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Linking sandwich pairs

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Abstract

Since the development of the calculus of variations there has been interest in finding critical points of functionals. This was intensified by the fact that for many equations arising in practice, the solutions are critical points. In searching for critical points, there is a distinct advantage if the functional G is semibounded. In this case one can find a Palais–Smale (PS) sequence

$$G(u_k) \longrightarrow c, \quad G'(u_k) \longrightarrow 0$$

or even a Cerami sequence

$$G(u_k) \longrightarrow c, \quad (1 + \|u_k\|)G'(u_k) \longrightarrow 0.$$

These sequences produce critical points if they have convergent subsequences. However, there is no clear method of finding critical points of functionals which are not semibounded. Linking subsets do provide such a method. They can produce a PS sequence provided they separate the functional. In the present paper we show that there are pairs of subsets that can produce Cerami-like sequences even though they do not separate the functional. All that is required is that the functional be bounded from above on one of the sets and bounded from below on the other, with no relationship needed between the bounds. This provides a distinct advantage in applications. We apply the method to several situations.

1. *Introduction*

Many problems arising in science and engineering call for the solving of the Euler equations of functionals, i.e., equations of the form

$$G'(u) = 0, \tag{1}$$

where $G(u)$ is a C^1 functional (usually representing the energy) arising from the given data. As an illustration, the equation

$$-\Delta u(x) = f(x, u(x))$$

is the Euler equation of the functional

$$G(u) = \frac{1}{2}\|\nabla u\|^2 - \int F(x, u(x))\, dx$$

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on an appropriate space, where

$$F(x,t) = \int_0^t f(x,s)\,ds, \tag{2}$$

and the norm is that of L^2. In such cases the solving of the Euler equations is tantamount to finding critical points of the corresponding functional. The classical approach was to look for maxima or minima. Extrema can exist only if the functional is semibounded. If one is looking for a minimum, a first approach is to consider a minimizing sequence, i.e., a sequence such that

$$G(u_k) \longrightarrow c, \tag{3}$$

where $c = \inf G$. One then has the task of showing that this sequence, or a subsequence of it, converges to a limit. If a subsequence converges, the limit will be a minimum.

However, when one wishes to solve Euler equations, one is not merely looking for extrema. One is interested in finding all critical points. In particular, how does one search for critical points when the functional is not semibounded? Is there anything that can be used to replace semiboundedness? A very successful approach has been to find linking subsets which separate the functional. By this we mean that we search for two subsets A, B such that

$$a_0 := \sup_A G \leqslant b_0 := \inf_B G \tag{4}$$

implies that there exists a Palais–Smale (PS) sequence satisfying

$$G(u_k) \longrightarrow a, \quad G'(u_k) \longrightarrow 0 \tag{5}$$

for some finite $a \geqslant b_0$. If this is true for every $G \in C^1$, we say that A links B. Sufficient conditions for sets to link as well as a rationale for the terminology will be given in the next section. Moreover, one can do even better. One can obtain a sequence satisfying

$$G(u_k) \longrightarrow c, \quad (1 + \|u_k\|)G'(u_k) \longrightarrow 0 \tag{6}$$

for $c = \inf G$. Such a sequence is called a **Cerami sequence.** It is easy to show that such a sequence has an even better chance of having a convergent subsequence. There are situations in which a PS sequence does not have a convergent subsequence, while a corresponding Cerami sequence does. The sets that link according to that definition produce not only PS sequences, but also Cerami sequences. A drawback of linking is that the sets must separate the functional, i.e., they must satisfy (4).

There is another approach which works when one cannot obtain linking sets which separate the functional. In this approach, one looks for suitable sets A, B such that the functional is bounded from below on one set and bounded from above on the other with no requirement on the relationship between the bounds. In other words, one looks for suitable subsets A, B of a Banach space E, which are such that

$$a_0 := \sup_A < \infty, \quad b_0 := \inf_B G > -\infty \tag{7}$$

holding for a given C^1 functional G on E implies the existence of a PS sequence

$$G(u_k) \longrightarrow c, \quad G'(u_k) \longrightarrow 0, \tag{8}$$

where $b_0 \leqslant c < \infty$. Obviously, it is much easier to verify (7) than it is to verify (4). If A, B are such that (7) always implies (8), we say that A, B form a **sandwich pair.** It was shown

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in [**11**] that sandwich pairs are not rare. General criteria for sets to form sandwich pairs were given there.

The purpose of this paper is to find sandwich pairs that produce sequences similar to Cerami sequences having advantages over ordinary PS sequences. We call such sets **linking sandwich pairs**. We present applications which take advantage of this fact. We exhibit cases in which PS sequences will not work.

One sandwich pair was studied initially (cf. [**7**, **8**, **14**]):

THEOREM 1. *Let N be a closed subspace of a Hilbert space E and let $M = N^{\perp}$. Assume that at least one of the subspaces M, N is finite dimensional. Let G be a C^1-functional on E such that*

$$m_0 := \inf_{w \in M} G(w) \neq -\infty \tag{9}$$

and

$$m_1 := \sup_{v \in N} G(v) \neq \infty. \tag{10}$$

Then there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \longrightarrow c,\ m_0 \leqslant c \leqslant m_1,\ G'(u_k) \longrightarrow 0. \tag{11}$$

It follows from this that M, N form a sandwich pair if one of them is finite dimensional. One of the results of this paper is:

THEOREM 2. *Under the hypotheses of Theorem* 1, *there are constants $c, C \in \mathbb{R}$ and sequences $\{v_k\} \subset \mathbb{R}$, $\{u_k\} \subset E$ such that $v_k \to \infty$ and*

$$G(u_k) \longrightarrow c,\ m_0 \leqslant c \leqslant m_1,\ (v_k + \|u_k\|)\|G'(u_k)\| \leqslant C. \tag{12}$$

The sequence (12) is not quite a Cerami sequence, but it is just as effective in most applications. Cerami sequences were introduced in [**3**]. Variations were given in [**2**, **5**, **6**].

Our abstract theory will be presented in Section 3 after we review some basic facts concerning linking. Our applications are given in Section 4. In Section 5 we present some results concerning differential equations in Banach space which are used in our proofs given in the last section.

2. *Linking*

Let E be a Banach space, and let Φ be the set of all continuous maps $\Gamma = \Gamma(t)$ from $E \times [0, 1]$ to E such that:

(1) $\Gamma(0) = I$, the identity map;

(2) For each $t \in [0, 1)$, $\Gamma(t)$ is a homeomorphism of E onto E and $\Gamma^{-1}(t) \in C(E \times [0, 1), E)$;

(3) $\Gamma(1)E$ is a single point in E and $\Gamma(t)A$ converges uniformly to $\Gamma(1)E$ as $t \to 1$ for each bounded set $A \subset E$;

(4) For each $t_0 \in [0, 1)$ and each bounded set $A \subset E$

$$\sup_{0 \leqslant t \leqslant t_0,\ u \in A} \{\|\Gamma(t)u\| + \|\Gamma^{-1}(t)u\|\} < \infty. \tag{13}$$

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We make the following:

Definition. For $A, B \subset E$ we say that A links B if:

(a) $A \cap B = \phi$;

(b) for each $\Gamma \in \Phi$ there is a $t \in (0, 1]$ such that

$$\Gamma(t)A \cap B \neq \phi.$$

Roughly speaking, this says that A links B if it cannot be slipped away from B by one of the mappings $\Gamma \in \Phi$. The importance of this concept appears in

THEOREM 3. *Let G be a C^1-functional on E, and let A, B be subsets of E such that A links B and*

$$a_0 := \sup_A G \leqslant b_0 := \inf_B G. \tag{14}$$

Assume that

$$a := \inf_{\Gamma \in \Phi} \sup_{0 \leqslant s \leqslant 1,\ u \in A} G(\Gamma(s)u) \tag{15}$$

is finite. Let $\psi(t)$ be a positive, nonincreasing, locally Lipschitz continuous function on $[0, \infty)$ such that

$$\int_0^\infty \psi(r)\, dr = \infty. \tag{16}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \longrightarrow a, \quad G'(u_k)/\psi(\|u_k\|) \longrightarrow 0. \tag{17}$$

COROLLARY 4. *Under the hypotheses of Theorem 3 there is a sequence $\{u_k\} \subset E$ such that*

$$G(u_k) \longrightarrow a, \quad (1 + \|u_k\|)G'(u_k) \longrightarrow 0. \tag{18}$$

Proof. We merely take

$$\psi(r) = \frac{1}{1 + r}$$

in Theorem 3.

Theorem 3 was proved in [**9**]. As a consequence, we have

COROLLARY 5. *Assume that, for each $\Gamma \in \Phi$, the set*

$$g_\Gamma := \{v = \Gamma(s)u : s \in (0, 1], u \in A, v \notin A, G(v) \geqslant a_0\} \tag{19}$$

is not empty. Then there is a sequence satisfying (17).

Proof. Let

$$B = \bigcup_{\Gamma \in \Phi} g_\Gamma. \tag{20}$$

Then $A \cap B = \phi$, and for each $\Gamma \in \Phi$, there are a $v \in B$, an $s \in (0, 1]$ and a $u \in A$ such that $v = \Gamma(s)u$. Thus $\Gamma(s)A \cap B \neq \phi$. This means that A links B. Since $a_0 \leqslant G(v)$ for

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all $v \in B$, we have $a_0 \leqslant b_0$. We can now apply Theorem 3 to conclude that a sequence satisfying (17) exists.

COROLLARY 6. *If $a < \infty$ and $a_0 \neq a$, then a sequence satisfying* (17) *exists.*

Proof. If $a_0 < a$, then for each $\Gamma \in \Phi$ there are a $u \in A,\ s \in (0, 1]$ such that $G(\Gamma(s)u) > a_0$. Clearly $v = \Gamma(s)u \notin A$. Thus the set g_Γ given by (19) is not empty. We can now apply Corollary 5.

3. *Criteria*

We now turn to our study of sandwich pairs. We are interested in obtaining structured sequences. We use Theorem 2 as a model.

Definition 7. We shall say that subsets A,B of a Banach space E form a **linking sandwich pair** if for every $G \in C^1(E, \mathbb{R})$ satisfying

$$a_0 := \sup_A G < \infty, \quad b_0 := \inf_B G > -\infty, \tag{21}$$

there are constants $c, C \in \mathbb{R}$ and sequences $\{\nu_k\} \subset \mathbb{R}$, $\{u_k\} \subset E$ such that $\nu_k \to \infty$ and

$$G(u_k) \longrightarrow c,\ b_0 \leqslant c \leqslant a_0,\ (\nu_k + \|u_k\|)\|G'(u_k)\| \leqslant C. \tag{22}$$

In this section we present sufficient conditions for sets to qualify as linking sandwich pairs. We have:

THEOREM 8. *Let G be a C^1-functional on E, and let $A, B \subset E$ be such that $d(A, B) > 0$, $A \subset \mathbf{B}_R$ links B, and*

$$a_0 := \sup_A G < \infty, \quad b_0 := \inf_B G > -\infty, \tag{23}$$

where

$$\mathbf{B}_R = \{u \in E : \|u\| \leqslant R\}.$$

Assume that a given by (15) *is finite. Let $\psi(t)$ be a locally Lipschitz continuous nonincreasing positive function satisfying*

$$\int_0^\infty \psi(t)\, dt = \infty \tag{24}$$

and

$$\int_0^d \psi(t)\, dt > a_0 - b_0, \tag{25}$$

where $d = d(A, B)$. Take ν so large that

$$\int_R^\nu \psi(t)\, dt > a_0 - b_0. \tag{26}$$

Let $\tilde{B} = B \cap B_\nu$. Then there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \longrightarrow c, \quad b_0 \leqslant c \leqslant a_0, \quad \|G'(u_k)\| \leqslant \psi(d(u_k, \tilde{B})). \tag{27}$$

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THEOREM 9. *Let G be a C^1-functional on E, and let $A, B \subset E$ be such that $d(A, B) > 0$, $A \subset \mathbf{B}_R$ links B, and*

$$a_0 := \sup_A G < \infty, \quad b_0 := \inf_B G > -\infty. \tag{28}$$

Assume that a given by (15) *is finite. Assume that there are constants $\varepsilon > 0$ and $\nu > 0$ such that $\nu\varepsilon < d$ and $(1-\varepsilon)\nu > R$. Then there are constants $c, \gamma_\varepsilon \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that*

$$G(u_k) \longrightarrow c, \quad b_0 \leqslant c \leqslant a_0, \quad (\nu + \|u_k\|)\|G'(u_k)\| \leqslant \gamma_\varepsilon. \tag{29}$$

THEOREM 10. *Let G be a C^1-functional on E, and let $A_k, B_k \subset E$ be such that $d_k = d(A_k, B_k) \to \infty$, $A_k \subset \mathbf{B}_{R_k}$ links B_k, and*

$$\sup_{A_k} G \leqslant a_0 < \infty, \quad \inf_{B_k} G \geqslant b_0 > -\infty. \tag{30}$$

Assume that

$$a_k := \inf_{\Gamma\in\Phi} \sup_{0\leqslant s\leqslant 1,\ u\in A_k} G(\Gamma(s)u) \tag{31}$$

is finite for each k. Assume that for some constant $\varepsilon > 0$ there is a positive sequence $\{\nu_k\}$ such that $\nu_k\varepsilon < d_k$ and $(1-\varepsilon)\nu_k > R_k$. Then there are constants $c, \gamma_\varepsilon \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \longrightarrow c, \quad b_0 \leqslant c \leqslant a_0, \quad (\nu_k + \|u_k\|)\|G'(u_k)\| \leqslant \gamma_\varepsilon. \tag{32}$$

COROLLARY 11. *Let N be a closed subspace of a Hilbert space E with complement M, and let $A^R = N\backslash\mathbf{B}_R$. Assume that one of the subspaces M, N is finite dimensional. Then A^R, M form a linking sandwich pair.*

THEOREM 12. *Let N be a finite dimensional subspace of a Banach space E, and let p be any point of N. Let F be a continuous map of E onto N such that $F = I$ on N and*

$$d(N \cap \partial\mathbf{B}_r, F^{-1}(p)) \longrightarrow \infty \text{ as } r \longrightarrow \infty.$$

If $R > \|p\|$, then $A^R = N\backslash\mathbf{B}_R$ and $F^{-1}(p)$ form a linking sandwich pair.

COROLLARY 13. *Let M be a closed subspace of a Hilbert space E with complement N, let v_0 is an element in N having unit norm, and let $\delta < R$ be positive numbers. Let $A = N\backslash\mathbf{B}_R$ and*

$$B = \{w \in M : \|w\| \geqslant \delta\} \cup \{sv_0 + w : w \in M, s \geqslant 0, \|sv_0 + w\| = \delta\}.$$

Assume that one of the subspaces M, N is finite dimensional. Then A, B form a linking sandwich pair.

4. *Applications*

In this section we assume that Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ sufficiently regular so that the Sobolev inequalities hold and the embedding of $H^{m,2}(\Omega)$ in $L^2(\Omega)$ iscompact (cf. [**1**]). Let A be a self-adjoint operator on $L^2(\Omega)$. We assume that $A \geqslant \lambda_0 > 0$

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and that

$$C_0^\infty(\Omega) \subset D := D(A^{1/2}) \subset H^{m,2}(\Omega)$$

for some $m > 0$, where $C_0^\infty(\Omega)$ denotes the set of test functions in Ω (i.e., infinitely differentiable functions with compact supports in Ω) and $H^{m,2}(\Omega)$ denotes the Sobolev space. If m is an integer, the norm in $H^{m,2}(\Omega)$ is given by

$$\|u\|_{m,2} := \left(\sum_{|\mu| \leqslant m} \|D^\mu u\|^2 \right)^{1/2}. \tag{33}$$

Here D^μ represents the generic derivative of order $|\mu|$ and the norm on the right-hand side of (33) is that of $L^2(\Omega)$. We shall not assume that m is an integer.

Let q be any number satisfying

$$\begin{aligned} 2 \leqslant q &< 2n/(n-2m), \quad 2m < n, \\ 2 \leqslant q &< \infty, \qquad\qquad n \leqslant 2m, \end{aligned}$$

and let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$. This means that $f(x,t)$ is continuous in t for a.e. $x \in \Omega$ and measurable in x for every $t \in \mathbb{R}$. We make the following assumptions:

(A) The function $f(x,t)$ satisfies

$$|f(x,t)| \leqslant V(x)^q(|t|^{q-1} + W(x))$$

and

$$f(x,t)/V(x)^q = o(|t|^{q-1}) \text{ as } |t| \longrightarrow \infty,$$

where $V(x) > 0$ is a function in $L^q(\Omega)$ such that

$$\|Vu\|_q \leqslant C\|u\|_D, \quad u \in D \tag{34}$$

and W is a function in $L^\infty(\Omega)$. Here

$$\|u\|_q := \left(\int_\Omega |u(x)|^q dx \right)^{1/q},$$
$$\|u\|_D := \|A^{1/2}u \tag{35}$$

and $q' = q/(q-1)$. (If $V(x)$ are bounded, then (34) will hold automatically by the Sobolev inequality. However, there are functions $V(x)$ which are unbounded and such that (34) holds.) With the norm (35), D becomes a Hilbert space. Define

$$F(x,t) := \int_0^t f(x,s)\, ds$$

and

$$G(u) := \|u\|_D^2 - 2\int_\Omega F(x,u)\, dx. \tag{36}$$

It is readily shown that G is a continuously differentiable functional on the whole of D (cf., e.g. [**10**]). Since the embedding of D in $L^2(\Omega)$ is compact, the spectrum of A consists of isolated eigenvalues of finite multiplicity

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_\ell < \cdots.$$

Let $\lambda_\ell,\ \ell > 0$, be one of these eigenvalues. (We take λ_0 to be an eigenvalue.)

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For each fixed k, let N_k denote the subspace of $D := D(A^{1/2})$ spanned by the eigenfunctions corresponding to $\lambda_0, \dots, \lambda_k$, and let $M_k = N_k^{\perp} \cap D$. Then $D = M_k \oplus N_k$. We define

$$\alpha_k := \max\{(Av, v) : v \in N_k, v \geqslant 0, \|v\| = 1\}, \tag{37}$$

where $\|v\|$ denotes the $L^2(\Omega)$ norm of v. We assume that A has an eigenfunction φ_0 of constant sign a.e. on Ω corresponding to the eigenvalue λ_0.

Next we define for $a \in \mathbb{R}$

$$\gamma_k(a) := \max\{(Av, v) - a\|v^-\|^2 \ : \ v \in N_k, \ \|v^+\| = 1\} \tag{38}$$

and

$$\Gamma_k(a) := \inf\{(Aw, w) - a\|w^-\|^2 : \ w \in M_k, \ \|w^+\| = 1\}, \tag{39}$$

where $u^{\pm} = \max\{\pm u, 0\}$.

Assume that

$$H(x, t) = 2F(x, t) - tf(x, t) \geqslant -W_1(x) \in L^1(\Omega), \quad x \in \Omega, \ t \in \mathbb{R} \tag{40}$$

and

$$H(x, t) \longrightarrow \infty \text{ a.e. as } |t| \longrightarrow \infty. \tag{41}$$

We have

THEOREM 14. *Take N to be the subspace of D spanned by the eigenspaces of A corresponding to the eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_\ell$. We take $M = N^{\perp} \cap D$. Assume that there are numbers a_1, a_2 such that $\alpha_\ell < a_1 \leqslant a_2$ and*

$$a_1(t^-)^2 + \gamma_\ell(a_1)(t^+)^2 - W_1(x) \leqslant 2F(x, t) \leqslant a_2(t^-)^2 + \Gamma_\ell(a_2)(t^+)^2 + W_2(x), \quad x \in \Omega, \ t \in \mathbb{R}, \tag{42}$$

and that (40) *and* (41) *hold. Then*

$$Au = f(x, u), \ u \in D \tag{43}$$

has at least one solution.

Proof. First, we note that

$$\sup_N G \leqslant B_1, \ \inf_M G \geqslant -B_2, \ B_j = \int_\Omega W_j(x)\, dx. \tag{44}$$

To see this, note that by (38) we have

$$\|v\|_D^2 \leqslant a_1\|v^-\|^2 + \gamma_\ell(a_1)\|v^+\|^2, \quad v \in N. \tag{45}$$

By (39) we have

$$a_2\|w^-\|^2 + \Gamma_\ell(a_2)\|w^+\|^2 \leqslant \|w\|_D^2, \quad w \in M. \tag{46}$$

Hence

$$G(v) \leqslant B_1, \quad v \in N,$$

and

$$G(w) \geqslant -B_2, \quad w \in M,$$

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by (42). By Theorem 2 we conclude that there are sequences $\nu_k \to \infty$ and $\{u_k\} \subset D$ such that

$$G(u_k) \longrightarrow c, \ -B_2 \leqslant c \leqslant B_1, \ (\nu_k + \|u_k\|_D)\|G'(u_k)\| \leqslant C. \tag{47}$$

In particular, we have

$$\|u_k\|_D^2 - 2\int_\Omega F(x, u_k)\,dx \longrightarrow c \tag{48}$$

and

$$\left|\|u_k\|_D^2 - (f(\cdot, x_k), u_k)\right| \leqslant K. \tag{49}$$

Consequently

$$\left|\int_\Omega H(x, u_k)\,dx\right| \leqslant K'. \tag{50}$$

If $\rho_k = \|u_k\|_D \to \infty$, let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Consequently there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω. In view of (48), we have

$$1 - 2\int_\Omega F(x, u_k)/\rho_k^2\,dx \longrightarrow 0. \tag{51}$$

But by (42), we have

$$2F(x, u_k)/\rho_k^2 \leqslant a_2(\tilde{u}_k^-)^2 + \Gamma_\ell(a_2)(\tilde{u}_k^+)^2 + W_2(x)/\rho_k^2. \tag{52}$$

In the limit this implies

$$1 \leqslant a_2\|\tilde{u}^-\|^2 + \Gamma_\ell(a_2)\|\tilde{u}^+\|^2.$$

This shows that $\tilde{u} \not\equiv 0$. Let Ω_0 be the subset of Ω on which $\tilde{u} \neq 0$. Then

$$|u_k(x)| = \rho_k|\tilde{u}_k(x)| \longrightarrow \infty, \quad x \in \Omega_0. \tag{53}$$

If $\Omega_1 = \Omega \setminus \Omega_0$, then we have

$$\int_\Omega H(x, u_k)\,dx = \int_{\Omega_0} + \int_{\Omega_1} \geqslant \int_{\Omega_0} H(x, u_k)\,dx - \int_{\Omega_1} W_1(x)\,dx \longrightarrow \infty. \tag{54}$$

This contradicts (50), and we see that $\rho_k = \|u_k\|_D$ is bounded. Once we know that the ρ_k are bounded we can apply [**10**, theorem 3·4·1] to obtain the desired conclusion.

Remark 15. It should be noted that the crucial element in the proof of Theorem 14 was (49). If we had been dealing with an ordinary Palais–Smale sequence, we could only conclude that

$$\|u_k\|_D^2 - (f(\cdot, u_k), u_k) = o(\rho_k),$$

which would imply only

$$\int_\Omega H(x, u_k)dx = o(\rho_k).$$

This would not contradict (54), and the argument would not go through.

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We also have

THEOREM 16. *The conclusion of Theorem* 14 *holds if in place of* (40), (41) *we assume*

$$H(x,t) \leqslant W_1(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R} \tag{55}$$

and

$$H(x,t) \longrightarrow -\infty \text{ a.e. as } |t| \longrightarrow \infty. \tag{56}$$

Proof. We use (55) and (56) to replace (54) with

$$\int_\Omega H(x,u_k)\,dx = \int_{\Omega_0} + \int_{\Omega_1} \leqslant \int_{\Omega_0} H(x,u_k)\,dx + \int_{\Omega_1} W_1(x)\,dx \longrightarrow -\infty. \tag{57}$$

We then proceed as before.

We also have

THEOREM 17. *If we replace* (40) *and* (41) *with*

$$H(x,t) := 2F(x,t) - tf(x,t) \leqslant C(|t|^\alpha + 1) \tag{58}$$

and

$$\sigma(x) := \limsup_{|t|\to\infty} H(x,t)/|t|^\alpha < 0 \quad a.e. \tag{59}$$

for some positive $\alpha < q$, *then*

$$Au = f(x,u),\ u \in D \tag{60}$$

has at least one solution.

Proof. As before, we have

$$G(v) \leqslant B_1, \quad v \in N,$$

and

$$G(w) \geqslant -B_2, \quad w \in M.$$

In view of these inequalities we can now apply Corollary 11 to conclude that there are sequences $\{\nu_k\} \subset \mathbb{R}$, $\{u_k\} \subset D$ such that $\nu_k \to \infty$ and

$$G(u_k) \longrightarrow c,\ -B_1 \leqslant c < B_2,\ (\nu_k + \|u_k\|)\,\|G'(u_k)\| \leqslant C. \tag{61}$$

Let $\rho_k = \|u_k\|_D$. If $\rho_k \to \infty$, then

$$G(u_k) = \rho_k^2 - 2\int_\Omega F(x,u_k)\,dx \longrightarrow c \tag{62}$$

and

$$(G'(u_k), u_k)/2 = \rho_k^2 - \int_\Omega f(x,u_k)u_k\,dx \longrightarrow 0.$$

Hence,

$$\left|\int_\Omega H(x,u_k)\,dx\right| \leqslant C. \tag{63}$$

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Let $\tilde{u_k} = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Thus, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω. By (58) and (59)

$$\limsup \int_\Omega H(x, u_k)\, dx/\rho_k^\alpha \leqslant \int_\Omega \limsup[H(x, u_k)/|u_k|^\alpha]|\tilde{u}_k|^\alpha\, dx$$
$$= \int_\Omega \sigma(x)|\tilde{u}|^\alpha\, dx.$$

Since $\sigma(x) < 0$ a.e. in Ω, the last two statements imply that $\tilde{u} \equiv 0$. However, we see from (62) that In view of (48), we have

$$1 - 2\int_\Omega F(x, u_k)/\rho_k^2\, dx \longrightarrow 0. \tag{64}$$

But by (42), we have

$$2F(x, u_k)/\rho_k^2 \leqslant a_2(\tilde{u}_k^-)^2 + \Gamma_\ell(a_2)(\tilde{u}_k^+)^2 + W_2(x)/\rho_k^2. \tag{65}$$

In the limit this implies

$$1 \leqslant a_2\|\tilde{u}^-\|^2 + \Gamma_\ell(a_2)\|\tilde{u}^+\|^2.$$

This shows that $\tilde{u} \not\equiv 0$. This contradiction tells us that the ρ_k must be bounded. We can now apply [**10**, theorem 3·4·1] to conclude that there is a $u \in D$ satisfying

$$G(u) = c,\ G'(u) = 0. \tag{66}$$

The proof is complete.

THEOREM 18. *The conclusion of Theorem* 14 *holds if we replace* (40) *and* (41) *with*

$$H(x, t) := 2F(x, t) - tf(x, t) \geqslant -C(|t|^\alpha + 1) \tag{67}$$

and

$$\sigma(x) := \limsup_{|t|\to\infty} H(x, t)/|t|^\alpha > 0 \quad a.e. \tag{68}$$

for some positive $\alpha < q$.

The nonquadraticity condition (59) was first introduced by Costa and Magalhães [**4**].
We now exhibit conditions which will guarantee nontrivial solutions. We have

THEOREM 19. *Assume that* $F(x, t)$ *satisfies*

$$a_1(t^-)^2 + \gamma_\ell(a_1)(t^+)^2 - W_1(x) \leqslant 2F(x, t)$$
$$\leqslant a_2(t^-)^2 + \mu(t^+)^2, \quad x \in \Omega,\ t \in \mathbb{R} \tag{69}$$

for numbers a_1, a_2 *satisfying* $\alpha_\ell < a_1 \leqslant a_2$, *where* W_1 *is a function in* $L^1(\Omega)$ *and* $\mu < \Gamma_\ell(a_2)$. *We also assume* (40), (41)

$$2F(x, t) \leqslant \lambda_\ell t^2, \quad |t| \leqslant \delta \tag{70}$$

for some $\delta > 0$, *and the functions in* $E(\lambda_\ell)$ *are bounded. Then* (43) *has a nontrivial solution.*

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Proof. By (38)

$$\|v\|_D^2 \leqslant a_1\|v^-\|^2 + \gamma_\ell(a_1)\|v^+\|^2, \quad v \in N, \tag{71}$$

and by (39) we have

$$a_2\|w^-\|^2 + \Gamma_\ell(a_2)\|w^+\|^2 \leqslant \|w\|_D^2, \quad w \in M. \tag{72}$$

Hence

$$G(v) \leqslant B_1, \quad v \in N. \tag{73}$$

Since $\mu < \Gamma_\ell(a_2)$, we see by continuity that there is an $\varepsilon > 0$ such that

$$\mu < (1-\varepsilon)\Gamma_\ell\left(\frac{a_2}{1-\varepsilon}\right).$$

Hence,

$$\begin{aligned} G(w) &\geqslant \varepsilon\|w\|_D^2 + (1-\varepsilon)\left[\Gamma_\ell\left(\frac{a_2}{1-\varepsilon}\right) - \frac{\mu}{1-\varepsilon}\right]\|w^+\|^2 \\ &\geqslant \varepsilon\|w\|_D^2, \quad w \in M \end{aligned} \tag{74}$$

by (69).

We note that (70) implies

$$G(w+y) \geqslant \varepsilon'\|w\|^2, \quad w \in M_\ell,\ y \in E(\lambda_\ell)$$

for $\|w+y\|$ sufficiently small. To see this, let $u = w + y$. Note that there is a $\rho > 0$ such that

$$\|y\|_D \leqslant \rho \Longrightarrow |y(x)| \leqslant \delta/2, \quad y \in E(\lambda_\ell).$$

Now suppose u satisfies

$$\|u\|_D \leqslant \rho \text{ and } |u(x)| \geqslant \delta \tag{75}$$

for some $x \in \Omega$. Then for those $x \in \Omega$ satisfying (75) we have

$$\delta \leqslant |u(x)| \leqslant |w(x)| + |y(x)| \leqslant |w(x)| + (\delta/2).$$

Hence

$$|y(x)| \leqslant \delta/2 \leqslant |w(x)|$$

and consequently,

$$|u(x)| \leqslant 2|w(x)| \tag{76}$$

for all such x. Now we have by hypothesis (A)

$$\begin{aligned} G(u) &\geqslant \|u\|_D^2 - \lambda_\ell \int_{|u|<\delta} u^2 dx - C\int_{|u|>\delta} (|Vu|^q + |V^q u|W)\, dx \\ &\geqslant \|u\|_D^2 - \lambda_\ell\|u\|^2 - C'\int_{|u|>\delta} |Vu|^q\, dx \\ &\geqslant \|w\|_D^2 - \lambda_\ell\|w\|^2 - C''\int_{2|w|>\delta} |Vw|^q\, dx \\ &\geqslant \left(1 - \frac{\lambda_\ell}{\lambda_{\ell+1}} - C'''\|w\|_D^{q-2}\right)\|w\|_D^2 \end{aligned}$$

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in view of the fact that $\|y\|_D^2 = \lambda_\ell \|y\|^2$,

$$|F(x,t)| \leqslant C(|Vt|^q + |V^q t|),$$

and (76) holds. Moreover, the following alternative holds:

Either:

(a) there is an infinite number of eigenfunctions $y \in E(\lambda_\ell) \setminus \{0\}$ such that

$$Ay = f(x,y) = \lambda_\ell y; \tag{77}$$

or:

(b) for each $\rho > 0$ sufficiently small, there is an $\varepsilon > 0$ such that

$$G(w) \geqslant \varepsilon, \quad \|w\|_D = \rho,\ w \in M_{\ell-1}. \tag{78}$$

Since option (a) solves our problem, we may assume that option (b) holds. Let $v_0 \in E(\lambda_\ell)$, and let $\varphi(t) \in C^1(\mathbb{R})$ be such that

$$0 \leqslant \varphi(t) \leqslant 1,\ \varphi(0) = 1$$

and

$$\varphi(t) = 0, \quad |t| \geqslant 1.$$

Let

$$F(v + w + sv_0) = v + [s + \delta - \delta\varphi(\|w\|^2/\delta^2)]v_0, \quad v \in N_{\ell-1},\ w \in M,\ s \in \mathbb{R}. \tag{79}$$

Take $A = N$, $B = F^{-1}(\delta v_0)$. By (73), (74), (78) we see that (21) holds with $b_0 > 0$ and $a_0 = B_1$. We can now apply Corollary 13 to conclude that there are sequences $\{\nu_k\} \subset \mathbb{R}$, $\{u_k\} \subset D$ such that $\nu_k \to \infty$ and

$$G(u_k) \longrightarrow c,\ b_0 \leqslant c < B_1,\ (\nu_k + \|u_k\|)\ \|G'(u_k)\| \leqslant C. \tag{80}$$

The proof proceeds as in the proof of Theorem 14 providing a solution of (43) satisfying

$$G(u) = c,\ b_0 \leqslant c \leqslant B_1.$$

Since $b_0 > 0$ and $G(0) = 0$, we see that the solution is nontrivial.

We also have:

THEOREM 20. *The conclusion of Theorem* 19 *holds if in place of* (40), (41) *we assume*

$$H(x,t) \leqslant W_1(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R} \tag{81}$$

and

$$H(x,t) \longrightarrow -\infty \text{ a.e. as } |t| \longrightarrow \infty. \tag{82}$$

THEOREM 21. *The conclusion of Theorem* 19 *holds if we replace* (40) *and* (41) *with*

$$H(x,t) := 2F(x,t) - tf(x,t) \leqslant C(|t|^\alpha + 1) \tag{83}$$

and

$$\sigma(x) := \limsup_{|t|\to\infty} H(x,t)/|t|^\alpha < 0 \quad a.e. \tag{84}$$

for some positive $\alpha < q$.

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THEOREM 22. *The conclusion of Theorem* 19 *holds if we replace* (40) *and* (41) *with*

$$H(x,t) := 2F(x,t) - tf(x,t) \geqslant -C(|t|^{\alpha} + 1) \tag{85}$$

and

$$\sigma(x) := \limsup_{|t|\to\infty} H(x,t)/|t|^{\alpha} > 0 \quad a.e. \tag{86}$$

for some positive $\alpha < q$.

5. *Ordinary differential equations*

In proving Theorem 3 we shall make use of various extensions of Picard's theorem in a Banach space. Some are well known.

THEOREM 23. *Let* X *be a Banach space, and let*

$$B_0 = \{x \in X : \|x - x_0\| \leqslant R_0\}$$

and

$$I_0 = \{t \in \mathbb{R} : |t - t_0| \leqslant T_0\}.$$

Assume that $g(t,x)$ *is a continuous map of* $I_0 \times B_0$ *into* X *such that*

$$\|g(t,x) - g(t,y)\| \leqslant K_0\|x - y\|, \quad x, y \in B_0,\ t \in I_0 \tag{87}$$

and

$$\|g(t,x)\| \leqslant M_0, \quad x \in B_0,\ t \in I_0. \tag{88}$$

Let T_1 *be such that*

$$T_1 \leqslant \min(T_0,\ R_0/M_0), \quad K_0T_1 < 1. \tag{89}$$

Then there is a unique solution $x(t)$ *of*

$$\frac{dx(t)}{dt} = g(t, x(t)), \quad |t - t_0| \leqslant T_1, \quad x(t_0) = x_0. \tag{90}$$

LEMMA 24. *Let* $\gamma(t)$ *and* $\rho(t)$ *be continuous functions on* $[0,\infty)$, *with* $\gamma(t)$ *nonnegative and* $\rho(t)$ *positive. Assume that*

$$\int_{u_0}^{\infty} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\,ds, \tag{91}$$

where $t_0 < T$ *and* $u_0 \geqslant 0$. *Then there is a unique solution of*

$$u'(t) = \gamma(t)\rho(u(t)), \quad t \in [t_0, T), \quad u(t_0) = u_0 \tag{92}$$

which is positive in (t_0, T) *and depends continuously on* u_0.

Proof. One can separate variables to obtain

$$W(u) = \int_{u_0}^{u} \frac{d\tau}{\rho(\tau)} = \int_{t_0}^{t} \gamma(s)\,ds.$$

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The function $W(u)$ is differentiable and increasing in $\mathbb{R}$, positive in (u_0, ∞), depends continuously on u_0 and satisfies

$$W(u) \longrightarrow L = \int_{u_0}^{\infty} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\, ds, \quad \text{as } u \longrightarrow \infty.$$

Thus, for each $t \in [t_0, T)$ there is a unique $u \in [u_0, \infty)$ such that

$$u = W^{-1}\left(\int_{t_0}^{t} \gamma(s)\, ds\right)$$

is the unique solution of (92), and it depends continuously on u_0.

LEMMA 25. *Let $\gamma(t)$ and $\rho(t)$ be continuous functions on $[0, \infty)$, with $\gamma(t)$ nonnegative and $\rho(t)$ positive. Assume that*

$$\int_{m}^{u_0} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\, ds, \tag{93}$$

where $t_0 < T$ and $m < u_0$ are given positive numbers. Then there is a unique solution of

$$u'(t) = -\gamma(t)\rho(u(t)), \quad t \in [t_0, T), \quad u(t_0) = u_0 \tag{94}$$

which is $\geqslant m$ in $[t_0, T)$ and depends continuously on u_0.

Proof. One can separate variables to obtain

$$W(u) = \int_{u}^{u_0} \frac{d\tau}{\rho(\tau)} = \int_{t_0}^{t} \gamma(s)\, ds.$$

The function $W(u)$ is differentiable and decreasing in $\mathbb{R}$, positive in $[m, u_0]$, depends continuously on u_0 and satisfies

$$W(u) \longrightarrow L = \int_{m}^{u_0} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\, ds, \quad \text{as } u \longrightarrow m.$$

Thus, for each $t \in [t_0, T)$ there is a unique $u \in [m, u_0]$ such that

$$u = W^{-1}\left(\int_{t_0}^{t} \gamma(s)\, ds\right)$$

is the unique solution of (94), and it depends continuously on u_0.

THEOREM 26. *Assume, in addition to the hypotheses of Theorem 23, that*

$$\|g(t, x)\| \leqslant \gamma(t)\rho(\|x - v\|), \quad x \in B_0,\ t \in I_0, \tag{95}$$

where $\gamma(t)$ and $\rho(t)$ satisfy the hypotheses of Lemma 24 with $T = t_0 + T_1$ and v is a fixed element of X. Let $u(t)$ be the positive solution of

$$u'(t) = \gamma(t)\rho(u(t)), \quad t \in [t_0, T), \quad u(t_0) = u_0 \geqslant \|x_0 - v\|, \tag{96}$$

provided by that lemma. Then the unique solution of (90) satisfies

$$\|x(t) - v\| \leqslant u(t), \quad t \in [t_0, T). \tag{97}$$

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Proof. Assume that there is a $t_1 \in [t_0, T)$ such that

$$u(t_1) < \|x(t_1) - v\|.$$

For $\varepsilon > 0$, let $u_\varepsilon(t)$ be the solution of

$$u'(t) = [\gamma(t) + \varepsilon]\rho(u(t)), \quad t \in [t_0, T), \quad u(t_0) = u_0. \tag{98}$$

By Lemma 24, a solution exists for $\varepsilon > 0$ sufficiently small. Let

$$w(t) = \|x(t) - v\| - u_\varepsilon(t).$$

Then, we may take ε sufficiently small so that

$$w(t_0) \leqslant 0, \quad w(t_1) > 0.$$

Let t_2 be the largest number in $[t_0, t_1)$ such that $w(t_2) = 0$ and

$$w(t) > 0, \quad t \in (t_2, t_1].$$

For $h > 0$ sufficiently small, we have

$$\frac{w(t_2 + h) - w(t_2)}{h} > 0.$$

Consequently,

$$D^+ w(t_2) \geqslant 0.$$

But

$$\begin{aligned} D^+ w(t_2) &= D^+\|x(t_2) - v\| - u'_\varepsilon(t_2) \\ &\leqslant \|x'(t_2)\| - u'_\varepsilon(t_2) \\ &= \|g(t_2, x(t_2))\| - [\gamma(t_2) + \varepsilon]\rho(u_\varepsilon(t_2)) \\ &\leqslant \gamma(t_2)\rho(\|x(t_2) - v\|) - [\gamma(t_2) + \varepsilon]\rho(u_\varepsilon(t_2)) \\ &= -\varepsilon\rho(u_\varepsilon(t_2)) \\ &< 0. \end{aligned} \tag{99}$$

This contradiction proves the theorem.

THEOREM 27. *Let $g(t, x)$ be a continuous map from $\mathbb{R} \times H$ to H, where H is a Banach space. Assume that for each point $(t_0, x_0) \in \mathbb{R} \times H$, there are constants $K, b > 0$ such that*

$$\|g(t, x) - g(t, y)\| \leqslant K\|x - y\|, \quad |t - t_0| < b, \ \|x - x_0\| < b, \ \|y - x_0\| < b. \tag{100}$$

Assume also that

$$\|g(t, x)\| \leqslant \gamma(t)\rho(\|x - v\|), \quad x \in H, \ t \in [t_0, T_M), \tag{101}$$

where $T_M \leqslant \infty$, and $\gamma(t)$, $\rho(t)$ satisfy the hypotheses of Lemma 24 with ρ nondecreasing and v a fixed element of H. Then for each $x_0 \in H$ there is a unique solution $x(t)$ of the equation

$$\frac{dx(t)}{dt} = g(t, x(t)), \quad t \in [t_0, T_M), \quad x(t_0) = x_0. \tag{102}$$

Moreover, $x(t)$ depends continuously on x_0 and satisfies

$$\|x(t) - v\| \leqslant u(t), \quad t \in [t_0, T_M), \tag{103}$$

where $u(t)$ is the solution of (92) *in that interval satisfying $u(t_0) = u_0 \geqslant \|x_0 - v\|$.*

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Proof. By Theorems 23 and 26 there is an interval $[t_0, t_0 + m]$, $m > 0$, in which a unique solution of

$$\frac{dx(t)}{dt} = g(t, x(t)), \quad t \in [t_0, t_0 + m], \quad x(t_0) = x_0 \tag{104}$$

exists and satisfies

$$\|x(t) - v\| \leqslant u(t), \quad t \in [t_0, t_0 + m], \tag{105}$$

where $u(t)$ is the unique solution of

$$u'(t) = \gamma(t)\rho(u(t)), \quad t \in [t_0, T_M), \quad u(t_0) = u_0 = \|x_0 - v\|. \tag{106}$$

Let $T \leqslant T_M$ be the supremum of all numbers $t_0 + m$ for which this holds. If $t_1 < t_2 < T$, then the solution in $[t_0, t_2]$ coincides with that in $[t_0, t_1]$, since such solutions are unique. Thus a unique solution of (104) satisfying (105) exists for each $t_0 < t < T$. Moreover, we have

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} g(t, x(t))\, dt.$$

Consequently,

$$\begin{aligned} \|x(t_2) - x(t_1)\| &\leqslant \int_{t_1}^{t_2} \|g(t, x(t))\|\, dt \\ &\leqslant \int_{t_1}^{t_2} \gamma(t)\rho(\|x(t) - v\|)\, dt \\ &\leqslant \int_{t_1}^{t_2} \gamma(t)\rho(u(t))\, dt \\ &= u(t_2) - u(t_1). \end{aligned}$$

Assume that $T < T_M$. Let t_k be a sequence such that $t_0 < t_k < T$ and $t_k \to T$. Then

$$\|x(t_k) - x(t_j)\| \leqslant u(t_k) - u(t_j) \longrightarrow 0.$$

Thus $\{x(t_k)\}$ is a Cauchy sequence in H. Since H is complete, $x(t_k)$ converges to an element $x_1 \in H$. Since $\|x(t_k) - v\| \leqslant u(t_k)$, we see that $\|x_1 - v\| \leqslant u(T)$. Moreover, we note that

$$x(t) \longrightarrow x_1 \text{ as } t \longrightarrow T.$$

To see this, let $\varepsilon > 0$ be given. Then there is a k such that

$$\|x(t_k) - x_1\| < \varepsilon, \quad u(T) - u(t_k) < \varepsilon.$$

Then for $t_k \leqslant t < T$,

$$\begin{aligned} \|x(t) - x_1\| &\leqslant \|x(t) - x(t_k)\| + \|x(t_k) - x_1\| \\ &\leqslant u(t) - u(t_k) + \|x(t_k) - x_1\| < 2\varepsilon. \end{aligned}$$

We define $x(T) = x_1$. Then, we have a solution of (104) satisfying (105) in $[0, T]$. By Theorem 23, there is a unique solution of

$$\frac{dy(t)}{dt} = g(t, y(t)), \quad y(T) = x_1 \tag{107}$$

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satisfying $\|y(t) - v\| \leqslant u(t)$ in some interval $|t - T| < \delta$. By uniqueness, the solution of (107) coincides with the solution of (104) in the interval $(T - \delta, T]$. Define

$$\begin{aligned} z(t) &= x(t), \quad t_0 \leqslant t < T, \\ z(T) &= x_1, \\ z(t) &= y(t), \quad T < t \leqslant T + \delta. \end{aligned}$$

This gives a solution of (104) satisfying (105) in the interval $[t_0, T + \delta)$, contradicting the definition of T. Hence, $T = T_M$.

THEOREM 28. *Let ρ, γ satisfy the hypotheses of Lemma* 25, *with ρ locally Lipschitz continuous. Let $u(t)$ be the solution of* (94), *and let $h(t)$ be a continuous function satisfying*

$$h(t) \geqslant h(s) - \int_s^t \gamma(r)\rho(h(r))\,dr, \quad t_0 \leqslant s < t < T, \quad h(t_0) \geqslant u_0. \tag{108}$$

Then

$$u(t) \leqslant h(t), \quad t \in [t_0, T). \tag{109}$$

Proof. Assume that there is a point t_1 in the interval such that

$$h(t_1) < u(t_1).$$

Let

$$y(t) = u(t) - h(t), \quad t \in [t_0, T).$$

Then, $y(t_0) \leqslant 0$ and $y(t_1) > 0$. Let τ be the largest point $< t_1$ such that $y(\tau) = 0$. Then

$$y(t) > 0, \quad t \in (\tau, t_1]. \tag{110}$$

Moreover, by (94) and (108) we have

$$y(t) \leqslant -\int_\tau^t \gamma(s)[\rho(u(s)) - \rho(h(s))]\,ds \leqslant L\int_\tau^t y(s)\,ds, \tag{111}$$

where L is the Lipschitz constant for ρ at $u(\tau)$ times the maximum of γ in the interval. Let

$$w(t) = \int_\tau^t y(s)\,ds.$$

Then

$$[e^{-Lt}w(t)]' = e^{-Lt}[y(t) - Lw(t)] \leqslant 0, \quad t \in [\tau, t_1].$$

Consequently,

$$e^{-Lt}w(t) \leqslant e^{-L\tau}w(\tau) = 0, \quad t \in [\tau, t_1].$$

Hence,

$$y(t) \leqslant Lw(t) \leqslant 0, \quad t \in [\tau, t_1],$$

contradicting (110). This completes the proof.

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6. *The remaining proofs*

We can now prove Theorem 8.

Proof. If

$$a := \inf_{\Gamma\in\Phi} \sup_{0\leqslant s\leqslant 1,\ u\in A} G(\Gamma(s)u) \tag{112}$$

is finite and $a_0 < a$, then the conclusion of the theorem follows from Corollary 6. Otherwise, we must have $b_0 \leqslant a_0 = a$. If the theorem were not true, there would be a $\delta > 0$ such that

$$\|G'(u)\| \geqslant \psi(d(u, \tilde{B})) \tag{113}$$

would hold for all u in the set

$$Q = \{u \in E : b_0 - 3\delta \leqslant G(u) \leqslant a + 3\delta\}. \tag{114}$$

By reducing δ if necessary, we can find $\theta < 1,\ T > 0$ such that

$$a_0 - b_0 + 2\delta < \theta T, \quad T < \int_\delta^d \psi(s)\, ds,\ T \leqslant \int_R^v \psi(t)\, dt. \tag{115}$$

Let

$$Q_0 = \{u \in Q : b_0 - 2\delta \leqslant G(u) \leqslant a + 2\delta\} \tag{116}$$

$$Q_1 = \{u \in Q : b_0 - \delta \leqslant G(u) \leqslant a + \delta\} \tag{117}$$

and

$$Q_2 = E \setminus Q_0,\ \eta(u) = d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)]. \tag{118}$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\eta(u) = 1,\ u \in Q_1;\ \eta(u) = 0,\ u \in \overline{Q}_2;\ 0 < \eta(u) < 1,\ \text{otherwise.}$$

There is a locally Lipschitz continuous map $Y(u)$ of $\hat{E} = \{u \in E : G'(u) \neq 0\}$ into itself such that

$$\|Y(u)\| \leqslant 1,\ \theta\|G'(u)\| \leqslant (G'(u), Y(u)), \quad u \in \hat{E} \tag{119}$$

(cf., e.g., [**10**]). Let $\sigma(t)$ be the flow generated by

$$W(u) = -\eta(u)Y(u)\rho(d(u, \tilde{B})), \tag{120}$$

where $\rho(\tau) = 1/\psi(\tau)$. Since $\|W(u)\| \leqslant \rho(d(u, \tilde{B}))$ and $W(u)$ is locally Lipschitz continuous, $\sigma(t)$ exists for all $t \in [0, T]$ in view of Theorem 27. Since

$$\sigma(t)u - u = \int_0^t W(\sigma(\tau)u)\, d\tau, \tag{121}$$

we have

$$\|\sigma(t)u - \sigma(s)u\| \leqslant \int_s^t \rho(d(\sigma(r)u, \tilde{B}))\, dr.$$

If $u \in A,\ v \in \tilde{B}$, we have

$$h(s) = d(\sigma(s)u, \tilde{B}) \leqslant \|\sigma(s)u - v\| \leqslant \|\sigma(t)u - v\| + \int_s^t \rho(d(\sigma(r)u, \tilde{B}))\, dr.$$

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This implies,

$$h(s) \leqslant h(t) + \int_s^t \rho(h(r))\, dr. \tag{122}$$

By Lemma 25, if $u(t)$ is the solution of (94) with $\rho(t) = 1/\psi(t)$, $\gamma = 1$, $t_0 = 0$, $m = \delta$, and $u_0 = d$, then

$$u(t) \geqslant \delta, \quad t \in [0, T].$$

Thus, by Theorem 28,

$$h(t) \geqslant \delta, \quad t \in [0, T].$$

This implies

$$\sigma(t)A \cap \tilde{B} = \phi, \quad t \in [0, T]. \tag{123}$$

For $v \in A$, let $x(t) = \sigma(t)v - v$. Then

$$\|x'(t)\| \leqslant \rho(\|x(t) + v\|), \quad x(0) = 0.$$

Let $u(t)$ be the solution of (92) satisfying $u(0) = R$. Then by Theorem 27,

$$\|\sigma(t)v\| = \|x(t) + v\| \leqslant u(t), \quad 0 \leqslant t \leqslant T.$$

Now it follows from the choice of v and the fact that $A \subset \mathbf{B}_R$, that

$$\int_R^{u(T)} \psi(t)\, dt = T \leqslant \int_R^v \psi(t)\, dt.$$

Thus,

$$\|\sigma(t)v\| \leqslant v, \quad v \in A,\ t \in [0, T]$$

by Lemma 24 and Theorem 27. Hence,

$$\sigma(t)A \cap [B \backslash \tilde{B}] = \phi, \quad t \in [0, T]. \tag{124}$$

If we combine this with (123), we obtain

$$\sigma(t)A \cap B = \phi, \quad t \in [0, T]. \tag{125}$$

We also have

$$\begin{aligned} dG(\sigma(t)v)/dt &= (G'(\sigma), \sigma') = -\eta(\sigma)(G'(\sigma), Y(\sigma))\rho(d(\sigma, \tilde{B})) \\ &\leqslant -\theta\eta(\sigma)\|G'(\sigma)\|\rho(d(\sigma, \tilde{B})) \\ &\leqslant -\theta\eta(\sigma)\psi(d(\sigma, \tilde{B}))\rho(d(\sigma, \tilde{B})) \\ &= -\theta\eta(\sigma) \end{aligned} \tag{126}$$

in view of (113) and (119). Now suppose $v \in E_{a+\delta}$ is such that there is a $t_1 \in [0, T]$ for which $\sigma(t_1)v \notin Q_1$, where

$$E_\gamma = \{v \in E : G(v) \leqslant \gamma\}.$$

Then

$$G(\sigma(t_1)v) < b_0 - \delta,$$

since we cannot have $G(\sigma(t_1)v) > a_0 + \delta$ for $v \in E_{a+\delta}$ by (126). But this implies

$$G(\sigma(T)v) < b_0 - \delta. \tag{127}$$

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On the other hand, if $\sigma(t)v \in Q_1$ for all $t \in [0, T]$, then

$$G(\sigma(T)v) \leqslant G(v) - \theta \int_0^T dt \leqslant a_0 + \delta - \theta T < b_0 - \delta$$

by (115). Thus, (127) holds for $v \in E_{a+\delta}$. By the definition (15) of a, there is a $\Gamma \in \Phi$ such that

$$G(\Gamma(s)u) < a + \delta, \quad 0 \leqslant s \leqslant 1, \ u \in A. \tag{128}$$

Hence,

$$G(\sigma(T)\Gamma(s)u) < b_0 - \delta, \quad 0 \leqslant s \leqslant 1, \ u \in A.$$

This implies

$$\sigma(T)\Gamma(s)A \cap B = \phi, \quad 0 \leqslant s \leqslant 1. \tag{129}$$

Define

$$\Gamma_1(s) = \begin{cases} \sigma(2sT), & 0 \leqslant s \leqslant \frac{1}{2}, \\ \sigma(T)\Gamma(2s-1), & \frac{1}{2} < s \leqslant 1. \end{cases}$$

Clearly, $\Gamma_1 \in \Phi$. But (125) and (129) imply

$$\Gamma_1(s)A \cap B = \phi,$$

contradicting the fact that A links B. Hence, there is a sequence satisfying (27). This completes the proof.

Proof of Theorem 9. Take

$$\gamma_\varepsilon = \frac{a_0 - b_0}{\ln(1 + \varepsilon/3)}$$

and

$$\psi(t) = \frac{\gamma_\varepsilon}{2\nu + t}.$$

Then $\psi(t)$ satisfies (24), (25) and (26). Consequently, there is a sequence satisfying

$$G(u_k) \longrightarrow c, \quad b_0 \leqslant c \leqslant a_0, \quad (2\nu + d(u_k, \tilde{B}))\|G'(u_k)\| \leqslant \gamma_\varepsilon.$$

Since

$$\|u_k\| \leqslant \nu + d(u_k, \tilde{B}),$$

we have

$$(\nu + \|u_k\|)\|G'(u_k)\| \leqslant \gamma_\varepsilon.$$

The proof is complete.

Proof of Theorem 10. Take γ_ε as above and

$$\psi_k(t) = \frac{\gamma_\varepsilon}{2\nu_k + t}.$$

Then $\psi_k(t)$ satisfies (24), (25) and (26) for each k. Then there is a sequence satisfying

$$G(u_k) \longrightarrow c, \quad b_0 \leqslant c \leqslant a_0, \quad (2\nu_k + d(u_k, \tilde{B}_k))\|G'(u_k)\| \leqslant \gamma_\varepsilon.$$

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Since

$$\|u_k\| \leqslant \nu_k + d(u_k, \tilde{B}_k),$$

we have

$$(\nu_k + \|u_k\|)\|G'(u_k)\| \leqslant \gamma_\varepsilon.$$

Apply Theorem 9.

Proof of Corollary 11. For any $R > 0$, $A_R = N \bigcap \partial \mathbf{B}_R$ links M by [**10**, corollary 13·5]. We can now apply Theorem 10. Take $\varepsilon = 1/3$, and let $\{R_k\} \to \infty$. Take $A_k = N \bigcap \partial \mathbf{B}_{R_k}$ and $B_k = M$ in Theorem 10. Then $d_k = R_k$ and $\nu_k = 3R_k$. We can take $C = a_0 - b_0/\ln(10/9)$.

Proof of Theorem 12. A_R links $B = F^{-1}(p)$ for each $R > \|p\|$ ([**10**, proposition 2·6·2]). Moreover, $d(A_R, B) > 0$. For, if $u_k \in A_R$, $v_k \in B$ and $\|u_k - v_k\| \to 0$, then there is a renamed subsequence such that $u_k \to u_0 \in A_R$. Hence, $v_k \to u_0$. Consequently, $p = F(v_k) \to F(u_0) = u_0$. Since this cannot happen, we can now apply Theorem 10.

Proof of Corollary 13. We note that A_R links B for R sufficiently large ([**10**, corollary 13·6]). Moreover, $d(A_R, B) > 0$. We can take $B = F^{-1}(\delta v_0)$, where F is given by (79). Apply Theorem 10.

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