}(T) \subset J_{q-i-1}$ for all $0 \le i < q$, the proposition follows.

Subcase 1.2. Assume that $V_0 \ne U_0$. We claim that we can find a universal constant $\sigma > 0$ and

$$a < a_0 < a_1 < a_2 < \cdots < a_N \le b < a_{N+1}$$

such that:

- $f^r(a_i) = a_{i+1}$ for all $i = 0, 1, \ldots, N$;
- $|a_0 - a_1| \ge \sigma|a_0 - a_N|$ and $|a_{N-1} - a_N| \ge \sigma|a_0 - a_N|$;
- $(a_0, a_{N+1}) \supset K$.

Assuming the claim for the moment, let us prove the proposition. Since $f^r|U_0$ has a critical point, by Proposition 3, $Sf^r < 0$ holds on $[a_0, a_N]$. By the Yoccoz lemma, we obtain that

$$|a_i - a_{i-1}| \asymp |a_0 - a_N|/\min(i + 1, N + 1 - i)^2 \tag{1}$$

holds for all $i = 1, 2, \ldots, N$. Let k be minimal such that $T \subset [a_0, a_k]$. Recall that we are assuming that $q \ge 2$. Hence $k \le N$, and for $1 \le i \le q - 2$ we have

$$f^{ir}(T) \subset [a_i, a_{k+i}] \subset [a_0, a_N].$$

By (1), we obtain that

$$\sum_{i=1}^{q-2}\left(\frac{|f^{ir}(T)|}{|a_0 - a_N|}\right)^{1+\alpha} \le \sum_{i=1}^{q-2}\left(\frac{|a_i - a_{k+i}|}{|a_0 - a_N|}\right)^{1+\alpha}$$

is bounded from above by a constant. Since $|a_0 - a_N| \le |U_r| = (1 + 2\tau)|I|$, the proposition follows.

Let us prove the claim. First we notice that one of the boundary points of V_0, say b, is a critical point of f^r. By Proposition 3,

$$|f^r(b) - b| \ge \kappa|U_0|.$$

Write $K = (x_0, x_1)$. Let us prove that there exists a constant $\kappa_0 > 0$ such that

$$\max_{x\in[a,x_0]} |f^r(x) - x| \ge \kappa_0|U_0|. \tag{2}$$

In fact, if a is also a critical point of f^r, then (2) holds by Proposition 3. If we assume that we are not in this case, then $a \in \partial U_0$ and $G = (f^r(a), f^r(x_0))$ contains a component of $U_r \setminus I$, hence $|G| \ge \tau|I|$. In the following we apply an argument from [**5**]. Let $H = (1/3)G$, and let $\{H_j\}_{j=0}^r$ be the chain with $H_r = H$ and $H_0 \subset (a, x_0)$. We

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may assume that $|H_0|/|U_0|$ is not too small, for otherwise (2) obviously holds. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be an affine homeomorphism such that $\sigma(H_0) = (0, 1)$, and consider the map $\varphi := \sigma \circ f^r \circ \sigma^{-1} : (0, 1) \to \sigma(H)$. By [**5**, Proposition 5.1], φ is C^3 close to a map in the Epstein class $\mathcal{E}_{1/2}$. By [**5**, Lemma 5.3], it follows that $\sup_{y \in H_0} |f^r(y) - y|/|H_0|$ is bounded away from zero. In particular (2) holds.

Now let us take $a_0 \in [a, x_0]$ such that $|f^r(a_0) - a_0| \geq \kappa_0 |U_0|$. Since f^r has no attracting fixed point, we have

$$f^r(b) > b + \kappa |U_0|, \quad f^r(a_0) > a_0 + \kappa_0 |U_0|.$$

Let N be maximal such that

$$a_0 < f^r(a_0) < \cdots < f^{rN}(a_0) \leq b.$$

Then $f^{r(N+1)}(a_0) \in (b, f^r(b)]$. Write $a_i = f^{ir}(a_0)$. Since $|(f^r)'|$ is uniformly bounded on U_0, and since $|f^r(b) - b|/|U_0|$ is bounded away from zero, $|a_{N-1} - a_N|/|U_0|$ is also bounded away from zero. The claim follows.

Case 2. Assume that $f^r : K \to f^r(K)$ has a fixed point p. Since f has no attracting fixed point, $f^r|K$ has a unique fixed point. Denote by I^-, I^+ the left and right components of $I \setminus \{p\}$, respectively. Then I^i is a nice interval that has a return domain J^i that contains p in its boundary. In the case that $(f^r)'(p) > 0$, $J_i = J \cap I^i$; otherwise, $J_i \subset J \cap I^i$. The return time s of J_i to I_i is either r or $2r$ according as $f^r|K$ is monotone increasing or decreasing. Let K_1 be the maximal subinterval of K such that $f^r(K_1) \subset K$, and let $K_1^i = K_1 \cap J^i$, $i \in \{+, -\}$. Then $f^s|K_1^i$ is monotone and has no fixed point. For $0 \leq js < (q-1)r$, $f^{js}(T \cap I^i) \subset K_1^i$. Applying Case 1, we obtain the desired estimate. □

4. *Proof of the Main Lemma*

Let I be a nice interval and J a return domain of I. For $x \in J$, we say that the first return map $R_I : J \to I$ is *non-central* with respect to x if $R_I(x) \notin J$. The following theorem is [**8**, Theorem A].

PROPOSITION 4. *For each $f \in \mathcal{F}^3$, there exists a constant $\tau > 0$ such that the following holds. Let $I_0 \supset I_1 \supset \cdots \supset I_{n+1}$ be a sequence of intervals, and let $x \in I_{n+1}$ with the following properties:*

- *I_0 is nice;*
- *for each $i = 0, 1, \ldots, n$, I_{i+1} is a return domain of I_n;*
- *there exist integers $m < \hat{m} < n$ such that both $R_{I_m} : I_{m+1} \to I_m$ and $R_{I_{\hat{m}}} : I_{\hat{m}+1} \to I_{\hat{m}}$ are non-central with respect to x.*

Then I_{n+1} is τ-well inside I_0.

Proof of the Main Lemma. Let $\alpha > 0$ and let $(V, \{T_j\}_{j=0}^s)$ be an admissible pair with $|V|$ small. It suffices to prove the following.

(∗) There exist universal constants $C > 0$ and $\tau > 0$ such that for any component I of $\mathrm{D}(V) \cup V$, there exists $0 \leq s' = s'(I) \leq s$ such that

$$\sum_{s' < j \leq s, T_j \subset I} |T_j|^{1+\alpha} \leq C |I|^{1+\alpha},$$

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and such that one of the following holds:
(i) $s' = 0$; or
(ii) $T_{s'}$ is contained in a nice interval I' with $(1+2\tau)I' \subset I$.
In fact, let $s_0 = \max_I s'(I)$. Then either $s_0 = 0$ or T_{s_0} is well inside a pull back of V. Moreover,

$$\sum_{j=s_0+1}^{s} |T_j|^{1+\alpha} = \sum_I \sum_{s_0<j\le s;\ T_j\subset I} |T_j|^{1+\alpha} \le \sum_I \sum_{s'(I)<j\le s;\ T_j\subset I} |T_j|^{1+\alpha}$$
$$\le \sum_I C|I|^{1+\alpha} \le C\rho(V)^\alpha \sum_I |I| \le C\rho(V)^\alpha.$$

To prove $(*)$, let us fix a component I of $\mathrm{D}(V) \cup V$. Certainly we may assume that $T_j \subset I$ for some $0 \le j \le s$. We shall define inductively a sequence of nice intervals $J_k \subset J_{k-1} \subset \cdots \subset J_1 \subset J_0 = I$ and integers $0 \le s_k < s_{k-1} < \cdots < s_1 < s_0 \le s$ such that $T_{s_j} \subset J_j$ for all $0 \le j \le k$.

Let $J_0 = I$ and $s_0 \le s$ be maximal such that $T_{s_0} \subset J_0$. Suppose that we have defined $J_i \subset J_{i-1} \subset \cdots \subset J_0$ and $0 \le s_i < s_{i-1} < \cdots < s_0$ such that $T_{s_j} \subset J_j$ for all $0 \le j \le i$. Let $0 \le s_{i+1} < s_i$ be maximal such that $T_{s_{i+1}} \subset J_i$, and define $J_{i+1} \subset J_i$ to be the return domain of J_i that contains $T_{s_{i+1}}$; if there is no such s_{i+1}, then let $k = i$.

Let $R_I|J_1 = f^r$. Let $m \le k$ be maximal such that

$$s_0 - s_1 = s_1 - s_2 = \cdots = s_{m-1} - s_m = r.$$

Statement. There exists $m_0 \le m$ such that

$$\sum_{i=0}^{m_0} |T_{s_i}|^{1+\alpha} \le C|I|^{1+\alpha}, \tag{3}$$

and such that one of the following holds:
(a) $m_0 = m$; or
(b) $T_{s_{m_0}}$ is contained in a nice interval I' with $(1+2\kappa)I' \subset I$;
where $C > 0$, $\kappa > 0$ are constants.

Assuming the statement for the moment, let us prove that either (i) or (ii) holds.

If (b) holds, then (ii) holds with $s' = s_{m_0}$ and we are done. So let us assume that (a) holds. If $m = k$, then $f^i(T) \cap J_k = \emptyset$ for all $0 < i < s_k$, so $f^{s_k} : T \to f^{s_k}(T)$ is a first landing into J_k, therefore the intervals $f(T)$, $f^2(T)$, $\ldots$, $f^{s_k}(T)$ are pairwise disjoint, hence

$$\sum_{1\le j\le s_k, T_j\subset I} |T_j|^{1+\alpha} = \sum_{1\le j\le s_k, T_j\subset I} |T_j||T_j|^\alpha \le |I|^\alpha \sum_{1\le j\le s_k, T_j\subset I} |T_j| \le |I|^{1+\alpha}.$$

Thus (i) holds with $s' = 0$ in this case. Assume now that $m < k$. Then $s_m - s_{m+1} > r$. Let $\hat{m} \in \{m+1, m+2, \ldots, k\}$ be maximal such that

$$s_m - s_{m+1} = s_{m+1} - s_{m+2} = \cdots = s_{\hat{m}-1} - s_{\hat{m}}.$$

Applying the statement, replacing I by J_m, we obtain that $m \le \hat{m}_0 \le \hat{m}$ such that

$$\sum_{i=m}^{\hat{m}_0} |T_{s_i}|^{1+\alpha} \le C|J_m|^{1+\alpha} \le C|I|^{1+\alpha},$$

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and such that one of the following holds:

(â) $\hat{m}_0 = \hat{m}$; or

(b̂) $T_{s_{\hat{m}_0}}$ is contained in a nice interval $\hat{I}'$ with $(1+2\kappa)\hat{I}' \subset J_m \subset I$.

If (b̂) holds, then (ii) holds with $s' = s_{\hat{m}_0}$ and we are done. If (â) holds and $\hat{m} = k$, then we obtain (i) as above. Suppose that (â) holds and $\hat{m} < k$. Then for an arbitrary $x \in T_{s_{\hat{m}+1}}$, $R_{I_m} : I_{m+1} \to I_m$ and $R_{I_{\hat{m}}} : I_{\hat{m}+1} \to I_{\hat{m}}$ are both non-central with respect to x. Thus, by Proposition 4, $J_{\hat{m}+1}$ is uniformly well inside I. Thus (ii) holds with $s' = s_{\hat{m}+1}$ and $I' = J_{\hat{m}+1}$.

The rest of the paper is devoted to the proof of the statement. Let us first assume that

$$(1+2\tau)J_1 \supset J_0 = I, \tag{4}$$

where τ is the constant in Proposition 3.

If $f^r : J_1 \to I$ is monotone, then by Proposition 2, (a) holds with $m_0 = m$. So let us assume that $f^r : J_1 \to I$ has a turning point. Provided that $|I|$ is small enough, $f^r|J_1$ is a composition of folding maps, so $f^r(\partial J_1)$ consists of one point in the boundary of I. To fix the notation, let us assume that f^r maps ∂J_1 to the left endpoint of I. Let c, c' be the left and right most turning points of $f^r|J_1$, respectively. Let L, L' be the left and right components of $J_1 \setminus [c, c']$. Note that $f^r(L) = f^r(L') = f^r(J_1)$.

First suppose that $f^r|J_1$ has no fixed point. Then $f^r(x) < x$ for all $x \in J_1$. Denote all the critical points of f^r in J_1 by $c_1 < c_2 < \cdots < c_\mu$. The number μ is uniformly bounded. Since $f^r|J_1$ moves every point to its left, applying Proposition 2 in each piece (c_j, c_{j+1}), we obtain that (a) holds with $m_0 = m$.

Assume now that $f^r|J_1$ has a fixed point. Since f^r has no attracting fixed point, $f^r(c) > c$. Thus $f^r|J_1$ has an orientation-reversing fixed point p with $p > c$. The pre-images of the fixed point p under $f^r|J_1$ divide the interval J_1 into finitely many subintervals J^i, $1 \le i \le \nu$, labeled from left to right. Note that each interval $\mathrm{int}(J^i)$ is a nice interval. Let $Q = \mathrm{int}(J^1)$, $Q' = \mathrm{int}(J^\nu)$ be the left and right most subintervals, respectively. Then f^r is monotone on Q and Q', and $f^r(Q) = f^r(Q') = (a_0, p)$, where a_0 is the left endpoint of I.

Since $(1+2\tau)J_1 \supset I$, by Proposition 3, $f^r|J_1$ has a uniformly bounded derivative, and $|f^r(c) - c|/|J_1|$ is bounded away from zero. Therefore $|c-p|/|J_1|$ is bounded away from zero since $f^r(c, p) = (f^r(c), p)$. Since $|f^r(Q)| = |f^r(Q')| = |a_0 - p| > |c-p|$, $|Q|/|J_1| = |Q'|/|J_1|$ is bounded away from zero. Thus each $\mathrm{int}(J^i)$, $1 < i < \nu$, is uniformly well inside J_1.

Let $K \ni p$ be the maximal subinterval of J_1 such that $f^r|K$ is diffeomorphic. If $T_{s_i} \ni p$ for all $1 \le i \le m$, then $T_{s_i} \subset K$ for all $1 \le i \le m$. By Proposition 2, we obtain that (a) holds with $m_0 = m$. So assume that there exists a minimal integer $1 \le i_0 \le m$ such that $T_{s_{i_0}} \not\ni p$. Then by the same argument, we have that $\sum_{i=1}^{i_0-1} |f^{s_i}(T)|^{1+\alpha} \le C|I|^{1+\alpha}$. For all $i_0 < i \le m$, T_{s_i} is contained in one of the intervals J^j, $1 \le j \le \nu$. If $T_{s_{i_0+1}} \subset J^j$ for some J^j distinct from Q and Q', then (b) holds with $m_0 = i_0 + 1$. Otherwise, let $i_1 \le m$ be maximal such that $f^{s_i}(T) \subset Q \cup Q'$ for all $i_0 < i \le i_1$. Since $f^r(Q) \cap Q' = f^r(Q') \cap Q' = \emptyset$, we have $T_{s_i} \subset Q$ for all $i_0 < j < i_1 - 1$. By Proposition 2, $\sum_{i=i_0+1}^{i_1} |T_{s_i}|^{1+\alpha} \le C|I|^{1+\alpha}$ for some universal constant $C > 0$. If $i_1 = m$, then (a) holds with $m_0 = m$. Otherwise, (b) holds with $m_0 = s_{i_1+1}$.

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So far, we have proved the statement under the assumption (4). The general case can be treated as follows. Let $n_0 \le m$ be maximal such that $(1+2\tau)J_n \not\supset J_{n-1}$ for all $1 \le n \le n_0$. Then $|J_{n-1}| \ge (1+\tau)|J_n|$ for all $1 \le n \le n_0$. Hence

$$\sum_{i=0}^{n_0} |T_{s_i}|^{1+\alpha} \le \sum_{i=0}^{n_0} |J_i|^{1+\alpha} \le |I|^{1+\alpha} \sum_{i=0}^{\infty} (1+\tau)^{-i}.$$

Thus (a) holds if $n_0 = m$. Otherwise, $n_0 < m$ and $(1+2\tau)J_{n_0+1} \supset J_{n_0}$. Replacing J_0, J_1 by J_{n_0}, J_{n_0+1} in the above argument, we complete the proof. □

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References

[1] E. de Faria and W. de Melo. Rigidity of critical circle mappings. I. *J. Eur. Math. Soc.* **1**(4) (1999), 339–392.
[2] O. Kozlovski. How to get rid of the negative Schwarzian condition. *Ann. of Math.* (2) **152**(3) (2000), 743–762.
[3] W. de Melo and S. van Strien. *One-dimensional Dynamics*. Springer, Berlin, 1993.
[4] S. Li and W. Shen. On C^r conjugacy of S-unimodal maps. *Nonlinearity* **19** (2006), 1629–1634.
[5] S. Li and W. Shen. Hausdorff dimension of Cantor attractors in one-dimensional dynamics. *Invent. Math.* **171** (2008), 345–387.
[6] W. Shen and M. Todd. A C^k real Koebe principle. *Fund. Math.* **185**(1) (2005), 61–69.
[7] M. Todd. Distortion bounds for $C^{2+\eta}$ unimodal maps. *Fund. Math.* **193** (2007), 37–77.
[8] S. van Strien and E. Vargas. Real bounds, ergodicity and negative Schwarzian for multimodal maps. *J. Amer. Math. Soc.* **17** (2004), 749–782; **20** (2007), 267–268 (erratum).

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