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ASYMPTOTIC ERROR DISTRIBUTION OF THE EULER SCHEME FOR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS WITH LOCALLY LIPSCHITZ COEFFICIENTS

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Abstract

This paper focuses mainly on the Euler scheme of stochastic delay differential equations with locally Lipschitz coefficients. The convergence in probability of the Euler scheme and the corresponding weak limit process of the normalized error process are derived. Furthermore, this paper also considers a class of specific degenerate stochastic delay equations and obtains the associated weak limit process for the stronger error process. The error parameter of this stronger error process for such a degenerate system is n instead of $\sqrt{n}$ in the normalized error process. This causes substantial challenges in the analysis and proofs and the weak limit process also becomes more complicated and involves some additional terms. This result is new and interesting even for the non-delay case.

Keywords: SDDEs; Euler schemes; locally Lipschitz condition; weak convergence; normalized error process

2020 Mathematics Subject Classification: Primary 60H10
Secondary 60H35; 65C30

1. Introduction and motivation

Let us consider the following scalar stochastic delay differential equation (SDDE)

$$\mathrm{d}X_t = b(X_t, X_{\delta(t)})\,\mathrm{d}t + \sigma(X_t, X_{\delta(t)})\,\mathrm{d}W_t, \quad t \geq 0 \tag{1}$$

with the initial data $\xi_{[-\tau,0]} \in C([-\tau, 0];\mathbb{R})$ (in this paper, $x_{[a,b]}$ represents the function x from $[a, b]$ to $\mathbb{R}$), where $\delta(t) = t/q_0 - \tau$ for some positive integer q_0 represents the variable time delay, and $b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are measurable functions. When $\tau = 0$, this equation is also called the stochastic pantograph equation and has wide applications in applied mathematics and engineering, see [2, 12, 17, 18, 30]. Since most stochastic differential equations (SDEs) and SDDEs have no explicit solutions, numerical methods play an important role in examining these equations (see [24]).

Many kinds of numerical schemes such as the Euler–Maruyama (EM) scheme, the Milstein-type scheme, and the implicit EM have been widely studied to approximate SDEs and SDDEs.

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By using the EM scheme, Mao and Sabanis [33] established strong convergence of SDEs with variable delay under the locally Lipschitz condition and the linear growth condition. This result was extended in [13] and the convergence order is obtained under locally Lipschitz and monotonicity conditions in the sense of almost sure. For higher-order numerical schemes, Küchler and Platen [25] examined the Itô–Taylor scheme for SDDEs with fixed delay, including the Euler-type, the Milstein-type, and the implicit schemes and obtained the convergence rate 1. Hu *et al.* [15] developed the strong Milstein approximation scheme to approximate SDDEs by using the Malliavin calculus and anticipating calculus.

By using the infinite-dimensional version of the Kolmogorov equation, Buckwar and Shardlow established the weak convergence for SDDEs by a class of smooth test functionals and showed that the convergence rate is proportional to the time step [7]. In [6], they, together with Kuske and Mohammed, proved that the EM scheme for SDDEs has weak order of convergence 1 by using the Malliavin calculus and the anticipating stochastic analysis techniques developed by Nualart and Pardoux [34]. In [3], Bao and Shao investigated the weak convergence of the EM schemes for path-dependent SDEs with irregular coefficients via Girsanov's transformation and the dimension-free Harnack inequality.

This paper aims to approximate (1) by using the one-step explicit Euler scheme X_t^n and investigate the limit distribution of the normalized error process $\sqrt{n}(X_t^n - X_t)$. For a class of specific degenerate stochastic delay equation (see (12)), the stronger error process $n(X_t^n - X_t)$ is also examined. In the past few decades, the limit distribution of the (normalized) error process has received increased attention owing to its broad applications in Monte Carlo methods and efficient algorithms for SDEs (see, for example, [1, 8–10, 14, 23). In [37], Rootzén demonstrated that the error in approximating stochastic integrals $\int_0^t g(W_s, s)\,\mathrm{d}W_s$ converges in distribution to the Brownian motion B_υ with the time scale

$$\upsilon = \frac{1}{2}\int_0^t \left(\frac{\partial g(W_s, s)}{\partial x}\right)^2 \mathrm{d}s.$$

Under the smoothness condition and the linear growth condition, [28] established the asymptotic distribution of the normalized error process of the Euler scheme for SDEs, which also implies the rate of the weak convergence of the EM scheme. Jacod and Protter [21] presented a necessary and sufficient condition of the weak convergence rate for the Euler scheme of SDEs driven by semimartingales. Yan further extended the work [21] to the Milstein-type scheme of SDEs driven by continuous semimartingales [39].

The above results about asymptotic error distributions of the EM schemes often require that the coefficients of SDEs satisfy the smoothness condition and the linear growth condition. Under non-globally Lipschitz continuous coefficients, Hutzenthaler *et al.* showed that the explicit EM scheme diverges in finite time under both strong and weak convergence [16]. When the coefficients are locally Lipschitz continuous, [35] established the asymptotic distribution of the normalized error process of the Euler scheme for SDEs. However, that paper required that the diffusion coefficient σ is non-negative and σ^{-2} is locally integrable, which excludes many simple cases such as $\sigma(x) = x$ or $\sigma(x) = -1$. To the best of the authors' knowledge, no result is known about the asymptotic properties of the normalized error processes of the EM scheme for SDDEs. This paper will fill this gap and derive the asymptotic error distribution of the EM scheme for SDDEs under locally Lipschitz coefficients. Moreover, for a class of special degenerate SDDEs with fixed time delay, this paper also examines the asymptotic properties of the stronger error process $n(X_t^n - X_t)$. The key contributions of this paper are highlighted as follows.

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- This paper extends the results in [35] to SDDEs with variable time delay and significantly relaxes the conditions on the diffusion coefficient. The presence of variable delay $\delta(t)$ implies that this SDDE is non-autonomous, which also introduces additional difficulties and requires more delicate calculations and proofs. When $\delta(t) = 0$, our results reduce to those in [35] under weaker coefficient conditions. The framework in [35] does not cover the case where b is locally Lipschitz and $\sigma = -1$, which shows that our results can be applied to more systems.
- For a class of special degenerate SDDEs with fixed delay, this paper further studies the asymptotic properties of the stronger error process $n(X_t^n - X_t)$, in which the error parameter n (rather than the normalized $\sqrt{n}$). This causes the weak limit process to become more complicated and involves additional terms. Based on this asymptotic error distribution for degenerate SDDEs, this paper further obtains the weak convergence order 1 of the EM scheme, which is difficult when using existing methods. Compared with the results in [6], our work not only relaxes the restrictions for test functions but also avoids using the Malliavin calculus.

The remainder of the paper is organized as follows. Section 2 presents some necessary assumptions, notation, lemmas, and the main results. In Section 3, the detailed proof of the first theorem (Theorem 2.3) is provided. To prove this theorem, several useful lemmas are also presented in this section. Section 4 gives the proof of the second main theorem (Theorem 2.4) about the asymptotic error distribution of the stronger error process $n(X_t^n - X_t)$ for a class of degenerate SDDEs with fixed delay. In Section 5, two examples are given to illustrate our results. The appendix collects several auxiliary lemmas.

2. Assumptions and main results

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$ represent a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Here W represents the scalar Brownian motion defined on this probability space. Denote by $\mathbb{Z}_+$ the set of positive integers. Let $|\cdot|$ denote the Euclidean norm. Let A^* represent the transpose of matrix A. In this paper, C denotes a generic positive constant, whose value may change for different usage. Thus, $C + C = C$ and $CC = C$ are understood in an appropriate sense. Similarly, C_α denotes the generic positive constant depending on parameter α. We use λ to denote the Lebesgue measure. For a set A, A^C represents the complement of A. Let $\lfloor a \rfloor$ denote the largest integer not exceeding a. For two martingales X_t and Y_t, $[X, Y]_t$ denotes the cross-variation process between them. We use $Y_n \xrightarrow{\text{a.s.}} Y$ to represent that Y_n converges to Y pathwise or almost surely (a.s.), $Y_n \xrightarrow{\text{P}} Y$ to denote that Y_n converges to Y in probability, and $Y_n \Rightarrow Y$ to represent that Y_n converges to Y weakly. Let us also present the definition of the stable convergence; see, for example, [21].

Defintion 2.1. *Let $\{Y_n\}_{n\geq 0}$ be a sequence of random variables taking values in a Polish space E, defined on the same probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Here Y_n is said to converge stably in law to Y (denoted by $Y^n \xRightarrow{\mathcal{S}} Y$) if Y is an E-valued random variable defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathrm{P}})$ of $(\Omega, \mathcal{F}, \mathrm{P})$ and*

$$\lim_{n\to\infty} \mathrm{E}\big(Uf(Y_n)\big) = \tilde{\mathrm{E}}\big(Uf(Y)\big)$$

for any bounded continuous function $f : E \to \mathbb{R}$ and any bounded measurable random variable U.

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The following Lemmas 2.1 and 2.2 are taken from [1, 27].

Lemma 2.1. *Let V^n, V, Y^n, and Y be defined on $(\Omega, \mathcal{F}, \mathrm{P})$, and with values on the metric space E. If $V^n \xrightarrow{\mathrm{P}} V$ and $Y^n \overset{\mathcal{S}}{\Rightarrow} Y$, then $(V^n, Y^n) \overset{\mathcal{S}}{\Rightarrow} (V, Y)$.*

A direct conclusion from this lemma is $(V, Y^n) \overset{\mathcal{S}}{\Rightarrow} (V, Y)$.

Lemma 2.2 *Let $(V^n)_{n\geq 1}$ and V be defined on the same filtered probability space. If (V^n, Y^n) converges to (V, Y) in distribution and semimartingale sequence $(Y^n)_{n\geq 1}$ is uniformly tight, then Y is a semimartingale with respect to the filtration generated by the limit process (V, Y) and, moreover,*

$$\left(V^n, Y^n, \int V^n_- \, \mathrm{d}Y^n\right) \Rightarrow \left(V, Y, \int V_- \, \mathrm{d}Y\right).$$

This sequence $(Y^n)_{n\geq 1}$ is called a 'good' sequence. If $(V^n, Y^n) \xrightarrow{P} (V, Y)$, then the above triple also converges in probability.

For more details on the 'good' sequence and its properties, see also [28]. To proceed, let us impose the following assumptions on the coefficients b and σ.

(H1) *Locally Lipschitz condition*: b and σ are locally Lipschitz continuous, i.e. for any $R > 0$ and $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq R$, there exists a constant $C_R > 0$ such that

$$|b(x_1, x_2) - b(y_1, y_2)| + |\sigma(x_1, x_2) - \sigma(y_1, y_2)| \leq C_R(|x_1 - x_2| + |y_1 - y_2|).$$

(H2) *Khasminskii's test*: there is a function $V \in C^2(\mathbb{R})$ with $\lim_{|x|\to\infty} V(x) = \infty$ and a positive constant L such that

$$b(x, y)V'(x) + \frac{1}{2}\sigma^2(x, y)V''(x) \leq L(1 + V(x) + V(y)),$$

where $V'(x)$ and $V''(x)$ represent the first and second derivatives of $V(x)$, respectively.

Let us present the following theorem, see also [32].

Theorem 2.1. *Under assumptions **(H1)** and **(H2)**, (1) admits a unique global solution for any given initial data $\xi_{[-\tau,0]} \in C([-\tau, 0]; \mathbb{R})$.*

Let $h = \tau/n \in (0, 1]$ for some integer n large enough and define $\lfloor t \rfloor_h = \lfloor t/h \rfloor h$. Let $t_k = kh$ for any $k = -n, -n+1, \ldots, \lfloor T/h \rfloor$, define the discrete Euler scheme for (1) on $[0, T]$ as follows: for $k = -n, -n+1, \ldots, 0$,

$$X^n_{t_k} = \xi_{t_k},$$

and for $k = 0, 1, \ldots, \lfloor T/h \rfloor$,

$$X^n_{t_{k+1}} = X^n_{t_k} + b\big(X^n_{t_k}, X^n_{\lfloor \delta(t_k) \rfloor_h}\big)h + \sigma\big(X^n_{t_k}, X^n_{\lfloor \delta(t_k) \rfloor_h}\big)\Delta W_k, \tag{2}$$

where $\Delta W_k = W_{t_{k+1}} - W_{t_k}$.

The corresponding continuous-time Euler scheme is given as follows: for $t \geq 0$,

$$X^n_t = X^n_{\lfloor t \rfloor_h} + b\big(X^n_{\lfloor t \rfloor_h}, X^n_{\lfloor \delta(\lfloor t \rfloor_h) \rfloor_h}\big)(t - \lfloor t \rfloor_h) + \sigma\big(X^n_{\lfloor t \rfloor_h}, X^n_{\lfloor \delta(\lfloor t \rfloor_h) \rfloor_h}\big)(W_t - W_{\lfloor t \rfloor_h}), \tag{3}$$

where $X^n_t = \xi_t$ for $t \in [-\tau, 0]$.

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Remark 2.1. The step size τ/n implies the delayed point will fall exactly on the discrete grid points (see also [25, 33, 38]). For any $t \in [kh, (k+1)h)$ for some $k \geq 0$,

$$\lfloor \delta(\lfloor t \rfloor_h) \rfloor_h = \left\lfloor \frac{kh}{q_0} \right\rfloor_h - \tau = \left\lfloor \frac{k}{q_0} \right\rfloor h - \tau \quad \text{and} \quad \lfloor \delta(t) \rfloor_h = \left\lfloor \frac{t}{q_0} \right\rfloor_h - \tau = \left\lfloor \frac{t/h}{q_0} \right\rfloor h - \tau.$$

Let $\tilde{k} = k - \lfloor k/q_0 \rfloor q_0$, then $0 \leq \tilde{k} < q_0$. It is obvious that

$$\left\lfloor \frac{k}{q_0} \right\rfloor \leq \left\lfloor \frac{t/h}{q_0} \right\rfloor < \left\lfloor \frac{k+1}{q_0} \right\rfloor = \left\lfloor \frac{k}{q_0} \right\rfloor + \left\lfloor \frac{\tilde{k}+1}{q_0} \right\rfloor \leq \left\lfloor \frac{k}{q_0} \right\rfloor + 1,$$

hence

$$\lfloor \delta(\lfloor t \rfloor_h) \rfloor_h = \lfloor \delta(t) \rfloor_h = \left\lfloor \frac{k}{q_0} \right\rfloor h - \tau.$$

In other words, $\lfloor \delta(t) \rfloor_h$ is the grid point closest to $\delta(t)$.

2.1. Convergence in probability

For all $q \in \mathbb{Z}_+$ satisfying $\sup_{-\tau \leq t \leq 0} |\xi_t| < q - 1$, let

$$D^q = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \vee |y| \leq q\}.$$

Define

$$b^q(x, y) = b(x, y)\varphi^q(x, y) \quad \text{and} \quad \sigma^q(x, y) = \sigma(x, y)\varphi^q(x, y), \tag{4}$$

where the truncated function $\varphi^q \in C^\infty(\mathbb{R} \times \mathbb{R}; [0, 1])$ satisfies

$$\varphi^q(x, y) = \begin{cases} 1, & (x, y) \in D^q, \\ 0, & (x, y) \in \mathbb{R}^2 \backslash D^{q+1}. \end{cases}$$

It is easy to verify that b^q and σ^q are global Lipschitz continuous if b and σ are locally Lipschitz continuous. This implies that under assumption **(H1)**, there exists a unique global solution X_t^q for the truncated SDDE

$$\mathrm{d}X_t^q = b^q\big(X_t^q, X_{\delta(t)}^q\big)\,\mathrm{d}t + \sigma^q\big(X_t^q, X_{\delta(t)}^q\big)\,\mathrm{d}W_t, \quad t \geq 0 \tag{5}$$

with initial $X_t^q = \xi_t$ for $t \in [-\tau, 0]$. Similar to (3), let $X_t^{n,q}$ denote the corresponding continuous-time Euler approximation of (5).

Theorem 2.2. *Let assumptions **(H1)** and **(H2)** hold, and assume further that ξ is Hölder continuous with index $\gamma > 0$. Then for any $\beta \in [0, \gamma/2)$ and $\varepsilon > 0$,*

$$\lim_{n \to \infty} \mathrm{P}\left(n^\beta \sup_{0 \leq t \leq T} |X_t^n - X_t| > \varepsilon \right) = 0, \quad \textit{for all } T > 0.$$

Proof. Note that b_q and σ_q are globally Lipschitz continuous. Applying a similar approach to [33] gives that for each $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathrm{P}\left(n^\beta \sup_{0 \leq t \leq T} |X_t^{n,q} - X_t^q| > \varepsilon \right) = 0. \tag{6}$$

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For any $q > \sup_{-\tau \le t \le 0} |\xi_t| + 1$, define

$$\tau_q = \inf\{t \ge 0 : |X_t| \ge q\}, \quad \tau_q^n = \inf\{t \ge 0 : |X_t^n| \ge q\}.$$

This implies that for all $t \in [0, T]$, $X_{t\wedge\tau_q} = X^q_{t\wedge\tau_q}$ and $X^n_{t\wedge\tau_q^n} = X^{n,q}_{t\wedge\tau_q^n}$ almost surely. Hence,

$$\sup_{0\le t\le T\wedge\tau_q\wedge\tau_q^n} |X_t(\omega) - X_t^n(\omega)| = \sup_{0\le t\le T\wedge\tau_q\wedge\tau_q^n} |X_t^q(\omega) - X_t^{n,q}(\omega)|. \tag{7}$$

Then we have

$$\begin{aligned} \mathrm{P}\Big(n^\beta \sup_{0\le t\le T} |X_t - X_t^n| > \varepsilon\Big) &\le \mathrm{P}\Big(n^\beta \sup_{0\le t\le T} |X_t - X_t^n| > \varepsilon,\, \tau_q \wedge \tau_q^n > T\Big) + \mathrm{P}(\tau_q \wedge \tau_q^n \le T) \\ &\le \mathrm{P}\Big(n^\beta \sup_{0\le t\le T} |X_t^q - X_t^{n,q}| > \varepsilon\Big) + \mathrm{P}(\tau_q \le T) + \mathrm{P}\big(\tau_q^n \le T\big). \end{aligned} \tag{8}$$

Under assumptions **(H1)** and **(H2)**, one observes that (1) admits a non-explosive solution, which yields

$$\lim_{q\to\infty} \mathrm{P}\big(\tau_q \le T\big) = 0. \tag{9}$$

Note that $\big|X^n_{t\wedge\tau_q^n}\big| = \big|X^{n,q}_{t\wedge\tau_q^n}\big| \le q$, $\big|X_{t\wedge\tau_{q-1}}\big| = \big|X^q_{t\wedge\tau_{q-1}}\big| \le q-1$ and $\big|X^n_{\tau_q^n}\big| = \big|X^{n,q}_{\tau_q^n}\big| = q$, $\big|X_{\tau_{q-1}}\big| = \big|X^q_{\tau_{q-1}}\big| = q-1$. Therefore, we have

$$\Big\{\omega\colon \sup_{0\le t\le T} |X_t^n| \ge q\Big\} = \Big\{\omega\colon \sup_{0\le t\le T} |X_t^{n,q}| \ge q\Big\}$$

and

$$\Big\{\omega\colon \sup_{0\le t\le T} |X_t| \ge q-1\Big\} = \Big\{\omega\colon \sup_{0\le t\le T} |X_t^q| \ge q-1\Big\}.$$

These, together with the definition of τ_q^n, imply

$$\begin{aligned} \mathrm{P}\big(\tau_q^n \le T\big) &= \mathrm{P}\Big(\sup_{0\le t\le T} |X_t^n| \ge q\Big) = \mathrm{P}\Big(\sup_{0\le t\le T} |X_t^{n,q}| \ge q\Big) \\ &\le \mathrm{P}\Big(\sup_{0\le t\le T} |X_t^{n,q} - X_t^q| > 1\Big) + \mathrm{P}\Big(\sup_{0\le t\le T} |X_t| \ge q-1\Big). \end{aligned}$$

According to convergence of the EM scheme under the Lipschitz condition [32, 33],

$$\limsup_{n\to\infty} \mathrm{P}\Big(\sup_{0\le t\le T} |X_t^{n,q} - X_t^q| > 1\Big) = 0.$$

Letting $n \to \infty$ and then letting $q \to \infty$ gives

$$\lim_{q\to\infty} \limsup_{n\to\infty} \mathrm{P}\left(\tau_q^n \le T\right) \le \lim_{q\to\infty} \mathrm{P}\Big(\sup_{0\le t\le T} |X_t| \ge q-1\Big) = 0. \tag{10}$$

Then it follows from (6) and (8) that

$$\limsup_{n\to\infty} \mathrm{P}\Big(n^\beta \sup_{0\le t\le T} |X_t - X_t^n| > \varepsilon\Big) = 0,$$

which gives the desired result. □

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Remark 2.2. These proof techniques are mainly from [35] and can establish the convergence under the local Lipschitz condition. This theorem extends the convergence in probability from SDEs in [35] to SDDEs.

By using similar techniques, the pointwise convergence of the SDDEs under the locally Lipschitz condition can be obtained. More precisely, if for every $\varepsilon > 0$, one has $|X_t^{n,q} - X_t^q| \le C_\varepsilon(\omega) n^{-\beta+\varepsilon}$ for some non-negative random variable $C_\varepsilon(\omega)$ and β, then there exists a non-negative random variable $C_\varepsilon^1(\omega)$ such that $|X_t^n - X_t| \le C_\varepsilon^1(\omega) n^{-\beta+\varepsilon}$ for every $\varepsilon > 0$. This idea has already been used in [20] for SDEs.

2.2. Asymptotic error distribution for 'non-degenerate' SDDEs

For $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, let $\partial f(x, y)/\partial x$ and $\partial f(x, y)/\partial y$ denote the derivatives with respect to x and y, respectively. Let D_f denote the set where f is not differentiable and D_f^k denote the projection of D_f on the kth component for $k = 1, 2$. For $(x_0, y_0) \in D_b \cup D_\sigma$, let us redefine the derivatives of b and σ at (x_0, y_0) to be zero. To estimate the asymptotic error distribution of the normalized error processes, we further impose the following assumptions.

(H3) The initial data ξ is Lipschitz continuous.

(H4) $\sigma^2(x, y) > 0$ for $(x, y) \in D_b \cup D_\sigma$.

Theorem 2.3. *Under assumptions **(H1)–(H4)**, the normalized error process $U_t^n := \sqrt{n}\big(X_t^n - X_t\big)$ converges weakly to U_t for $t \in [0, T]$, where U_t is the unique solution of the linear SDDE*

$$\begin{aligned} U_t = {} & \int_0^t \frac{\partial b(X_s, X_{\delta(s)})}{\partial x} U_s \, \mathrm{d}s + \int_0^t \frac{\partial b(X_s, X_{\delta(s)})}{\partial y} U_{\delta(s)} \cdot I_{\{\delta(s)>0\}} \, \mathrm{d}s \\ & + \int_0^t \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x} U_s \, \mathrm{d}W_s + \int_0^t \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y} U_{\delta(s)} \cdot I_{\{\delta(s)>0\}} \, \mathrm{d}W_s \\ & - \int_0^t \sqrt{\frac{\tau}{2}} \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x} \sigma(X_s, X_{\delta(s)}) \, \mathrm{d}B_s^1 \\ & - \int_0^t \sqrt{\frac{f(s)}{s}} \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y} \sigma(X_{\delta(s)}, X_{\delta(\delta(s))}) \, \mathrm{d}B_s^2, \quad t \in [0, T] \end{aligned} \tag{11}$$

with $U_t = 0$ for $t \in [-\tau, 0]$, where $f(s) = (\tau q_0 \delta(s)/2) I_{\{\delta(s)\ge 0\}}$, B^1 and B^2 are two independent Brownian motions defined on an extension $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}})_{t\ge 0}, \bar{\mathrm{P}})$ of the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathrm{P})$, and independent of W.

The proof of Theorem 2.3 is presented in Section 3. We now present two corollaries and a remark to illustrate this theorem clearly.

Corollary 2.1. *Set $D = \{x = (x_1, x_2)^* \in \mathbb{R}^2 : \partial\sigma(x)/\partial x_1 \ne 0$ or $\partial\sigma(x)/\partial x_2 \ne 0\}$. Even if b, σ, δ and ξ are smooth enough and $\lambda(D^C) = 0$, the optimal rate of strong uniform convergence of the Euler schemes defined by (2) is still $1/\sqrt{n}$.*

Proof. Assume $\mathrm{E}[\sup_{0\le t\le T} |X_t^n - X_t|] \le C/n^{1/2+\epsilon}$ holds for some $\epsilon > 0$. The Chebyshev inequality leads to

$$\mathrm{P}\Big(n^{1/2} \sup_{0\le t\le T} |X_t^n - X_t| > C\Big) \to 0$$

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as $n \to \infty$, which further implies

$$\mathrm{P}\Big(\sup_{0\le t\le T}|U_t^n| > C\Big) \le \mathrm{P}\Big(n^{1/2}\sup_{0\le t\le T}|X_t^n - X_t| > C\Big) \to 0,$$

that is, $U_{[0,T]}^n \xrightarrow{\mathrm{P}} 0$ as $n \to \infty$. This contradicts the fact that $U_t \not\equiv 0$ due to $\partial\sigma(x)/\partial x_1 \neq 0$ or $\partial\sigma(x)/\partial x_2 \neq 0$ for $(x_1, x_2)^* \in D$ with $\lambda(D^C) = 0$. This proof is complete. □

The following corollary can be regarded as the weak convergence of the SDDE under relaxed conditions on the test function, with a convergence rate of $n^{-1/2}$.

Corollary 2.2. *Assume assumptions **(H3)** and **(H4)** hold. If b and σ satisfy the Lipschitz condition, then for any f satisfying $|f(x) - f(y)| \le C(1 + |x|^p + |y|^p)|x - y|$ for some $C, p \ge 0$ and $\mathrm{P}(X_T \in D_f) = 0$,*

$$\sqrt{n}\big[\mathrm{E}f(X_T^n) - \mathrm{E}f(X_T)\big] \to \bar{\mathrm{E}}\big[f'(X_T)U_T\big],$$

where $\bar{\mathrm{E}}$ is the expectation with respect to the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}})_{t\ge 0}, \bar{\mathrm{P}})$ in Theorem 2.3.

Proof. It is easy to observe that

$$\sqrt{n}\big(f(X_T^n) - f(X_T)\big) = f'(X_T)U_T^n + \left[\frac{f(X_T^n) - f(X_T)}{X_T^n - X_T} - f'(X_T)\right]U_T^n, \quad \mathrm{P}-\text{a.s.}$$

Note that the Lipschitz condition implies that b and σ also satisfy the linear growth condition, which yields that

$$\mathrm{E}\left[\sup_{0\le t\le T}|X_t|^{2(p+1)}\right] \le C \quad \text{and} \quad \mathrm{E}\left[\sup_{0\le t\le T}|U_t^n|^2\right] \le C,$$

which imply $f'(X_T)U_T^n$ is uniformly integrable by the condition $|f(x) - f(y)| \le C(1 + |x|^p + |y|^p)|x - y|$. In terms of (11), the Lipschitz conditions on b and σ give

$$\begin{aligned}\mathrm{E}\left[\sup_{0\le t\le T}|U_t|^p\right] \le& C_p\left(\int_0^T \mathrm{E}|U_s|^p\,\mathrm{d}s + \int_0^T \mathrm{E}\left[\sup_{0\le u\le s}|U_u|^p\right]\mathrm{d}s + \mathrm{E}\left(\int_0^T |U_s|^2\,\mathrm{d}s\right)^{p/2}\right)\\ &+ C_p\mathrm{E}\left(\int_0^T |U_{\delta(s)}|^2\,\mathrm{d}s\right)^{p/2} + C_p\mathrm{E}\left(\int_0^T |X_s|^2 + |X_{\delta(s)}|^2\,\mathrm{d}s\right)^{p/2}\\ \le& C_p\left(\int_0^T \mathrm{E}|U_s|^p\,\mathrm{d}s + \int_0^T \mathrm{E}\left[\sup_{0\le u\le s}|U_u|^p\right]\mathrm{d}s\right) + C_p,\end{aligned}$$

where $C_p > 0$ is a constant. Applying the Grönwall inequality yields

$$\mathrm{E}\left[\sup_{0\le t\le T}|U_t|^p\right] \le C_p < \infty,$$

which implies that $f'(X_T)U_T$ is integrable. Applying the Hölder inequality gives

$$\mathrm{E}\left|\left[\frac{f(X_T^n) - f(X_T)}{X_T^n - X_T} - f'(X_T)\right]U_T^n\right| \le (\mathrm{E}(U_T^n)^2)^{1/2}\left[\mathrm{E}\left(\frac{f(X_T^n) - f(X_T)}{X_T^n - X_T} - f'(X_T)\right)^2\right]^{1/2},$$

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which tends to zero as $n \to \infty$ since $X_T^n \xrightarrow{\text{a.s.}} X_T$. By [4, Theorem 4.1], we also have $U_T^n \overset{\mathcal{S}}{\Rightarrow} U_T$ for any convergent subsequence. This, together with Lemma 2.1, implies $f'(X_T)U_T^n \Rightarrow f'(X_T)U_T$. Furthermore, in light of [5, Theorem 3.5], one deduces

$$\sqrt{n}\big[\mathrm{E}f(X_T^n) - Ef(X_T)\big] \to \bar{\mathrm{E}}\big[f'(X_T)U_T\big].$$

This completes the proof. □

Remark 2.3. When b and σ are globally Lipschitz continuous, it has been established that $\mathrm{E}\big[\sup_{0\le t\le T}|X_t^n - X_t|\big] \le C/\sqrt{n}$, that is, the strong convergence order is $1/2$, see, for example, [33]. This implies that $\mathrm{E}\big[\sup_{0\le t\le T}\sqrt{n}|X_t^n - X_t|\big] \le C$, which indicates that the normalized error parameter is $\sqrt{n}$. This observation motivates us to consider the following question: for some special SDDEs, if their Euler schemes achieve first-order convergence, could we examine the asymptotic error distribution for the stronger error process $n\big(X_t^n - X_t\big)$? This motivates us to investigate the following degenerate SDDEs.

2.3. Asymptotic error distribution for degenerate SDDEs

Let us consider the following two-dimensional degenerate SDDEs:

$$\begin{cases} \mathrm{d}X_t^1 = b_1\big(X_t, X_{t-\tau}^1\big)\,\mathrm{d}t, \\ \mathrm{d}X_t^2 = b_2\big(X_t, X_{t-\tau}^1\big)\,\mathrm{d}t + \sigma\big(X_t^1, X_{t-\tau}^1\big)\,\mathrm{d}W_t, \end{cases} \tag{12}$$

with initial data $X_t^1 = \xi_t^1$ for $t \in [-\tau, 0]$, $X_0^2 = x_0 \in \mathbb{R}$, where $X_t = (X_t^1, X_t^2)$, $b_1, b_2 : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are measurable functions.

Remark 2.4. Such an equation is considered for the following reasons. First, this equation can describe some important systems such as the Hamiltonian system with time delay and noise. Second, since the diffusion coefficient σ only depends on the first component and its delay, this diffusion coefficient can be treated as a constant. It is well known that the Euler scheme for such SDEs has the strong convergence rate 1 when the diffusion coefficient is constant since the Euler scheme coincides with the Milstein scheme. This motivates us to consider the asymptotic error distribution for the stronger error process $n\big(X_t^n - X_t\big)$. Of course, here the fixed delay is considered only due to the restriction of computation and proof techniques.

Let $C_b^k(\mathbb{R}^d \times \mathbb{R}; \mathbb{R})$ denote the family of k-times continuously differentiable functions $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, bounded with their derivatives up to order k. For $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, let $\partial f(x, y)/\partial x_i$ denote the derivative with respect to the ith component of x and $\partial f(x, y)/\partial y$ represent the derivative with respect to y. Also denote

$$\nabla_x f(x, y) = \left(\frac{\partial f(x, y)}{\partial x_1}, \frac{\partial f(x, y)}{\partial x_2}\right) \quad \text{and} \quad \nabla_{xx} f(x, y) = \begin{pmatrix} \dfrac{\partial^2 f(x, y)}{\partial x_1 \partial x_1} & \dfrac{\partial^2 f(x, y)}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f(x, y)}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f(x, y)}{\partial x_2 \partial x_2} \end{pmatrix}.$$

Without loss of generality, assume $0 < \tau \le 1$. Similar to (3), let X_t^n denote the continuous-time Euler scheme of (12), that is, for $t \ge 0$,

$$\begin{cases} X_t^{n,1} = X_{\lfloor t\rfloor_h}^{n,1} + b_1\big(X_{\lfloor t\rfloor_h}^n, X_{\lfloor t-\tau\rfloor_h}^{n,1}\big)(t - \lfloor t\rfloor_h), \\ X_t^{n,2} = X_{\lfloor t\rfloor_h}^{n,2} + b_2\big(X_{\lfloor t\rfloor_h}^n, X_{\lfloor t-\tau\rfloor_h}^{n,1}\big)(t - \lfloor t\rfloor_h) + \sigma\big(X_{\lfloor t\rfloor_h}^{n,1}, X_{\lfloor t-\tau\rfloor_h}^{n,1}\big)(W_t - W_{\lfloor t\rfloor_h}), \end{cases} \tag{13}$$

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where $X_t^n = (X_t^{n,1}, X_t^{n,2})^*$ and $X_t^{n,1} = \xi_t$ for $t \in [-\tau, 0]$.

Then let us present the main theorem of this subsection.

Theorem 2.4. *Assume $b_i \in C_b^2(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R})$ for $i = 1, 2$, $\sigma \in C_b^2(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ and $\xi \in C_b^1(\mathbb{R}; \mathbb{R})$. Then $V_t^n := n(X_t^n - X_t)$ converges weakly to $V_t = (V_t^1, V_t^2)^*$, where V_t^i is the weak solution of the following SDDE:*

$$
\begin{aligned}
V_t^i = {} & \int_0^t \langle \nabla_x b_i(X_s, X_{s-\tau}^1), V_s \rangle \, \mathrm{d}s + \int_0^t \frac{\partial b_i(X_s, X_{s-\tau}^1)}{\partial y} V_{s-\tau}^1 \cdot I_{\{s>\tau\}} \, \mathrm{d}s \\
& - \int_0^t \frac{\partial b_i(X_s, X_{s-\tau}^1)}{\partial y} b_1(X_{s-\tau}, X_{s-2\tau}^1) \cdot I_{\{s>\tau\}} \, \mathrm{d}\bar{N}_s^1 \\
& - \frac{\tau}{2} \int_0^t \frac{\partial b_i(X_s, X_{s-\tau}^1)}{\partial y} \frac{\mathrm{d}\xi_{s-\tau}^1}{\mathrm{d}s} \cdot I_{\{s \le \tau\}} \, \mathrm{d}s \\
& - \int_0^t \langle \nabla_x b_i(X_s, X_{s-\tau}^1), b(X_s, X_{s-\tau}^1) \rangle \, \mathrm{d}\bar{M}_s^1 - \int_0^t \frac{\partial b_i(X_s, X_{s-\tau}^1)}{\partial x_2} \sigma_i(X_s^1, X_{s-\tau}^1) \, \mathrm{d}\bar{M}_s^2 \\
& - \frac{\tau}{4} \int_0^t \frac{\partial^2 b_i(X_s, X_{s-\tau}^1)}{\partial x_2 \partial x_2} \sigma^2(X_s^1, X_{s-\tau}^1) ds + \int_0^t \frac{\partial \sigma_i(X_s^1, X_{s-\tau}^1)}{\partial x_1} V_s^1 \, \mathrm{d}W_s \\
& + \int_0^t \frac{\partial \sigma_i(X_s^1, X_{s-\tau}^1)}{\partial y} V_{s-\tau}^1 \cdot I_{\{s>\tau\}} \, \mathrm{d}W_s - \int_0^t \frac{\partial \sigma_i(X_s^1, X_{s-\tau}^1)}{\partial x_1} b_1(X_s, X_{s-\tau}^1) \, \mathrm{d}\bar{M}_s^3 \\
& - \int_0^t \frac{\sigma_i(X_s^1, X_{s-\tau}^1)}{\partial y} b_1(X_{s-\tau}, X_{s-2\tau}^1) \cdot I_{\{s>\tau\}} \, \mathrm{d}\bar{N}_s^2 - \frac{\tau}{2} \int_0^t \frac{\sigma_i(X_s^1, X_{s-\tau}^1)}{\partial y} \frac{\mathrm{d}\xi_{s-\tau}^1}{\mathrm{d}s} \cdot I_{\{s \le \tau\}} \, \mathrm{d}s
\end{aligned}
$$

with $V_t^i = 0$ for $t \in [-\tau, 0]$, where $\sigma_1 = 0$, $\sigma_2 = \sigma$, $b = (b_1, b_2)^*$, and

$$
\bar{M}_t := (\bar{M}_t^1, \bar{M}_t^2, \bar{M}_t^3) = \left(\frac{t\tau}{2}, \frac{\tau}{\sqrt{12}} B_t + \frac{\tau}{2} W_t, -\frac{\tau}{\sqrt{12}} B_t + \frac{\tau}{2} W_t \right),
$$

$$
\bar{N}_t := (\bar{N}_t^1, \bar{N}_t^2) = \left(\frac{(t-\tau)\tau}{2}, -\sqrt{\frac{\tau^2(t-\tau)}{12t}} B_t + \sqrt{\frac{\tau^2(t-\tau)}{4t}} W_t \right) \cdot I_{\{t>\tau\}},
$$

and B is a standard Brownian motion independent of W.

Remark 2.5. Compared with Theorem 2.3, Theorem 2.4 considers the stronger error process $n(X_t^n - X_t)$, which implies that the proof techniques and computations will become more complicated since the second-order error term $n(X_s^n - X_{\lfloor s \rfloor}^n)^*(X_s^n - X_{\lfloor s \rfloor}^n)$ needs to be estimated. Moreover, the limiting distribution also depends on the derivative of the initial function in Theorem 2.4.

The proof of Theorem 2.4 is presented in Section 4. Let us present the following corollary and remark to illustrate this theorem.

Corollary 2.3. *Under the conditions of Theorem 2.4, for any f satisfying $|f(x) - f(y)| \le C(1 + |x|^p + |y|^p)|x - y|$ for some $C, p \ge 0$ and $\mathrm{P}(X_T \in D_f) = 0$,*

$$
n\big[\mathrm{E} f(X_T^n) - \mathrm{E} f(X_T)\big] \to \bar{\mathrm{E}}\big[\nabla f(X_T) V_T\big],
$$

where $\nabla f(x) = (\partial f(x)/\partial x_1, \partial f(x)/\partial x_2)$ and $\bar{\mathrm{E}}$ is the expectation with respect to the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}})_{t \ge 0}, \bar{\mathrm{P}})$.

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Remark 2.6. Under the conditions of Theorem 2.4, it is obvious that Theorem 2.3 still holds for the standard error process $U_t^n = \sqrt{n}(X_t^n - X_t)$ of (12), that is, U_t^n converges weakly to U_t with the help of Lemma A.2, where U_t is the weak solution to the following SDDE

$$U_t^i = \sum_{j=1}^{2} \int_0^t U_s^j \frac{\partial b_i(X_s, X_{s-\tau}^1)}{\partial x_j}\,\mathrm{d}s + \int_0^t U_{s-\tau}^1 \frac{\partial b_i(X_s, X_{s-\tau}^1)}{\partial y} \cdot I_{\{s>\tau\}}\,\mathrm{d}s$$
$$+ \int_0^t U_s^1 \frac{\partial \sigma_i(X_s^1, X_{s-\tau}^1)}{\partial x_1}\,\mathrm{d}W_s + \int_0^t U_{s-\tau}^1 \frac{\partial \sigma_i(X_s^1, X_{s-\tau}^1)}{\partial y} \cdot I_{\{s>\tau\}}\,\mathrm{d}W_s, \quad t \in [0, T] \quad (14)$$

with $U_t^i = 0$ for $t \in [-\tau, 0]$. Since (14) is linear and it is easy to observe that $U_t \equiv (0, 0)^*$ is its unique solution, which implies that $\sqrt{n}(X_t^n - X_t)$ converges to 0. The main reason is that the Euler scheme of (12) coincides with the Milstein scheme and holds the convergence order 1, while the parameter of the standard error process is $\sqrt{n}$.

In general, for the Euler scheme with an order of convergence of 1/2, to examine the asymptotic error distribution of the standard error process $\sqrt{n}(X_t^n - X_t)$, it is sufficient to estimate the first-order expansions of b and σ since the second-order expansions can be omitted due to the fact that $\sqrt{n}\mathrm{E}\big[\sup_{-\tau\le s\le T}|X_s - X_s^n|^2\big] \le Cn^{-1/2}$ tends to zero. This also shows that the initial data have no effect on the asymptotic error distribution of this system since the initial function is Lipschitz continuous. For (12), since the Euler scheme coincides with the Milstein scheme and holds convergence rate 1, it is necessary to take the second-order expansion into account, and the initial data will affect the asymptotic error distribution. This is the reason why the error process V_t^n uses the stronger scaling parameter n in Theorem 2.4. Even for (12) without time delay, this result is still new and interesting. Of course, Corollary 2.3 also implies that the weak convergence rate of the EM scheme for the degenerate stochastic system (12) is 1.

3. Proof of Theorem 2.3

To prove Theorem 2.3, let us first present the following four lemmas. The first lemma establishes the well-posedness of (11). For the normalized error process $U_t^{n,q} = \sqrt{n}(X_t^{n,q} - X_t^q)$ with the truncation, the second lemma proves that if $U_{[0,T]}^{n,q}$ converges weakly to $U_{[0,T]}^q$, then it follows that $U_{[0,T]}^n$ converges weakly to $U_{[0,T]}$, where $U_{[0,T]}^q$ is the truncation of $U_{[0,T]}$. With this lemma at hand, in order to prove Theorem 2.3, it suffices to prove $U_{[0,T]}^{n,q} \Rightarrow U_{[0,T]}^q$ as $n \to \infty$. This implies that we only need to prove Theorem 2.3 under the global Lipschitz condition. For notational convenience, we omit q in the last two lemmas. The third lemma establishes the tightness of $U_{[0,T]}^n$. In the fourth lemma, the weak limit of $U_{[0,T]}^n$ is characterized.

Lemma 3.1. *Under assumptions **(H1)–(H3)**, (11) is well-posed.*

Proof. Let X_t denote a unique strong solution to (1). Note that (11) is a linear non-autonomous SDDE with respect to U_t. In view of [31, Theorem 3.1], it is sufficient to show

$$\mathrm{P}\bigg(\int_0^t \bigg|\frac{\partial b(X_s, X_{\delta(s)})}{\partial x}\bigg|\,\mathrm{d}s < \infty\bigg) = 1, \quad \mathrm{P}\bigg(\int_0^t \bigg|\frac{\partial b(X_s, X_{\delta(s)})}{\partial y}\bigg|\,\mathrm{d}s < \infty\bigg) = 1$$

and

$$\mathrm{P}\bigg(\int_0^t \bigg|\frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x}\bigg|^2\,\mathrm{d}s < \infty\bigg) = 1, \quad \mathrm{P}\bigg(\int_0^t \bigg|\frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y}\bigg|^2\,\mathrm{d}s < \infty\bigg) = 1.$$

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Recalling that b is locally Lipschitz and $\partial b(x, y)/\partial x = 0$ for $(x, y) \in D_b$, then for any $R > 0$,

$$\Big\{ \sup_{0\le s\le t} |X_s| \le R \Big\} \subseteq \Big\{ \Big| \int_0^t \frac{\partial b(X_s, X_{\delta(s)})}{\partial x} \Big| \, \mathrm{d}s < \infty \Big\}.$$

This, together with the fact that $\lim_{R\to\infty} \mathrm{P}\{\tau_R < T\} = 0$, implies that

$$\mathrm{P}\Big(\int_0^t \Big| \frac{\partial b(X_s, X_{\delta(s)})}{\partial x} \Big| \, \mathrm{d}s < \infty \Big) \ge \lim_{R\to\infty} \mathrm{P}\left(\sup_{0\le s\le t} |X_s| \le R \right) = 1 - \lim_{R\to\infty} \mathrm{P}\{\tau_R < T\} = 1.$$

Similarly, we have

$$\mathrm{P}\Big(\int_0^t \Big| \frac{\partial b(X_s, X_{\delta(s)})}{\partial y} \Big| \, \mathrm{d}s < \infty \Big) = 1, \quad \mathrm{P}\Big(\int_0^t \Big| \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x} \Big|^2 \, \mathrm{d}s < \infty \Big) = 1,$$

and

$$\mathrm{P}\Big(\int_0^t \Big| \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y} \Big|^2 \, \mathrm{d}s < \infty \Big) = 1.$$

In light of $\partial\sigma(X_s, X_{\delta(s)})/\partial x < \infty$ and $\partial\sigma(X_s, X_{\delta(s)})/\partial y < \infty$ almost surely, and $f(s) = 0$ for $s \in [0, q_0\tau]$, it is easy to verify that the following two stochastic integrals

$$\int_0^t \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x} \sigma(X_s, X_{\delta(s)}) \, \mathrm{d}B_s^1 \quad \text{and} \quad \int_0^t \sqrt{\frac{f(s)}{s}} \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y} \sigma(X_{\delta(s)}, X_{\delta(\delta(s))}) \, \mathrm{d}B_s^2$$

are well-defined. Hence, SDDE (11) is well-posed. □

Recall the definition of b^q and σ^q in (4). For $t \in [0, T]$, let U_t^q be the solution to the following linear SDDE:

$$\begin{aligned} U_t^q = & \int_0^t \frac{\partial b^q\big(X_s^q, X_{\delta(s)}^q\big)}{\partial x} U_s^q \, \mathrm{d}s + \int_0^t \frac{\partial b^q\big(X_s^q, X_{\delta(s)}^q\big)}{\partial y} U_{\delta(s)}^q \cdot I_{\{\delta(s)>0\}} \, \mathrm{d}s \\ & + \int_0^t \frac{\partial \sigma^q\big(X_s^q, X_{\delta(s)}^q\big)}{\partial x} U_s^q \, \mathrm{d}W_s + \int_0^t \frac{\partial \sigma^q\big(X_s^q, X_{\delta(s)}^q\big)}{\partial y} U_{\delta(s)}^q \cdot I_{\{\delta(s)>0\}} \, \mathrm{d}W_s \\ & - \int_0^t \sqrt{\frac{\tau}{2}} \frac{\partial \sigma^q\big(X_s^q, X_{\delta(s)}^q\big)}{\partial x} \sigma^q\big(X_s^q, X_{\delta(s)}^q\big) \, \mathrm{d}B_s^1 \\ & - \int_0^t \sqrt{\frac{f(s)}{s}} \frac{\partial \sigma^q\big(X_s^q, X_{\delta(s)}^q\big)}{\partial y} \sigma^q\big(X_{\delta(s)}^q, X_{\delta(\delta(s))}^q\big) \, \mathrm{d}B_s^2 \end{aligned} \tag{15}$$

with $U_s^q = 0$ for $s \in [-\tau, 0]$. The following lemma indicates that the convergence of $U_{[0,T]}^{n,q} \Rightarrow U_{[0,T]}^q$ yields the convergence of $U_{[0,T]}^n \Rightarrow U_{[0,T]}$.

Lemma 3.2. *Let assumptions **(H1)–(H4)** hold. For any $q > \sup_{-\tau\le t\le 0} |\xi_t| + 1$, if $U_{[0,T]}^{n,q} \Rightarrow U_{[0,T]}^q$ as $n \to \infty$, then $U_{[0,T]}^n \Rightarrow U_{[0,T]}$ as $n \to \infty$.*

Proof. First, let us prove that $U_t^{n,q}$ and U_t^q are truncations of U_t^n and U_t, respectively. More precisely,

$$\mathrm{E}\left[\sup_{0\le t\le T} |U_t - U_t^q| \cdot I_{\{\tau_q \ge T\}} \right] = 0, \quad \mathrm{E}\left[\sup_{0\le t\le T} |U_t^n - U_t^{n,q}| \cdot I_{\{\tau_q \wedge \tau_q^n \ge T\}} \right] = 0,$$

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where τ_q, τ_q^n were defined in Theorem 2.2. Using the Itô formula yields

$$\begin{aligned}
|U^q_{t\wedge\tau_q} - U_{t\wedge\tau_q}|^2 \le{}& M^u_{t\wedge\tau_q} + C\int_0^{t\wedge\tau_q} (U^q_s - U_s)\left[\frac{\partial b^q\big(X^q_s, X^q_{\delta(s)}\big)}{\partial x}U^q_s - \frac{\partial b(X_s, X_{\delta(s)})}{\partial x}U_s\right]\mathrm{d}s \\
&+ C\int_0^{t\wedge\tau_q} (U^q_s - U_s)\left[\frac{\partial b^q\big(X^q_s, X^q_{\delta(s)}\big)}{\partial y}U^q_{\delta(s)} - \frac{\partial b(X_s, X_{\delta(s)})}{\partial y}U_{\delta(s)}\right]\mathrm{d}s \\
&+ C\int_0^{t\wedge\tau_q} \left|\frac{\partial \sigma^q\big(X^q_s, X^q_{\delta(s)}\big)}{\partial x}U^q_s - \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x}U_s\right|^2\mathrm{d}s \\
&+ C\int_0^{t\wedge\tau_q} \left|\frac{\partial \sigma^q\big(X^q_s, X^q_{\delta(s)}\big)}{\partial y}U^q_{\delta(s)} - \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y}U_{\delta(s)}\right|^2 I_{\{\delta(s)>0\}}\mathrm{d}s \\
&+ C\int_0^{t\wedge\tau_q} \frac{\tau}{2}\left|\frac{\partial \sigma^q\big(X^q_s, X^q_{\delta(s)}\big)}{\partial x}\sigma^q\big(X^q_s, X^q_{\delta(s)}\big) - \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x}\sigma(X_s, X_{\delta(s)})\right|^2\mathrm{d}s \\
&+ C\int_0^{t\wedge\tau_q} \frac{f(s)}{s}\left|\frac{\partial \sigma^q\big(X^q_s, X^q_{\delta(s)}\big)}{\partial y}\sigma^q\big(X^q_{\delta(s)}, X^q_{\delta(\delta(s))}\big)\right. \\
&\left. - \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y}\sigma(X_{\delta(s)}, X_{\delta(\delta(s))})\right|^2\mathrm{d}s \\
=:{}& M^u_{t\wedge\tau_q} + \sum_{j=1}^{6} Z^j_{t\wedge\tau_q},
\end{aligned} \tag{16}$$

where M^u_t is local martingale with $M^u_0 = 0$. Noting that $\partial b^q\big(X^q_s, X^q_{\delta(s)}\big)/\partial x = \partial b(X_s, X_{\delta(s)})/\partial x$ on $\{\tau_q \ge s\}$, we have

$$\begin{aligned}
Z^1_{t\wedge\tau_q} \le{}& C\int_0^t \left|\big(U^q_s - U_s\big)\left(\frac{\partial b^q\big(X^q_s, X^q_{\delta(s)}\big)}{\partial x} - \frac{\partial b(X_s, X_{\delta(s)})}{\partial x}\right)U^q_s\right| I_{\{\tau_q\ge s\}}\mathrm{d}s \\
&+ C\int_0^t \left|\big(U^q_s - U_s\big)\frac{\partial b(X_s, X_{\delta(s)})}{\partial x}(U^q_s - U_s)\right| I_{\{\tau_q\ge s\}}\mathrm{d}s \\
\le{}& C\int_0^t |\big(U^q_s - U_s\big)|^2\left|\frac{\partial b(X_s, X_{\delta(s)})}{\partial x}\right| I_{\{\tau_q\ge s\}}\mathrm{d}s.
\end{aligned}$$

Similar techniques to $Z^1_{t\wedge\tau_q}$ yield that $Z^j_{t\wedge\tau_q}$, $j = 2, 3, 4, 5, 6$, satisfy similar inequalities. It follows that

$$|U^q_{t\wedge\tau_q} - U_{t\wedge\tau_q}|^2 \le C\int_0^t \sup_{0\le u\le s}\big(U^q_{u\wedge\tau_q} - U_{u\wedge\tau_q}\big)^2\,\mathrm{d}A_s + M^u_{t\wedge\tau_q},$$

where

$$A_t = \int_0^t \left[\left|\frac{\partial b(X_s, X_{\delta(s)})}{\partial x}\right| + \left|\frac{\partial b(X_s, X_{\delta(s)})}{\partial y}\right| + \left|\frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x}\right| + \left|\frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y}\right|\right] I_{\{\tau_q\ge s\}}\,\mathrm{d}s.$$

Applying the stochastic Grönwall inequality (see [11, Corollary 5.2]) gives

$$\mathrm{E}\left[\sup_{0\le t\le T}|U^q_{t\wedge\tau_q} - U_{t\wedge\tau_q}|\right] = 0,$$

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which implies

$$\mathrm{E}\left[\sup_{0\le t\le T}|U_t-U_t^q|\cdot I_{\{\tau_q\ge T\}}\right]=0. \tag{17}$$

Similar techniques give

$$\mathrm{E}\left[\sup_{0\le t\le T}|U_t^n-U_t^{n,q}|\cdot I_{\{\tau_q\wedge\tau_q^n\ge T\}}\right]=0. \tag{18}$$

Now let us prove $U^n_{[0,T]}\Rightarrow U_{[0,T]}$ if $U^{n,q}_{[0,T]}\Rightarrow U^q_{[0,T]}$. For any bounded Lipschitz continuous functional f with the Lipschitz constant L, by (17) and (18),

$$\begin{aligned}
|\mathrm{E}[f(U^n_{[0,T]})-f(U_{[0,T]})]| &\le |\mathrm{E}[f(U^n_{[0,T]})-f(U^{n,q}_{[0,T]})]|+|\mathrm{E}[f(U^{n,q}_{[0,T]})-f(U^q_{[0,T]})]|\\
&\quad+|\mathrm{E}[f(U^q_{[0,T]})-f(U_{[0,T]})]|\\
&\le \left|\mathrm{E}\left(f(U^n_{[0,T]})-f(U^{n,q}_{[0,T]})\right)\cdot I_{\{\tau_q\wedge\tau_q^n>T\}}\right|\\
&\quad+\left|\mathrm{E}\left(f(U^n_{[0,T]})-f(U^{n,q}_{[0,T]})\right)\left(I_{\{\tau_q^n\le T\}}+I_{\{\tau_q\le T\}}\right)\right|\\
&\quad+\left|\mathrm{E}[f(U^{n,q}_{[0,T]})-f(U^q_{[0,T]})]\right|+\left|\mathrm{E}\left(f(U^q_{[0,T]})-f(U_{[0,T]})\right)\cdot I_{\{\tau_q>T\}}\right|\\
&\quad+\left|\mathrm{E}\left(f(U^q_{[0,T]})-f(U_{[0,T]})\right)\cdot I_{\{\tau_q\le T\}}\right|\\
&\le L\mathrm{E}\Big[\sup_{0\le t\le T}|U_t^n-U_t^{n,q}|\cdot I_{\{\tau_q\wedge\tau_q^n>T\}}\Big]+\left|\mathrm{E}[f(U^{n,q}_{[0,T]})-f(U^q_{[0,T]})]\right|\\
&\quad+2M\mathrm{E}\left[I_{\{\tau_q^n\le T\}}+I_{\{\tau_q\le T\}}\right]+L\mathrm{E}\Big[\sup_{0\le t\le T}|U_t^q-U_t|\cdot I_{\{\tau_q>T\}}\Big]\\
&\quad+2M\mathrm{E}I_{\{\tau_q\le T\}}\\
&\le \left|\mathrm{E}[f(U^{n,q}_{[0,T]})-f(U^q_{[0,T]})]\right|+4M(\mathrm{P}(\tau_q\le T)+\mathrm{P}(\tau_q^n\le T)),
\end{aligned}$$

where $M=\sup_{x\in C([0,T];\mathbb{R})}|f(x)|$. Recall that $U^{n,q}_{[0,T]}$ converges weakly to $U^q_{[0,T]}$. Letting $n\to\infty$ gives

$$\limsup_{n\to\infty}|\mathrm{E}[f(U^n_{[0,T]})-f(U_{[0,T]})]|\le 4M\mathrm{P}(\tau_q\le T)+4M\limsup_{n\to\infty}\mathrm{P}(\tau_q^n\le T).$$

In light of (9) and (10), letting $q\to\infty$ gives the desired result. □

Lemma 3.3. *Let assumption **(H3)** hold. Assume further that b and σ are Lipschitz continuous, then $\{U^n_{[0,T]}\}_{n\ge1}$ is tight in $C([0,T];\mathbb{R})$.*

Proof. For each $n\ge1$, $U_t^n=0$ for $t\in[-\tau,0]$. By using a similar approach to [33], it follows that

$$\mathrm{E}\left[\sup_{0\le t\le T}|U_t^n|^4\right]\le C_T \tag{19}$$

and

$$\mathrm{E}\left[\sup_{0\le t\le T}|X^n_{\lfloor t\rfloor_h}-X_t^n|^4\right]\le C_T n^{-2}.$$

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By the Burkholder–Davis–Gundy inequality, Hölder's inequality and assumption **(H3)**, for any $t, t+s \in [0, T]$ with $s \in [0, 1]$,

$$\begin{aligned}
\mathrm{E}|U^n_{t+s} - U^n_t|^4 &= \mathrm{E}n^2|X^n_{t+s} - X^n_t - (X_{t+s} - X_t)|^4 \\
&\le \mathrm{E}n^2\bigg[\int_t^{t+s} b\big(X^n_{\lfloor u\rfloor_h}, X^n_{\lfloor \delta(\lfloor u\rfloor_h)\rfloor_h}\big) - b(X_u, X_{\delta(u)})\,\mathrm{d}u \\
&\quad + \int_t^{t+s} \sigma\big(X^n_{\lfloor u\rfloor_h}, X^n_{\lfloor \delta(\lfloor u\rfloor_h)\rfloor_h}\big) - \sigma(X_u, X_{\delta(u)})\,\mathrm{d}W_u\bigg]^4 \\
&\le Cn^2s^3\mathrm{E}\int_t^{t+s} |X^n_{\lfloor u\rfloor_h} - X_u|^4 + |X^n_{\lfloor \delta(\lfloor u\rfloor_h)\rfloor_h} - X_{\delta(u)}|^4\,\mathrm{d}u \\
&\quad + Cn^2s\int_t^{t+s} |X^n_{\lfloor u\rfloor_h} - X_u|^4 + |X^n_{\lfloor \delta(\lfloor u\rfloor_h)\rfloor_h} - X_{\delta(u)}|^4\,\mathrm{d}u \\
&\le Cs\int_t^{t+s} \mathrm{E}\bigg[\bigg(\sup_{0\le u\le T} n^2|X^n_u - X_u|^4\bigg) + \bigg(\sup_{0\le u\le T} n^2|X^n_{\lfloor u\rfloor_h} - X^n_u|^4\bigg) \\
&\quad + n^2(\delta(u) - \lfloor\delta(\lfloor u\rfloor_h)\rfloor_h)^4\bigg]\,\mathrm{d}u \\
&\le Cs^2.
\end{aligned}$$

According to the tightness criterion (see, for example, [22, Theorem 4.11]), $\{U^n_{[0,T]}\}_{n\ge1}$ is tight in $C([0, T]; \mathbb{R})$. □

In the next lemma, we characterize the limit process of $U^n_{[0,T]}$.

Lemma 3.4. *Let assumptions **(H3)** and **(H4)** hold. Further assume that b and σ are Lipschitz continuous, then $U^n_{[0,T]} \Rightarrow U_{[0,T]}$ as $n \to \infty$, where U_t is the weak solution to (11).*

Proof. Let us divide this proof into the following two steps.

Step 1: Expressing U^n_t as a linear integral equation with respect to U^n_t and $U^n_{\delta(s)}$. Note that $\lfloor\delta(\lfloor t\rfloor_h)\rfloor_h = \lfloor\delta(t)\rfloor_h$. By (3),

$$\begin{aligned}
U^n_t = \sqrt{n}\big(X^n_t - X_t\big) &= \sqrt{n}\bigg[\int_0^t b\big(X^n_{\lfloor s\rfloor_h}, X^n_{\lfloor\delta(s)\rfloor_h}\big) - b(X_s, X_{\delta(s)})\,\mathrm{d}s\bigg] \\
&\quad + \sqrt{n}\bigg[\int_0^t \big(\sigma\big(X^n_{\lfloor s\rfloor_h}, X^n_{\lfloor\delta(s)\rfloor_h}\big) - \sigma(X_s, X_{\delta(s)})\big)\,\mathrm{d}W_s\bigg] \\
&=: U^{n,1}_t + U^{n,2}_t.
\end{aligned}$$

Note that for any $x, y \in \mathbb{R}$, when $x = y$, $b(x, \cdot) - b(y, \cdot) = 0$. When $x \ne y$, we can rewrite $b(x, \cdot) - b(y, \cdot)$ as $\frac{b(x,\cdot)-b(y,\cdot)}{x-y}(x - y)$. Similar ideas can deal with the term $b(\cdot, x) - b(\cdot, y)$. Therefore, we can rewrite $U^{n,1}_t$ as

$$\begin{aligned}
U^{n,1}_t &= \int_0^t \sqrt{n}\big[b\big(X^n_s, X_{\delta(s)}\big) - b(X_s, X_{\delta(s)}) + b\big(X^n_s, X^n_{\delta(s)}\big) - b\big(X^n_s, X_{\delta(s)}\big)\big]\,\mathrm{d}s \\
&\quad - \int_0^t \sqrt{n}\big[b\big(X^n_s, X^n_{\delta(s)}\big) - b\big(X^n_{\lfloor s\rfloor_h}, X^n_{\delta(s)}\big) + b\big(X^n_{\lfloor s\rfloor_h}, X^n_{\delta(s)}\big) - b\big(X^n_{\lfloor s\rfloor_h}, X^n_{\lfloor\delta(s)\rfloor_h}\big)\big]\,\mathrm{d}s \\
&= \int_0^t \big[b\big(X^n_s, X_{\delta(s)}\big) - b(X_s, X_{\delta(s)})\big]\frac{U^n_s}{X^n_s - X_s}\,\mathrm{d}s
\end{aligned}$$

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$$
\begin{aligned}
&+\int_0^t I_{\{\delta(s)>0\}}\big[b\big(X_s^n, X_{\delta(s)}^n\big)-b\big(X_s^n, X_{\delta(s)}\big)\big]\frac{U_{\delta(s)}^n}{X_{\delta(s)}^n-X_{\delta(s)}}ds\\
&-\int_0^t \big[b\big(X_s^n, X_{\delta(s)}^n\big)-b\big(X_{\lfloor s\rfloor_h}^n, X_{\delta(s)}^n\big)\big]\frac{\sqrt{n}\big(X_s^n-X_{\lfloor s\rfloor_h}^n\big)}{X_s^n-X_{\lfloor s\rfloor_h}^n}\,\mathrm{d}s\\
&-\int_0^t I_{\{\delta(s)>0\}}\big[b\big(X_{\lfloor s\rfloor_h}^n, X_{\delta(s)}^n\big)-b\big(X_{\lfloor s\rfloor_h}^n, X_{\lfloor\delta(s)\rfloor_h}^n\big)\big]\frac{\sqrt{n}\big(X_{\delta(s)}^n-X_{\lfloor\delta(s)\rfloor_h}^n\big)}{X_{\delta(s)}^n-X_{\lfloor\delta(s)\rfloor_h}^n}\,\mathrm{d}s\\
&-\int_0^t I_{\{\delta(s)\le 0\}}\big[b\big(X_{\lfloor s\rfloor_h}^n, X_{\delta(s)}^n\big)-b\big(X_{\lfloor s\rfloor_h}^n, X_{\lfloor\delta(s)\rfloor_h}^n\big)\big]\,\mathrm{d}s, \qquad (20)
\end{aligned}
$$

where we used the fact that $X_{\delta(s)}^n = X_{\delta(s)}$ for $\delta(s)\le 0$ in second equality. Similarly,

$$
\begin{aligned}
U_t^{n,2} &= \int_0^t \big[\sigma\big(X_s^n, X_{\delta(s)}\big)-\sigma(X_s, X_{\delta(s)})\big]\frac{U_s^n}{X_s^n-X_s}\,\mathrm{d}W_s\\
&+\int_0^t I_{\{\delta(s)>0\}}\big[\sigma\big(X_s^n, X_{\delta(s)}^n\big)-\sigma\big(X_s^n, X_{\delta(s)}\big)\big]\frac{U_{\delta(s)}^n}{X_{\delta(s)}^n-X_{\delta(s)}}\,\mathrm{d}W_s\\
&-\int_0^t \big[\sigma\big(X_s^n, X_{\delta(s)}^n\big)-\sigma\big(X_{\lfloor s\rfloor_h}^n, X_{\delta(s)}^n\big)\big]\frac{\sqrt{n}\big(X_s^n-X_{\lfloor s\rfloor_h}^n\big)}{X_s^n-X_{\lfloor s\rfloor_h}^n}\,\mathrm{d}W_s\\
&-\int_0^t I_{\{\delta(s)>0\}}\big[\sigma\big(X_{\lfloor s\rfloor_h}^n, X_{\delta(s)}^n\big)-\sigma\big(X_{\lfloor s\rfloor_h}^n, X_{\lfloor\delta(s)\rfloor_h}^n\big)\big]\frac{\sqrt{n}\big(X_{\delta(s)}^n-X_{\lfloor\delta(s)\rfloor_h}^n\big)}{X_{\delta(s)}^n-X_{\lfloor\delta(s)\rfloor_h}^n}\,\mathrm{d}W_s\\
&-\int_0^t I_{\{\delta(s)\le 0\}}\big[\sigma\big(X_{\lfloor s\rfloor_h}^n, X_{\delta(s)}^n\big)-\sigma\big(X_{\lfloor s\rfloor_h}^n, X_{\lfloor\delta(s)\rfloor_h}^n\big)\big]\,\mathrm{d}W_s. \qquad (21)
\end{aligned}
$$

In terms of (3), for $s>0$,

$$
X_s^n - X_{\lfloor s\rfloor_h}^n = b\big(X_{\lfloor s\rfloor_h}^n, X_{\lfloor\delta(s)\rfloor_h}^n\big)(s-\lfloor s\rfloor_h)+\sigma\big(X_{\lfloor s\rfloor_h}^n, X_{\lfloor\delta(s)\rfloor_h}^n\big)(W_s-W_{\lfloor s\rfloor_h}). \qquad (22)
$$

Note that $\delta(s)>0$ implies $\lfloor\delta(s)\rfloor_h\ge 0$. Then, for $\delta(s)>0$,

$$
\begin{aligned}
X_{\delta(s)}^n - X_{\lfloor\delta(s)\rfloor_h}^n &= b\big(X_{\lfloor\delta(s)\rfloor_h}^n, X_{\lfloor\delta(\lfloor\delta(s)\rfloor_h)\rfloor_h}^n\big)(\delta(s)-\lfloor\delta(s)\rfloor_h)\\
&\quad+\sigma\big(X_{\lfloor\delta(s)\rfloor_h}^n, X_{\lfloor\delta(\lfloor\delta(s)\rfloor_h)\rfloor_h}^n\big)(W_{\delta(s)}-W_{\lfloor\delta(s)\rfloor_h})\\
&=: A_s^1(\delta(s)-\lfloor\delta(s)\rfloor_h)+A_s^2(W_{\delta(s)}-W_{\lfloor\delta(s)\rfloor_h}). \qquad (23)
\end{aligned}
$$

Let $I_A^1 = I_A$ and $I_A^2 = I_{A^C}$, where A^C represents the complement of A. Then from (20) to (23), it follows that

$$
\begin{aligned}
U_t^{n,1} &= \sum_{m=1}^2\int_0^t I_{\{(X_s,X_{\delta(s)})\in D_b\}}^m\big(b\big(X_s^n, X_{\delta(s)}\big)-b(X_s, X_{\delta(s)})\big)\frac{U_s^n}{X_s^n-X_s}\,\mathrm{d}s\\
&+\sum_{m=1}^2\int_0^t I_{\{(X_s,X_{\delta(s)})\in D_b\}}^m\cdot I_{\{\delta(s)>0\}}\big(b(X_s^n, X_{\delta(s)}^n)-b\big(X_s^n, X_{\delta(s)}\big)\big)\frac{U_{\delta(s)}^n}{\big(X_{\delta(s)}^n-X_{\delta(s)}\big)}\,\mathrm{d}s\\
&-\sum_{m=1}^2\int_0^t I_{\{(X_s,X_{\delta(s)})\in D_b\}}^m\big(b\big(X_s^n, X_{\delta(s)}^n\big)-b\big(X_{\lfloor s\rfloor_h}^n, X_{\delta(s)}^n\big)\big)\frac{b\big(X_{\lfloor s\rfloor_h}^n, X_{\lfloor\delta(s)\rfloor_h}^n\big)}{X_s^n-X_{\lfloor s\rfloor_h}^n}\,\mathrm{d}M_s^{n,1}
\end{aligned}
$$

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$$
\begin{aligned}
&- \sum_{m=1}^{2} \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_b\}} \big(b\big(X^n_s, X^n_{\delta(s)}\big) - b\big(X^n_{\lfloor s \rfloor_h}, X^n_{\lfloor \delta(s) \rfloor_h}\big)\big) \frac{\sigma\big(X^n_{\lfloor s \rfloor_h}, X^n_{\lfloor \delta(s) \rfloor_h}\big)}{X^n_s - X^n_{\lfloor s \rfloor_h}} \, \mathrm{d}M^{n,2}_s \\
&- \sum_{m=1}^{2} \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_b\}} \big(b\big(X^n_{\lfloor s \rfloor_h}, X^n_{\delta(s)}\big) - b\big(X^n_{\lfloor s \rfloor_h}, X^n_{\lfloor \delta(s) \rfloor_h}\big)\big) \frac{A^1_s}{X^n_{\delta(s)} - X^n_{\lfloor \delta(s) \rfloor_h}} \, \mathrm{d}N^{n,1}_s \\
&- \sum_{m=1}^{2} \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_b\}} \big(b\big(X^n_{\lfloor s \rfloor_h}, X^n_{\delta(s)}\big) - b\big(X^n_{\lfloor s \rfloor_h}, X^n_{\lfloor \delta(s) \rfloor_h}\big)\big) \frac{A^2_s}{X^n_{\delta(s)} - X^n_{\lfloor \delta(s) \rfloor_h}} \, \mathrm{d}N^{n,2}_s \\
&- \sum_{m=1}^{2} \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_b\}} I_{\{\delta(s) \le 0\}} \big(b\big(X^n_{\lfloor s \rfloor_h}, X^n_{\delta(s)}\big) - b\big(X^n_{\lfloor s \rfloor_h}, X^n_{\lfloor \delta(s) \rfloor_h}\big)\big) \, \mathrm{d}s \\
=: &\sum_{i=1}^{7} \sum_{m=1}^{2} F^{n,m,i}_t,
\end{aligned}
$$

and

$$
\begin{aligned}
U^{n,2}_t = &\sum_{m=1}^{2} \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_\sigma\}} \big(\sigma(X^n_s, X^n_{\delta(s)}) - \sigma\big(X^n_s, X_{\delta(s)}\big)\big) \frac{U^n_s}{X^n_s - X_s} \, \mathrm{d}W_s \\
&+ \sum_{m=1}^{2} \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_\sigma\}} \cdot I_{\{X_{\delta(s)} > 0\}} \big(\sigma(X^n_s, X^n_{\delta(s)}) - \sigma\big(X^n_{\lfloor s \rfloor_h}, X^n_{\delta(s)}\big)\big) \frac{U^n_{\delta(s)}}{X^n_{\delta(s)} - X_{\delta(s)}} \, \mathrm{d}W_s \\
&- \sum_{m=1}^{2} \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_\sigma\}} \big(\sigma\big(X^n_s, X^n_{\delta(s)}\big) - \sigma\big(X^n_{\lfloor s \rfloor_h}, X^n_{\delta(s)}\big)\big) \frac{b\big(X^n_{\lfloor s \rfloor_h}, X^n_{\lfloor \delta(s) \rfloor_h}\big)}{X^n_s - X^n_{\lfloor s \rfloor_h}} \, \mathrm{d}M^{n,3}_s \\
&- \sum_{m=1}^{2} \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_\sigma\}} \big(\sigma\big(X^n_s, X^n_{\delta(s)}\big) - \sigma\big(X^n_{\lfloor s \rfloor_h}, X^n_{\delta(s)}\big)\big) \frac{\sigma\big(X^n_{\lfloor s \rfloor_h}, X^n_{\lfloor \delta(s) \rfloor_h}\big)}{X^n_s - X^n_{\lfloor s \rfloor_h}} \, \mathrm{d}M^{n,4}_s \\
&- \sum_{m=1}^{2} \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_\sigma\}} \big(\sigma\big(X^n_{\lfloor s \rfloor_h}, X^n_{\delta(s)}\big) - \sigma\big(X^n_{\lfloor s \rfloor_h}, X^n_{\lfloor \delta(s) \rfloor_h}\big)\big) \frac{A^1_s}{X^n_{\delta(s)} - X^n_{\lfloor \delta(s) \rfloor_h}} \, \mathrm{d}N^{n,3}_s \\
&- \sum_{m=1}^{2} \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_\sigma\}} \big(\sigma\big(X^n_{\lfloor s \rfloor_h}, X^n_{\delta(s)}\big) - \sigma\big(X^n_{\lfloor s \rfloor_h}, X^n_{\lfloor \delta(s) \rfloor_h}\big)\big) \frac{A^2_s}{X^n_{\delta(s)} - X^n_{\lfloor \delta(s) \rfloor_h}} \, \mathrm{d}N^{n,4}_s \\
&- \sum_{m=1}^{2} \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_\sigma\}} I_{\{\delta(s) \le 0\}} \big(\sigma\big(X^n_{\lfloor s \rfloor_h}, X^n_{\delta(s)}\big) - \sigma\big(X^n_{\lfloor s \rfloor_h}, X^n_{\lfloor \delta(s) \rfloor_h}\big)\big) \, \mathrm{d}W_s \\
=: &\sum_{i=1}^{7} \sum_{m=1}^{2} G^{n,m,i}_t,
\end{aligned}
$$

where

$$
M^{n,1}_t = \int_0^t \sqrt{n}(s - \lfloor s \rfloor_h) \, \mathrm{d}s, \quad M^{n,2}_t = \int_0^t \sqrt{n}(W_s - W_{\lfloor s \rfloor_h}) \, \mathrm{d}s,
$$

$$
M^{n,3}_t = \int_0^t \sqrt{n}(s - \lfloor s \rfloor_h) \, \mathrm{d}W_s, \quad M^{n,4}_t = \int_0^t \sqrt{n}(W_s - W_{\lfloor s \rfloor_h}) \, \mathrm{d}W_s,
$$

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and

$$N_t^{n,1} = \int_0^t \sqrt{n}(\delta(s) - \lfloor \delta(s) \rfloor_h) \cdot I_{\{\delta(s)>0\}} \, \mathrm{d}s, \quad N_t^{n,2} = \int_0^t \sqrt{n}(W_{\delta(s)} - W_{\lfloor \delta(s) \rfloor_h}) \cdot I_{\{\delta(s)>0\}} \, \mathrm{d}s,$$
$$N_t^{n,3} = \int_0^t \sqrt{n}(\delta(s) - \lfloor \delta(s) \rfloor_h) \cdot I_{\{\delta(s)>0\}} \, \mathrm{d}W_s,$$
$$N_t^{n,4} = \int_0^t \sqrt{n}(W_{\delta(s)} - W_{\lfloor \delta(s) \rfloor_h}) \cdot I_{\{\delta(s)>0\}} \, \mathrm{d}W_s.$$

Step 2: Characterizing the limit equation of $U_{[0,T]}^n$. Define M^n and N^n by

$$M_t^n = \left(M_t^{n,1}, \ M_t^{n,2}, \ M_t^{n,3}, \ M_t^{n,4}\right), \quad N_t^n = \left(N_t^{n,1}, N_t^{n,2}, \ N_t^{n,3}, \ N_t^{n,4}\right). \tag{24}$$

By virtue of Lemmas 2.1 and A.2, applying [19, Theorem 4.1] gives

$$\left(X^n, W, M^n, N^n\right) \overset{\mathcal{S}}{\Rightarrow} (X, W, M, N)$$

as $n \to \infty$, where

$$M_t = \left(0, 0, 0, \sqrt{\frac{\tau}{2}} B_t^1\right), \quad N_t = \left(0, 0, 0, \sqrt{\frac{f(t)}{t}} B_t^2\right),$$

and $B^1 = \{B_t^1, t \geq 0\}$ and $B^2 = \{B_t^2, t \geq 0\}$ are two independent Bronwnian motions defined on the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}})_{t\geq 0}, \bar{\mathrm{P}})$, which is an extension of the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathrm{P})$. Meanwhile, B^1 and B^2 are also independent of W. Since $U_{[0,T]}^n$ is tight, the Prohorov theorem yields $\left(U^n, X^n, W, M^n, N^n\right)_{[0,T]}$ is also tight in $C([0, T];\mathbb{R}^{10})$. Hence, there exists a subsequence (for notational simplicity, index this subsequence by n also) such that

$$\left(U^n, X^n, W, M^n, N^n\right)_{[0,T]} \Rightarrow (\tilde{U}, X, W, M, N)_{[0,T]}.$$

By the Skorohod representation theorem (see, for example, [29, Theorem 3, p. 29]), it can be rewritten as

$$\left(U^n, X^n, W, M^n, N^n\right)_{[0,T]} \xrightarrow{\text{a.s.}} (\tilde{U}, X, W, M, N)_{[0,T]}. \tag{25}$$

Since (11) admits a unique solution, it remains to show that $\tilde{U}_t$ is the weak solution to (11). Define

$$\tilde{F}_t^{n,m,1} = F_t^{n,m,1} - \int_0^t I_{\{(X_s, X_{\delta(s)}) \in D_b\}}^m \frac{\partial b(X_s, X_{\delta(s)})}{\partial x} \tilde{U}_s \, \mathrm{d}s,$$
$$\tilde{F}_t^{n,m,2} = F_t^{n,m,2} - \int_0^t I_{\{(X_s, X_{\delta(s)}) \in D_b\}}^m \cdot I_{\{\delta(s)>0\}} \frac{\partial b(X_s, X_{\delta(s)})}{\partial y} \tilde{U}_{\delta(s)} \, \mathrm{d}s,$$
$$\tilde{F}_t^{n,m,z} = F_t^{n,m,z}, \quad \text{for } z = 3, 4, \ldots, 7,$$

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and

$$\tilde{G}_t^{n,m,1} = G_t^{n,m,1} - \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_\sigma\}} \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x} \tilde{U}_s \, \mathrm{d}W_s,$$

$$\tilde{G}_t^{n,m,2} = G_t^{n,m,2} - \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_\sigma\}} \cdot I_{\{X_{\delta(s)} > 0\}} \tilde{U}_{\delta(s)} \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y} dW_s,$$

$$\tilde{G}_t^{n,m,z} = G_t^{n,m,z}, \quad \text{for } z = 3, 5, 7,$$

$$\tilde{G}_t^{n,m,4} = G_t^{n,m,4} + \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_\sigma\}} \sqrt{\frac{\tau}{2}} \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x} \sigma(X_s, X_{\delta(s)}) \, \mathrm{d}B_s^1,$$

$$\tilde{G}_t^{n,m,6} = G_t^{n,m,6} + \int_0^t I^m_{\{(X_s, X_{\delta(s)}) \in D_\sigma\}} \sqrt{\frac{f(s)}{s}} \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y} \sigma(X_{\delta(s)}, X_{\delta(\delta(s))}) \, \mathrm{d}B_s^2.$$

From the definition of $\hat{U}_t$ in (A9),

$$\sum_{z=1}^{7} \sum_{m=1}^{2} \left(\tilde{F}_t^{n,m,z} + \tilde{G}_t^{n,m,z} \right) = U_t^n - \hat{U}_t. \tag{26}$$

Lemma A.3 shows that U_t^n converges in probability to $\hat{U}_t$. This, together with the fact that $U_{[0,T]}^n \xrightarrow{\text{a.s.}} \tilde{U}_{[0,T]}$, implies that $\hat{U}_{[0,T]} = \tilde{U}_{[0,T]}$ almost surely. This shows that $\hat{U}_t = \tilde{U}_t$ almost surely from (A9). This immediately leads to the desired result, namely that $U_{[0,T]}^n \Rightarrow U_{[0,T]}$ as $n \to \infty$, where U_t is the weak solution to Eq. (11). This proof is completed. □

Proof of Theorem 2.3. With Lemmas 3.2 and 3.4 at hand, it is easy to verify that the desired assertion $U_t^n \Rightarrow U_t$ holds. □

Remark 3.1. Furthermore, Theorem 2.3 can be extended to the multi-dimensional case and can be applied in more kinds of delay. For the multi-dimensional case, consider the following SDDE

$$\mathrm{d}X_t = b(X_t, X_{\delta(t)}) \, \mathrm{d}t + \sigma(X_t, X_{\delta(t)}) \, \mathrm{d}W_t, \quad t \in [0, T],$$

with initial value $X_t = \xi_t$, $t \in [-\tau, 0]$, where

$$b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \quad \sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times r}.$$

Replace assumption **(H4)** by the following.

(H4′) There exists k_0 such that for any $x \in D_{b_i} \cup D_{\sigma_{ik}}$, for any $l = 1, \ldots, r$, $\sigma_{k_0 l}^2(x) > 0$.

Then, we only need to re-estimate (A14). By the occupation time formula (see [36]),

$$\begin{aligned} \mathrm{E} \sum_{l=1}^{r} \int_0^T I^1_{\{(X_s, X_{\delta(s)}) \in D_{b_i}\}} \sigma_{k_0 l}(X_s, X_{\delta(s)})^2 \, \mathrm{d}s &= \mathrm{E} \int_0^T I^1_{\{(X_s, X_{\delta(s)}) \in D_{b_i}\}} \, \mathrm{d}[X^{k_0}, X^{k_0}]_s \\ &\le \mathrm{E} \int_0^T I^1_{\{X_s^{k_0} \in D_{b_i}^{k_0}\}} \, \mathrm{d}[X^{k_0}, X^{k_0}]_s \\ &= \mathrm{E} \int_{D_{b_i}^{k_0}} L_T^x \left(X^{k_0} \right) \mathrm{d}x = 0, \end{aligned}$$

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where $L_t^x(X^{k_0})$ represents the local time in x of the semimartingale X^{k_0}. We use the facts that $\lambda(D_{b_i}^{k_0})=0$ and $L_t^x(X^{k_0})<\infty$ for x in the sense of λ-a.s. By assumption (**H4'**), $\int_0^T I^1_{\{(X_s,X_{\delta(s)})\in D_{b_i}\}}\,\mathrm{d}s=0$ follows.

From the proof process of Theorem 2.3, it is easy to observe that Theorem 2.3 still holds for more general time delay in b. For example, $\delta(t)$ is only continuously differentiable.

4. Proof of Theorem 2.4

Although the proof of Theorem 2.4 is more complicated and subtler, it follows from similar techniques to Theorem 2.3. We only provide the key illustration and different estimations.

Proof of Theorem 2.4. In terms of (12) and (13), by using Taylor's formula, we have

$$
\begin{aligned}
&V_t^{n,2}\\
&=n\int_0^t\big(b_2\big(X^n_{\lfloor s\rfloor_h},X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)-b_2\big(X_s,X^1_{s-\tau}\big)\big)ds+n\int_0^t\big(\sigma\big(X^{n,1}_{\lfloor s\rfloor_h},X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)-\sigma\big(X^1_s,X^1_{s-\tau}\big)\big)\,\mathrm{d}W_s\\
&=\int_0^t\big\langle\nabla_x b_2\big(X_s,X^1_{s-\tau}\big),V^n_s\big\rangle\,\mathrm{d}s+\int_0^t\frac{1}{2}\big\langle\big(X^n_s-X_s\big),\nabla_{xx}b_2\big(\theta^{n,1}_s,X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)n\big(X^n_s-X_s\big)\big\rangle\,\mathrm{d}s\\
&\quad+\int_0^t\frac{\partial b_2\big(X^n_s,X^1_{s-\tau}\big)}{\partial y}V^{n,1}_{s-\tau}\cdot I_{\{s-\tau>0\}}\,\mathrm{d}s+\int_0^t\frac{1}{2}\frac{\partial^2 b_2\big(X^n_s,\theta^{n,2}_s\big)}{\partial y^2}n\big(X^{n,1}_{s-\tau}-X^1_{s-\tau}\big)^2\cdot I_{\{s-\tau>0\}}\,\mathrm{d}s\\
&\quad-\int_0^t\frac{\partial b_2\big(X^n_s,X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial y}n\big(X^{n,1}_{s-\tau}-X^1_{\lfloor s-\tau\rfloor_h}\big)\,\mathrm{d}s-\int_0^t\frac{1}{2}\frac{\partial^2 b_2\big(X^n_s,\theta^{n,4}_s\big)}{\partial y^2}n\big(X^{n,1}_{s-\tau}-X^1_{\lfloor s-\tau\rfloor_h}\big)^2\,\mathrm{d}s\\
&\quad-\int_0^t\big\langle\nabla_x b_2\big(X^n_{\lfloor s\rfloor_h},X^{n,1}_{\lfloor s-\tau\rfloor_h}\big),n\big(X^n_s-X^n_{\lfloor s\rfloor_h}\big)\big\rangle\,\mathrm{d}s\\
&\quad-\int_0^t\frac{1}{2}\big\langle\big(X^n_s-X^n_{\lfloor s\rfloor_h}\big),\nabla_{xx}b_2\big(\theta^{n,3}_s,X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)n\big(X^n_s-X^n_{\lfloor s\rfloor_h}\big)\big\rangle\,\mathrm{d}s\\
&\quad+\int_0^t\frac{\partial\sigma\big(X^1_s,X^1_{s-\tau}\big)}{\partial x_1}V^{n,1}_s\,\mathrm{d}W_s+\int_0^t\frac{1}{2}\frac{\partial^2\sigma\big(\bar\theta^{n,1}_s,X^1_{s-\tau}\big)}{\partial x_1\partial x_1}n\big(X^{n,1}_s-X^1_s\big)^2\,\mathrm{d}W_s\\
&\quad+\int_0^t\frac{\partial\sigma\big(X^{n,1}_s,X^1_{s-\tau}\big)}{\partial y}V^{n,1}_{s-\tau}\cdot I_{\{s-\tau>0\}}\,\mathrm{d}W_s\\
&\quad+\int_0^t\frac{1}{2}\frac{\partial^2\sigma\big(X^{n,1}_s,\bar\theta^{n,2}_s\big)}{\partial y^2}n\big(X^{n,1}_{s-\tau}-X^1_{s-\tau}\big)^2\cdot I_{\{s-\tau>0\}}\,\mathrm{d}W_s\\
&\quad-\int_0^t\frac{\partial\sigma\big(X^{n,1}_{\lfloor s\rfloor_h},X^{n,1}_{s-\tau}\big)}{\partial x_1}n\big(X^{n,1}_s-X^{n,1}_{\lfloor s\rfloor_h}\big)\,\mathrm{d}W_s-\int_0^t\frac{1}{2}\frac{\partial^2\sigma\big(\bar\theta^{n,3}_s,X^{n,1}_{s-\tau}\big)}{\partial x_1^2}n\big(X^{n,1}_s-X^{n,1}_{\lfloor s\rfloor_h}\big)^2\,\mathrm{d}W_s\\
&\quad-\int_0^t\frac{\partial\sigma\big(X^{n,1}_{\lfloor s\rfloor_h},X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial y}n\big(X^{n,1}_{s-\tau}-X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)\,\mathrm{d}W_s\\
&\quad-\int_0^t\frac{1}{2}\frac{\partial^2\sigma\big(X^n_{\lfloor s\rfloor_h},X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial y^2}n\big(X^{n,1}_{s-\tau}-X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)^2\,\mathrm{d}W_s,
\end{aligned}
\tag{27}
$$

where $\{\theta^{n,i}_s,i=1,2,3,4\}$ and $\{\bar\theta^{n,i}_s,i=1,2,3,4\}$ are from the Taylor expansion. More precisely, $\theta^{n,1}_s(\bar\theta^{n,1}_s)$ is valued between $X^n_s(X^{n,1}_s)$ and $X_s(X^1_s)$ and can be expressed as $\theta^{n,1}_s=X^n_s+\eta^1(X_s-X^n_s)$ and $\bar\theta^{n,1}_s=X^{n,1}_s+\bar\eta^1(X^1_s-X^{n,1}_s)$ for some $\eta^1,\bar\eta^1\in[0,1]$. Similarly, $\theta^{n,2}_s(\bar\theta^{n,2}_s)$ is valued between $X^{n,1}_{s-\tau}(X^{n,1}_{s-\tau})$ and $X^1_{s-\tau}(X^1_{s-\tau})$, $\theta^{n,3}_s(\bar\theta^{n,3}_s)$ is valued between $X^n_s(X^{n,1}_s)$ and

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$X^n_{\lfloor s\rfloor_h}(X^{n,1}_{\lfloor s\rfloor_h})$, $\theta^{n,4}_s(\bar\theta^{n,4}_s)$ is valued between $X^{n,1}_{s-\tau}(X^{n,1}_{s-\tau})$ and $X^{n,1}_{\lfloor s-\tau\rfloor_h}(X^{n,1}_{\lfloor s-\tau\rfloor_h})$, respectively. By virtue of (13),

$$
\begin{aligned}
V^{n,2}_t = {} & \int_0^t \big\langle \nabla_x b_2\big(X_s, X^1_{s-\tau}\big), V^n_s\big\rangle \,\mathrm{d}s + \int_0^t \frac{\partial b_2\big(X^n_s, X^1_{s-\tau}\big)}{\partial y} V^{n,1}_{s-\tau}\cdot I_{\{s-\tau>0\}}\,\mathrm{d}s \\
& - \int_0^t \frac{b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial y} b_1\big(X^n_{\lfloor s-\tau\rfloor_h}, X^{n,1}_{\lfloor s-2\tau\rfloor_h}\big)\,\mathrm{d}\bar N^{n,1}_s - \int_0^t \frac{b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial y}\,\mathrm{d}Z^n_s \\
& - \int_0^t \big\langle \nabla_x b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big), b\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)\big\rangle\,\mathrm{d}\bar M^{n,1}_s \\
& - \int_0^t \frac{\partial b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2}\sigma\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)\,\mathrm{d}\bar M^{n,2}_s \\
& - \frac{\tau}{4}\int_0^t \frac{\partial^2 b_2\big(X_s, X^1_{s-\tau}\big)}{\partial x_2\partial x_2}\sigma^2\big(X^1_s, X^1_{s-\tau}\big)\,\mathrm{d}s \\
& + \int_0^t \frac{\partial\sigma\big(X^1_s, X^1_{s-\tau}\big)}{\partial x_1} V^{n,1}_s\,\mathrm{d}W_s + \int_0^t \frac{\partial\sigma\big(X^{n,1}_s, X^1_{s-\tau}\big)}{\partial y} V^{n,1}_{s-\tau}\cdot I_{\{s-\tau>0\}}\,\mathrm{d}W_s \\
& - \int_0^t \frac{\partial\sigma\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{s-\tau}\big)}{\partial x_1} b_1\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)\,\mathrm{d}\bar M^{n,3}_s \\
& - \int_0^t \frac{\partial\sigma\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial y} b_1\big(X^n_{s-\tau}, X^{n,1}_{s-2\tau}\big)\,\mathrm{d}\bar N^{n,2}_s \\
& - \int_0^t \frac{\partial\sigma\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial y}\,\mathrm{d}Z^n_s + \alpha(n),
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
\alpha(n) = {} & \int_0^t \frac{1}{2}\big\langle \big(X^n_s - X_s\big), \nabla_{xx} b_2\big(\theta^{n,1}_s, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big) n\big(X^n_s - X_s\big)\big\rangle\,\mathrm{d}s \\
& + \int_0^t \frac{1}{2}\frac{\partial^2 b_2\big(X^n_s, \theta^{n,2}_s\big)}{\partial y^2} n\big(X^{n,1}_{s-\tau} - X^1_{s-\tau}\big)^2\cdot I_{\{s-\tau>0\}}\,\mathrm{d}s \\
& + \int_0^t \frac{1}{2}\frac{\partial^2\sigma\big(\bar\theta^{n,1}_s, X^1_{s-\tau}\big)}{\partial x_1\partial x_1} n\big(X^{n,1}_s - X^1_s\big)^2\,\mathrm{d}W_s \\
& + \int_0^t \frac{1}{2}\frac{\partial^2\sigma\big(X^{n,1}_s, \bar\theta^{n,2}_s\big)}{\partial y^2} n\big(X^{n,1}_{s-\tau} - X^1_{s-\tau}\big)^2\cdot I_{\{s-\tau>0\}}\,\mathrm{d}W_s \\
& - \int_0^t \frac{1}{2}\frac{\partial^2 b_2\big(X^n_s, \theta^{n,4}_s\big)}{\partial y^2} n\big(X^{n,1}_{s-\tau} - X^1_{\lfloor s-\tau\rfloor_h}\big)^2\,\mathrm{d}s \\
& - \bigg(\int_0^t \frac{\partial b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2}\sigma\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big) n(W_s - W_{\lfloor s\rfloor_h})\,\mathrm{d}s \\
& - \int_0^t \frac{\partial b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2}\sigma\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)\,\mathrm{d}\bar M^{n,2}_s \bigg)
\end{aligned}
$$

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$$
\begin{aligned}
&- \Bigg(\int_0^t \frac{1}{2} \big\langle \big(X_s^n - X_{\lfloor s \rfloor_h}^n\big)^*, \nabla_{xx} b_2\big(\theta_s^{n,3}, X_{\lfloor s-\tau \rfloor_h}^{n,1}\big) n\big(X_s^n - X_{\lfloor s \rfloor_h}^n\big) \big\rangle \, \mathrm{d}s \\
&- \frac{\tau}{4} \int_0^t \frac{\partial^2 b_2\big(X_s, X_{s-\tau}^1\big)}{\partial x_2 \partial x_2} \sigma^2\big(X_s^1, X_{s-\tau}^1\big) \, \mathrm{d}s \Bigg) \\
&- \int_0^t \frac{1}{2} \frac{\partial^2 \sigma\big(\bar{\theta}_s^{n,3}, X_{s-\tau}^{n,1}\big)}{\partial x_1^2} n\big(X_s^{n,1} - X_{\lfloor s \rfloor_h}^{n,1}\big)^2 \, \mathrm{d}W_s \\
&- \int_0^t \frac{1}{2} \frac{\partial^2 \sigma\big(X_{\lfloor s \rfloor_h}^n, X_{\lfloor s-\tau \rfloor_h}^{n,1}\big)}{\partial y^2} n\big(X_{s-\tau}^{n,1} - X_{\lfloor s-\tau \rfloor_h}^{n,1}\big)^2 \, \mathrm{d}W_s \\
=: &\sum_{j=1}^9 I_j. \qquad (29)
\end{aligned}
$$

and

$$
\bar{M}_t^{n,1} = \int_0^t n(s - \lfloor s \rfloor_h) \, \mathrm{d}s, \quad \bar{M}_t^{n,2} = \int_0^t n(\lfloor s \rfloor_h + h - s) \, \mathrm{d}W_s,
$$

$$
\bar{M}_t^{n,3} = \int_0^t n(s - \lfloor s \rfloor_h) \, \mathrm{d}W_s, \quad \bar{N}_t^{n,1} = \int_0^t n(s - \tau - \lfloor s - \tau \rfloor_h) \cdot I_{\{s-\tau>0\}} \, \mathrm{d}s,
$$

$$
\bar{N}_t^{n,2} = \int_0^t n(s - \tau - \lfloor s - \tau \rfloor_h) \cdot I_{\{s-\tau>0\}} \, \mathrm{d}W_s, \quad Z_t^n = \int_0^t n\big(\xi_{s-\tau}^1 - \xi_{\lfloor s-\tau \rfloor_h}^1\big) \cdot I_{\{s \le \tau\}} \, \mathrm{d}s. \quad (30)
$$

By Lemma A.4, $\alpha(n)$ converges to 0 in probability. In light of Lemmas 2.1 and A.5, as well as [19, Theorem 4.1], one derives

$$
\big(\bar{M}^{n,1}, \bar{M}^{n,2}, \bar{M}^{n,3}, \bar{N}^{n,1}, \bar{N}^{n,2}\big) \overset{\mathcal{S}}{\Rightarrow} (\bar{M}, \bar{N})
$$

and

$$
Z_t^n \to \frac{1}{2} \tau \big(\xi_{t \wedge \tau - \tau}^1 - \xi_{-\tau}^1\big)
$$

with probability 1. In addition, similar to the proof process of Lemma 3.3, V_t^n is tight in $C([0, T]; \mathbb{R}^2)$. Hence, by a proof process similar to Lemma 3.4, we obtain $V_t^{n,2} \Rightarrow V_t^2$. Similarly, there is $V_t^{n,1} \Rightarrow V_t^1$. This completes the proof. □

5. Examples

In this section we present two examples to illustrate Theorems 2.3 and 2.4, respectively.

Example 5.1. Consider

$$
\mathrm{d}X_t = \big(X_t - X_t^3\big) \, \mathrm{d}t + (X_t + X_{t/2-1}) \, \mathrm{d}W_t, \quad t \in [0, T], \qquad (31)
$$

with initial data $X_t = 1$ for $t \in [-1, 0]$. It is easily observed that all the conditions in Theorem 2.2 are satisfied. Let X_t^n be the Euler scheme for (31). Then $\sqrt{n}\big(X_t^n - X_t\big)$ converges weakly to U_t with $U_0 = 0$, which can be expressed as follows: for $t \in (0, 2]$,

$$
U_t = \int_0^t \big(1 - 3X_s^2\big) U_s \, \mathrm{d}s + \int_0^t U_s \, \mathrm{d}W_s - \int_0^t \frac{\sqrt{2}}{2} (X_s + 1) \, \mathrm{d}B_s
$$

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and for $t \geq 2$,

$$U_t = \int_0^t \big(1 - 3X_s^2\big)U_s \,\mathrm{d}s + \int_0^t U_s \,\mathrm{d}W_s + \int_2^t U_{s/2-1} \,\mathrm{d}W_s - \int_0^t \frac{\sqrt{2}}{2}(X_s + X_{s/2-1}) \,\mathrm{d}B_s - \int_2^t \sqrt{\frac{s-1}{2}}(X_s + X_{s/2-1}) \,\mathrm{d}\bar{B}_s,$$

where B and $\bar{B}$ are two independent Brownian motions independent of W.

Example 5.2. Consider the following equation:

$$\begin{cases} \mathrm{d}X_t^1 = \big(aX_{t-\tau}^1 + bX_t^2\big) \,\mathrm{d}t, \\ \mathrm{d}X_t^2 = \big(X_{t-\tau}^1 + X_t^2\big) \,\mathrm{d}t + X_t^1 \,\mathrm{d}W_t, \end{cases} \tag{32}$$

with initial data $X_t^1 = 1$ for $t \in [-\tau, 0]$, $X_0^2 = 1$, where a and b are constant.

It is easy to check all the conditions in Theorem 2.4 are satisfied. Let $(X_t^{n,1}, X_t^{n,2})^*$ be the Euler scheme of (32). Then $\big(n(X_t^{n,1} - X_t^1), n(X_t^{n,2} - X_t^2)\big)^*$ converge weakly to $(V_t^1, V_t^2)^*$, where $(V_t^1, V_t^2)^*$ with $(V_0^1, V_0^2)^* = (0, 0)^*$ is the solution of the following SDDE: for $t \in (0, \tau]$,

$$\begin{cases} V_t^1 = \int_0^t bV_s^2 \,\mathrm{d}s - \frac{\tau}{2}\int_0^t b\big(X_{s-\tau}^1 + X_s^2\big) \,\mathrm{d}s, \\ V_t^2 = \displaystyle\int_0^t V_s^2 \,\mathrm{d}s - \frac{\tau}{2}\int_0^t \big(X_{s-\tau}^1 + X_s^2\big) \,\mathrm{d}s - \frac{\tau}{\sqrt{12}}\int_0^t X_s^1 \,\mathrm{d}B_s - \frac{\tau}{2}\int_0^t X_s^1 \,\mathrm{d}W_s + \int_0^t V_s^1 \,\mathrm{d}W_s \\ \qquad + \dfrac{\tau}{\sqrt{12}}\displaystyle\int_0^t \big(aX_{s-\tau}^1 + bX_s^2\big) \,\mathrm{d}B_s - \frac{\tau}{2}\int_0^t \big(aX_{s-\tau}^1 + bX_s^2\big) \,\mathrm{d}W_s \end{cases}$$

and for $t \geq \tau$,

$$\begin{cases} V_t^1 = \displaystyle\int_0^t bV_s^2 \,\mathrm{d}s - \frac{\tau}{2}\int_0^t b\big(X_{s-\tau}^1 + X_s^2\big) \,\mathrm{d}s + \int_\tau^t aV_{s-\tau}^1 \,\mathrm{d}s - \frac{\tau}{2}\int_\tau^t a\big(aX_{s-2\tau}^1 + bX_{s-\tau}^2\big) \,\mathrm{d}s, \\ V_t^2 = \displaystyle\int_0^t V_s^2 \,\mathrm{d}s - \frac{\tau}{2}\int_0^t \big(X_{s-\tau}^1 + X_s^2\big) \,\mathrm{d}s - \frac{\tau}{\sqrt{12}}\int_0^t X_s^1 \,\mathrm{d}B_s - \frac{\tau}{2}\int_0^t X_s^1 \,\mathrm{d}W_s + \int_0^t V_s^1 \,\mathrm{d}W_s \\ \qquad + \dfrac{\tau}{\sqrt{12}}\displaystyle\int_0^t \big(aX_{s-\tau}^1 + bX_s^2\big) \,\mathrm{d}B_s - \frac{\tau}{2}\int_0^t \big(aX_{s-\tau}^1 + bX_s^2\big) \,\mathrm{d}W_s + \int_\tau^t V_{s-\tau}^1 \,\mathrm{d}s \\ \qquad - \dfrac{\tau}{2}\displaystyle\int_\tau^t \big(aX_{s-2\tau}^1 + bX_{s-\tau}^2\big) \,\mathrm{d}s, \end{cases}$$

where B and $\bar{B}$ are two independent Brownian motions independent of W.

In particular, if $a = b = 0$, then $X_t^1 \equiv 1$. This implies that $V_t^1 \equiv 0$, and for any $t \geq 0$,

$$V_t^2 = \int_0^t V_s^2 \,\mathrm{d}s - \frac{\tau}{2}\int_0^t \big(1 + X_s^2\big) \,\mathrm{d}s - \frac{\tau}{\sqrt{12}}B_t - \frac{\tau}{2}W_t.$$

Appendix A. Proofs of some auxiliary lemmas

Let us present the following results:

Lemma A.1. *Let $\{\eta_n\}_{n\geq 1}$ be a random sequence and assume that $\lim_{n\to\infty} \mathrm{E}\eta_n$ exists. If the variance $Var(\eta_n) \to 0$ as $n \to \infty$, then $\eta_n \xrightarrow{\mathrm{P}} \lim_{n\to\infty} \mathrm{E}\eta_n$.*

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Proof. Applying the Chebyshev inequality gives

$$\mathrm{P}\big(|\eta_n - \lim_{n\to\infty}\mathrm{E}\eta_n| > \varepsilon\big) \le \frac{\mathrm{E}\big(\eta_n - \lim_{n\to\infty}\mathrm{E}\eta_n\big)^2}{\varepsilon^2} \le \frac{\mathrm{Var}(\eta_n) + 2\big(\mathrm{E}\eta_n - \lim_{n\to\infty}\mathrm{E}\eta_n\big)^2}{\varepsilon^2} \to 0$$

as $n \to \infty$. This proof is complete. □

Lemma A.2. *Let M^n and N^n be defined by (24), the following limits hold:*

(a_1) $M_t^{n,\kappa} \xrightarrow{\mathrm{P}} 0$, $\kappa = 1, 2, 3$, (a_2) $N_t^{n,\kappa} \xrightarrow{\mathrm{P}} 0$, $\kappa = 1, 2, 3$,
(a_3) $\big[M^{n,4}, M^{n,4}\big]_t \xrightarrow{\mathrm{P}} \frac{t\tau}{2}$, (a_4) $\big[M^{n,4}, W\big]_t \xrightarrow{\mathrm{P}} 0$,
(a_5) $\big[N^{n,4}, N^{n,4}\big]_t \xrightarrow{\mathrm{P}} \frac{q_0\tau\delta(t)}{2} I_{\{\delta(t)>0\}}$, (a_6) $\big[N^{n,4}, W\big]_t \xrightarrow{\mathrm{P}} 0$,
(a_7) $\big[M^{n,4}, N^{n,4}\big]_t \xrightarrow{\mathrm{P}} 0$.

Proof. Obviously, $M_t^{n,1} \longrightarrow 0$ as $n \to \infty$ due to $s - \lfloor s\rfloor_h \le h \le \tau/n$. Using the formula of integration by parts for semimartingale and the Abel transformation yields

$$\begin{aligned}
\int_0^t (W_s - W_{\lfloor s\rfloor_h})\,\mathrm{d}s =& t\,W_t - \int_0^t s\,\mathrm{d}W_s - \sum_{l=0}^{\lfloor t/h\rfloor-1} \big((l+1)h - lh\big)W_{lh} - (t - \lfloor t\rfloor_h)W_{\lfloor t\rfloor_h} \\
=& tW_t - \int_0^t s\,\mathrm{d}W_s - \Big(\lfloor t\rfloor_h W_{\lfloor t\rfloor_h} - \sum_{l=0}^{\lfloor t/h\rfloor-1} (l+1)h(W_{(l+1)h} - W_{lh})\Big) \\
& - (t - \lfloor t\rfloor_h)W_{\lfloor t\rfloor_h} \\
=& \int_{\lfloor t\rfloor_h}^t t\,\mathrm{d}W_s - \int_0^t s\,\mathrm{d}W_s + \int_0^{\lfloor t\rfloor_h} \big(\lfloor s\rfloor_h + h\big)\,\mathrm{d}W_s \\
=& \int_{\lfloor t\rfloor_h}^t (t - h - \lfloor t\rfloor_h)\,\mathrm{d}W_s + \int_0^t \big(\lfloor s\rfloor_h + h - s\big)\,\mathrm{d}W_s. \qquad \text{(A1)}
\end{aligned}$$

Hence,

$$\mathrm{E}\big(M_t^{n,2}\big)^2 \le 2n(t - h - \lfloor t\rfloor_h)^2\mathrm{E}|W_t - W_{\lfloor t\rfloor_h}|^2 + 2\mathrm{E}\int_0^t n\big(\lfloor s\rfloor_h + h - s\big)^2\,\mathrm{d}s \le \frac{C}{n},$$

which yields $M_t^{n,2} \xrightarrow{\mathrm{P}} 0$ as $n \to \infty$. Note that

$$\mathrm{E}\big(M_t^{n,3}\big)^2 = \int_0^t n(s - \lfloor s\rfloor_h)^2\,\mathrm{d}s \le \frac{t\tau^2}{n}.$$

This, together with $\mathrm{E}M_t^{n,3} = 0$ and Lemma A.1, implies that $M_t^{n,3} \xrightarrow{\mathrm{P}} 0$ as $n \to \infty$. Hence, (a_1) holds.

The similar techniques yield that (a_2) holds.

Now let us prove (a_3)–(a_7) one by one. Note that

$$\big[M^{n,4}, M^{n,4}\big]_t = \int_0^t n(W_s - W_{\lfloor s\rfloor_h})^2\,\mathrm{d}s.$$

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Let $K = \lfloor t/h \rfloor$ and $h_1 = t - Kh$. It is obvious that $0 \le h_1 < h$. One has

$$\begin{aligned}
\mathrm{E}\big[M^{n,4}, M^{n,4}\big]_t &= \mathrm{E}\int_0^t n(W_s - W_{\lfloor s \rfloor_h})^2 \,\mathrm{d}s = \int_0^t n(s - \lfloor s \rfloor_h)\,\mathrm{d}s \\
&= \sum_{l=0}^{K-1} \int_{lh}^{(l+1)h} n(s - lh)\,\mathrm{d}s - \int_{Kh}^t n(t - Kh)\,\mathrm{d}s \\
&= \frac{nKh^2}{2} + \frac{n(t - Kh)^2}{2} \\
&= \frac{\tau Kh + nh_1^2}{2}. \qquad \text{(A2)}
\end{aligned}$$

Letting $n \to \infty$ gives

$$\mathrm{E}\big[M^{n,4}, M^{n,4}\big]_t \to \frac{t\tau}{2}. \qquad \text{(A3)}$$

Recall

$$\mathrm{E}(W_t - W_s)^{2k} = \prod_{i=1}^{k} (2i - 1)(t - s)^k$$

for any $t \ge s \ge 0$ and $k \in \mathbb{Z}_+$. According to the independent increment property of Brownian motion,

$$\begin{aligned}
\mathrm{Var}\big(\big[M^{n,4}, M^{n,4}\big]_t\big) &= \mathrm{Var}\bigg(\sum_{l=0}^{K-1} \int_{lh}^{(l+1)h} n(W_s - W_{\lfloor s \rfloor_h})^2\,\mathrm{d}s + \int_{Kh}^t n(W_s - W_{Kh})^2\,\mathrm{d}s\bigg) \\
&\le \sum_{l=0}^{K} \mathrm{E}\bigg(\int_{lh}^{(l+1)h} n(W_s - W_{\lfloor s \rfloor_h})^2\,\mathrm{d}s\bigg)^2 \\
&\le \sum_{l=0}^{K} \tau n \int_{lh}^{(l+1)h} \mathrm{E}\big(W_s - W_{\lfloor s \rfloor_h}\big)^4\,\mathrm{d}s \\
&\le Cn^{-1}. \qquad \text{(A4)}
\end{aligned}$$

This, together with (A3) and Lemma A.1, implies

$$\big[M^{n,4}, M^{n,4}\big]_t \xrightarrow{\mathrm{P}} \frac{t\tau}{2},$$

that is, (a_3) holds.

Similar to (A4), the independent increment property of Brownian motion implies

$$\begin{aligned}
\mathrm{E}[M^{n,4}, W]_t^2 &= \int_0^t \int_0^t n\mathrm{E}(W_s - W_{\lfloor s \rfloor_h})(W_u - W_{\lfloor u \rfloor_h})\,\mathrm{d}u\,\mathrm{d}s \\
&= \int_0^t \int_0^t n(s \wedge u - \lfloor u \rfloor_h) \cdot I_{\{\lfloor s \rfloor_h = \lfloor u \rfloor_h\}}\,\mathrm{d}u\,\mathrm{d}s \\
&\le \int_0^t \int_0^t n(u - \lfloor u \rfloor_h) \cdot I_{\{\lfloor s \rfloor_h = \lfloor u \rfloor_h\}}\,\mathrm{d}u\,\mathrm{d}s
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{l=0}^{K-1} \int_{lh}^{(l+1)h} \int_{lh}^{(l+1)h} (nu - nhl)\,\mathrm{d}u\,\mathrm{d}s + \int_{Kh}^{t} \int_{Kh}^{t} (nu - nhK)\,\mathrm{d}u\,\mathrm{d}s \\
&\le nKh^3 + \frac{1}{2} nhh_1^2 \\
&\le \tau t h + \frac{1}{2}\tau h_1^2. \qquad \text{(A5)}
\end{aligned}
$$

This, together with the Lyapunov inequality, implies $\mathrm{E}[M^{n,4}, W]_t \to 0$, which in turn yields $\mathrm{Var}\big([M^{n,4}, W]_t\big) \to 0$. Applying Lemma A.1 again gives

$$
\big[M^{n,4}, W\big]_t \xrightarrow{\mathrm{P}} 0,
$$

which implies (a_4).

Let us examine (a_5). Note that

$$
\big[N^{n,4}, N^{n,4}\big]_t = \int_0^t n(W_{\delta(s)} - W_{\lfloor \delta(s) \rfloor_h})^2 \cdot I_{\{\delta(s)>0\}}\,\mathrm{d}s,
$$

and $\delta(t) = t/q_0 - \tau$ for $q_0 \in \mathbb{Z}_+$. Then we arrive at

$$
\begin{aligned}
\mathrm{E}\big[N^{n,4}, N^{n,4}\big]_t &= \mathrm{E} \int_0^t n(W_{\delta(s)} - W_{\lfloor \delta(s) \rfloor_h})^2 \cdot I_{\{\delta(s)>0\}}\,\mathrm{d}s \\
&= \int_0^t n\big(\delta(s) - \lfloor \delta(s) \rfloor_h\big) \cdot I_{\{\delta(s)>0\}}\,\mathrm{d}s \\
&= \int_0^t n\left(\frac{s}{q_0} - \tau - \left\lfloor \frac{s/q_0 - \tau}{h} \right\rfloor h\right) \cdot I_{\{\delta(s)>0\}}\,\mathrm{d}s \\
&= \int_0^t n\left(\frac{s}{q_0 h} h - \left\lfloor \frac{s}{q_0 h} \right\rfloor h\right) \cdot I_{\{\delta(s)>0\}}\,\mathrm{d}s \\
&= \int_0^t \tau \left(\frac{s}{q_0 h} - \left\lfloor \frac{s}{q_0 h} \right\rfloor\right) \mathrm{d}s - \int_0^{q_0\tau} \tau \left(\frac{s}{q_0 h} - \left\lfloor \frac{s}{q_0 h} \right\rfloor\right) \mathrm{d}s. \qquad \text{(A6)}
\end{aligned}
$$

Taking $p = [t/q_0 h]$ and $h_2 = t - pq_0 h < q_0 h$, we have

$$
\begin{aligned}
\int_0^t \tau \left(\frac{s}{q_0 h} - \left\lfloor \frac{s}{q_0 h} \right\rfloor\right) \mathrm{d}s &= \sum_{l=0}^{p-1} \int_{lq_0 h}^{(l+1)q_0 h} \tau \left(\frac{s}{q_0 h} - l\right) \mathrm{d}s + \int_{pq_0 h}^{t} \tau \left(\frac{s}{q_0 h} - p\right) \mathrm{d}s \\
&= \sum_{l=0}^{p-1} \tau \left(\frac{q_0^2 h^2}{2q_0 h}(2l+1) - lq_0 h\right) + \tau \left(\frac{h_2^2 + 2h_2 pq_0 h}{2q_0 h} - ph_2\right) \\
&= \frac{\tau p q_0 h}{2} + \frac{\tau h_2^2}{2q_0 h} \to \frac{\tau t}{2}, \quad \text{as } n \to \infty. \qquad \text{(A7)}
\end{aligned}
$$

Hence, it follows from (A6) to (A7) that

$$
\begin{aligned}
\mathrm{E}\big[N^{n,4}, N^{n,4}\big]_t &= \mathrm{E} \int_0^t n(W_{\delta(s)} - W_{\lfloor \delta(s) \rfloor_h})^2 \cdot I_{\{\delta(s)>0\}}\,\mathrm{d}s \\
&\to \frac{\tau(t - q_0\tau)}{2} I_{\{\delta(t)>0\}} = \frac{q_0 \tau \delta(t)}{2} I_{\{\delta(t)>0\}}.
\end{aligned}
$$

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Similar to (A4), it is easy to see that

$$\begin{aligned}\operatorname{Var}\big(\big[N^{n,4}, N^{n,4}\big]_t\big) &\le \sum_{l=0}^{p} nq\tau \int_{lq_0h}^{(l+1)q_0h} \mathrm{E}\big(W_{\delta(s)} - W_{\lfloor \delta(\lfloor s\rfloor_h)\rfloor_h}\big)^4 \cdot I_{\{\delta(s)>0\}}\,\mathrm{d}s \\ &\le Cn^{-1}.\end{aligned}$$

Therefore, we obtain $[N^{n,4}, N^{n,4}]_t \xrightarrow{\mathrm{P}} (q_0\tau\delta(t)/2)I_{\{\delta(t)>0\}}$, that is, (a_5) holds.

Applying a similar technique to (a_4) gives (a_6).

Let us prove (a_7). Note that

$$\big[M^{n,4}, N^{n,4}\big]_t = \int_0^t n(W_s - W_{\lfloor s\rfloor_h})(W_{\delta(s)} - W_{\lfloor \delta(s)\rfloor_h}) \cdot I_{\{\delta(s)>0\}}\,\mathrm{d}s.$$

Noting that $h = \tau/n$, one has $\delta(s) = s/q_0 - \tau \le \lfloor s\rfloor_h$. It can be observed that

$$\mathrm{E}\big[M^{n,4}, N^{n,4}\big]_t = \mathrm{E}\int_0^t n(W_s - W_{\lfloor s\rfloor_h})(W_{\delta(s)} - W_{\lfloor \delta(s)\rfloor_h}) \cdot I_{\{\delta(s)>0\}} \cdot I_{\{\delta(u)>0\}}\,\mathrm{d}s = 0. \tag{A8}$$

This, together with the independent incremental property of Brownian motion, implies

$$\begin{aligned}\operatorname{Var}\big(\big[M^{n,4}, N^{n,4}\big]_t\big) &= \mathrm{E}\int_0^t\int_0^t n^2(W_s - W_{\lfloor s\rfloor_h})(W_{\delta(s)} - W_{\lfloor \delta(s)\rfloor_h})(W_u - W_{\lfloor u\rfloor_h}) \\ &\quad \times (W_{\delta(u)} - W_{\lfloor \delta(u)\rfloor_h})I_{\{\delta(s)>0\}}I_{\{\delta(u)>0\}}\,\mathrm{d}s\,\mathrm{d}u \\ &= \mathrm{E}\int_0^t\int_0^t n^2(W_s - W_{\lfloor s\rfloor_h})(W_u - W_{\lfloor u\rfloor_h})(W_{\delta(s)} - W_{\lfloor \delta(s)\rfloor_h}) \\ &\quad \times (W_{\delta(u)} - W_{\lfloor \delta(u)\rfloor_h})I_{\{\delta(s)>0\}}I_{\{\delta(u)>0\}}I_{\{\lfloor s\rfloor_h = \lfloor u\rfloor_h\}}\,\mathrm{d}s\,\mathrm{d}u,\end{aligned}$$

where we have used the fact that when $\lfloor s\rfloor_h > \lfloor u\rfloor_h$, $W_s - W_{\lfloor s\rfloor_h}$ is independent of $(W_{\delta(s)} - W_{\lfloor \delta(s)\rfloor_h})(W_u - W_{\lfloor u\rfloor_h})(W_{\delta(u)} - W_{\lfloor \delta(u)\rfloor_h})$ due to $s \ge \lfloor s\rfloor_h > u \vee \lfloor u\rfloor_h \vee \delta(s) \vee \lfloor \delta(s)\rfloor_h \vee \delta(u) \vee \lfloor \delta(u)\rfloor_h$ for $\delta(s) \wedge \delta(u) > 0$, and when $\lfloor s\rfloor_h < \lfloor u\rfloor_h$, $W_u - W_{\lfloor u\rfloor_h}$ is independent of $(W_{\delta(u)} - W_{\lfloor \delta(u)\rfloor_h})(W_s - W_{\lfloor s\rfloor_h})(W_{\delta(s)} - W_{\lfloor \delta(s)\rfloor_h})$. In addition, since $(W_s - W_{\lfloor s\rfloor_h})(W_u - W_{\lfloor u\rfloor_h})I_{\{\lfloor s\rfloor_h = \lfloor u\rfloor_h\}}$ is independent of $(W_{\delta(s)} - W_{\lfloor \delta(s)\rfloor_h})(W_{\delta(u)} - W_{\lfloor \delta(u)\rfloor_h})I_{\{\lfloor s\rfloor_h = \lfloor u\rfloor_h\}}$ for $\delta(s) \wedge \delta(u) > 0$, we have

$$\begin{aligned}\operatorname{Var}\big(\big[M^{n,4}, N^{n,4}\big]_t\big) &\le \int_0^t\int_0^t n^2(s \wedge u - \lfloor u\rfloor_h)(\delta(s \wedge u) - \lfloor \delta(u)\rfloor_h)I_{\{\delta(s)\wedge\delta(u)>0\}}I_{\{\lfloor s\rfloor_h = \lfloor u\rfloor_h\}}\,\mathrm{d}s\,\mathrm{d}u \\ &\le C\int_0^t\int_0^t n(u - \lfloor u\rfloor_h)I_{\{\lfloor s\rfloor_h = \lfloor u\rfloor_h\}}\,\mathrm{d}s\,\mathrm{d}u.\end{aligned}$$

Combining with (A5), this implies that $[M^{n,4}, N^{n,4}]_t \xrightarrow{P} 0$, that is, (a_7) holds. This completes this proof □.

Remark A.1. Let Z^n be a simple predictable process with $|Z^n| \le 1$ for all $n \ge 1$. By the Burkholder–Davis–Gundy inequality, we obtain

$$\mathrm{E}\left|\int_0^t Z^n_{s-}\,\mathrm{d}M^{n,4}_s\right| \le 4\sqrt{2T\tau},$$

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which, together with [26, Theorem 7.10], implies the uniform tightness of $M_s^{n,4}$. And using a similar approach further gives the uniform tightness of the other components of M^n and N^n. According to the result in [26], M^n and N^n are 'good' sequences.

Define

$$\begin{aligned}\hat{U}_t = &\int_0^t \frac{\partial b(X_s, X_{\delta(s)})}{\partial x}\tilde{U}_s\,\mathrm{d}s + \int_0^t \frac{\partial b(X_s, X_{\delta(s)})}{\partial y}\tilde{U}_{\delta(s)}\cdot I_{\{\delta(s)>0\}}\,\mathrm{d}s\\ &+ \int_0^t \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x}\tilde{U}_s\,\mathrm{d}W_s + \int_0^t \frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y}\tilde{U}_{\delta(s)}\cdot I_{\{\delta(s)>0\}}\,\mathrm{d}W_s\\ &- \int_0^t \sqrt{\frac{\tau}{2}}\frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial x}\sigma(X_s, X_{\delta(s)})\,\mathrm{d}B_s^1\\ &- \int_0^t \sqrt{\frac{f(s)}{s}}\frac{\partial \sigma(X_s, X_{\delta(s)})}{\partial y}\sigma(X_{\delta(s)}, X_{\delta(\delta(s))})\,\mathrm{d}B_s^2. \end{aligned} \tag{A9}$$

Let us present the following lemma.

Lemma A.3. *U_t^n converges in probability to $\hat{U}_t$.*

Proof. To prove this convergence, let us estimate every term in (26) one by one. Recall that $\big(U^n, X^n, W, M^n, N^n\big)_{[0,T]} \xrightarrow{\text{a.s.}} (\tilde{U}, X, W, M, N)_{[0,T]}$. In light of (19), using Fatou's lemma yields

$$\mathrm{E}\left[\sup_{0\le t\le T}|\tilde{U}_t|\right]^4 \le C_T. \tag{A10}$$

Since b and σ are Lipschitz continuous, applying a standard argument (see, for example, [33]) gives that for any $p>0$

$$\mathrm{E}\left[\sup_{0\le t\le T}|X_t|^p\right] < \infty, \quad \mathrm{E}\left[\sup_{0\le t\le T}|X_t^n|^p\right] < \infty. \tag{A11}$$

Due to $U_{[0,T]}^n \xrightarrow{\text{a.s.}} \tilde{U}_{[0,T]}$, there exists a $\mathcal{A}\subset\Omega$ with $\mathrm{P}(\mathcal{A})=1$ such that for any $\omega\in\mathcal{A}$,

$$\big(X^n, U^n\big)_{[0,T]} \to (X, \tilde{U})_{[0,T]}. \tag{A12}$$

Noting that b is Lipschitz continuous, it follows from Assumption **(H3)** that for any $t\in[0,T]$,

$$|\tilde{F}_t^{n,m,7}| \le \int_0^t C\sqrt{n}(\xi_{\delta(s)} - \xi_{\lfloor\delta(s)\rfloor_h})\,\mathrm{d}s \le Ct\sqrt{n}\max_{0\le s\le t}(\delta(s) - \lfloor\delta(s)\rfloor_h) \le Ct\frac{1}{\sqrt{n}}, \quad m=1,2,$$

which implies $\sup_{0\le t\le T}|\tilde{F}_t^{n,m,7}| \xrightarrow{\text{a.s.}} 0$ as $n\to\infty$. From the differentiability of b, we see that

$$\begin{aligned} &I^2_{\{(X_s,X_{\delta(s)})\in D_b\}}\cdot\left[\frac{U_s^n}{\big(X_s^n - X_s\big)}\Big(b\big(X_s^n, X_{\delta(s)}\big) - b(X_s, X_{\delta(s)})\Big)\right]\\ &\quad\to I^2_{\{(X_s,X_{\delta(s)})\in D_b\}}\cdot\tilde{U}_s\frac{\partial b(X_s, X_{\delta(s)})}{\partial x} \quad \text{as } n\to\infty. \end{aligned}$$

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Since U_s^n is a continuous process almost surely, the limit process $\tilde{U}_s$ is finite for any $\omega \in \mathcal{A}$. In addition, b satisfies the Lipschitz condition and is differentiable on D_b^C. Then for any fixed $\omega \in \mathcal{A}$, there exists a $K_{T,d}(\omega) < \infty$ such that for any $s \in [0, T]$

$$\left| \left[b(X_s^n(\omega), X_{\delta(s)}(\omega)) - b(X_s(\omega), X_{\delta(s)}(\omega)) \right] \frac{U_s^n(\omega)}{\left(X_s^n(\omega) - X_s(\omega)\right)} - \frac{\partial b(X_s(\omega), X_{\delta(s)}(\omega))}{\partial x} \tilde{U}_s(\omega) \right| I^2_{\{(X_s(\omega), X_{\delta(s)}(\omega)) \in D_b\}} \le K_{T,d}(\omega). \tag{A13}$$

Applying the dominated convergence theorem yields that for any $t \in [0, T]$,

$$\tilde{F}_t^{n,2,1} \xrightarrow{\text{a.s.}} 0.$$

With the help of (19) and (A10), in order to prove that $\sup_{0 \le t \le T} |\tilde{F}_t^{n,1,1}| \xrightarrow{\text{a.s.}} 0$, it is sufficient to prove $\int_0^T I^1_{\{(X_s, X_{\delta(s)}) \in D_b\}} ds = 0$. By the occupation time formula (see, for example, [36]), we obtain

$$\begin{aligned} \mathrm{E} \int_0^T I^1_{\{(X_s, X_{\delta(s)}) \in D_b\}} \sigma^2(X_s, X_{\delta(s)}) \, \mathrm{d}s &= \mathrm{E} \int_0^T I^1_{\{(X_s, X_{\delta(s)}) \in D_b\}} \, \mathrm{d}[X, X]_s \\ &\le \mathrm{E} \int_0^T I^1_{\{X_s \in D_b^1\}} \, \mathrm{d}[X, X]_s \\ &= \mathrm{E} \int_{D_b^1} L_T^x(X) \, \mathrm{d}x, \end{aligned} \tag{A14}$$

where $L_t^x(X)$ represents the local time in x of the semimartingale X. Note that

$$\mathrm{E} \int_0^T I^1_{\{X_s \in \mathbb{R}\}} \, \mathrm{d}[X, X]_s = \mathrm{E}[X, X]_T < \infty.$$

Then the occupation time formula implies that $L_t^x(X)$ is integral on $\mathbb{R} \times \Omega$. Recalling that $\lambda(D_b^1) = 0$, we therefore have $\mathrm{E} \int_{D_b^1} L_T^x(X) \, \mathrm{d}x = 0$. This, together with (A14), implies

$$\mathrm{E} \int_0^T I^1_{\{(X_s, X_{\delta(s)}) \in D_b\}} \sigma^2(X_s, X_{\delta(s)}) \, \mathrm{d}s = 0.$$

By assumption **(H4)**, it follows that

$$\int_0^T I^1_{\{(X_s, X_{\delta(s)}) \in D_b\}} \, \mathrm{d}s = 0, \tag{A15}$$

which further yields

$$\sup_{0 \le t \le T} |\tilde{F}_t^{n,1,1}| \xrightarrow{\text{a.s.}} 0. \tag{A16}$$

Applying a similar techniques to (A16) gives

$$\sup_{0 \le t \le T} |\tilde{F}_t^{n,m,2}| \xrightarrow{\text{a.s.}} 0.$$

Recall that $(M_t^{n,1}, M_t^{n,2}, N_t^{n,1}, N_t^{n,2}) \xrightarrow{\text{a.s.}} (0, 0, 0, 0)$ uniformly on $[0, T]$. This, together with the Lipschitz condition on b and σ, implies that for $z = 3, 4, 5, 6$,

$$\sup_{0 \le t \le T} |\tilde{F}_t^{n,m,z}| \xrightarrow{\text{a.s.}} 0.$$

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Now let us estimate $\tilde{G}_t^{n,m,z}$, $m=1,2$, $z=1,2,\dots,7$. Similar to $\tilde{F}_t^{n,m,7}$, it follows that

$$\mathrm{E}\left[\sup_{0\le t\le T}|\tilde{G}_t^{n,m,7}|\right]\le C\mathrm{E}\left[\int_0^T n(\delta(s)-\lfloor\delta(s)\rfloor_h)^2\,\mathrm{d}s\right]^{1/2}\le\frac{C\sqrt{T}}{\sqrt{n}}\longrightarrow 0,\quad \text{as } n\to\infty.$$

Applying the Burkholder–Davis–Gundy inequality yields

$$\begin{aligned}\mathrm{E}\left[\sup_{0\le t\le T}|\tilde{G}_t^{n,m,1}|\right]&\le C\mathrm{E}\left[\int_0^T I^m_{\{(X_s,X_{\delta(s)})\in D_\sigma\}}\left[\frac{U_s^n}{(X_s^n-X_s)}\big(\sigma(X_s^n,X_{\delta(s)})-\sigma(X_s,X_{\delta(s)})\big)\right.\right.\\&\quad\left.\left.-\tilde{U}_s\frac{\partial\sigma(X_s,X_{\delta(s)})}{\partial x}\right]^2\mathrm{d}s\right]^{1/2}\\&\le C\mathrm{E}\left[\int_0^T I^m_{\{(X_s,X_{\delta(s)})\in D_\sigma\}}\left[\big(\sigma(X_s^n,X_{\delta(s)})-\sigma(X_s,X_{\delta(s)})\big)\cdot\frac{(U_s^n-\tilde{U}_s)}{(X_s^n-X_s)}\right]^2\mathrm{d}s\right]^{1/2}\\&\quad+C\mathrm{E}\left[\int_0^T I^m_{\{(X_s,X_{\delta(s)})\in D_\sigma\}}|\tilde{U}_s|^2\left[\frac{\sigma(X_s^n,X_{\delta(s)})-\sigma(X_s,X_{\delta(s)})}{(X_s^n-X_s)}\right.\right.\\&\quad\left.\left.-\frac{\partial\sigma(X_s,X_{\delta(s)})}{\partial x}\right]^2\mathrm{d}s\right]^{1/2}\\&=:G^{n,m,1}(1)+G^{n,m,1}(2),\quad m=1,2.\end{aligned}\tag{A17}$$

Noting that σ is Lipschitz continuous, by the Lyapunov inequality,

$$G^{n,2,1}(1)\le C\left[\int_0^T\mathrm{E}\big(U_s^n-\tilde{U}_s\big)^2\,\mathrm{d}s\right]^{1/2}.$$

Recall that $U^n_{[0,T]}\xrightarrow{\text{a.s.}}\tilde{U}_{[0,T]}$. In light of (19), (A10), and by using the dominated convergence theorem, we obtain

$$G^{n,2,1}(1)\longrightarrow 0.\tag{A18}$$

Note that

$$I^2_{\{(X_s,X_{\delta(s)})\in D_\sigma\}}\frac{\sigma(X_s^n,X_{\delta(s)})-\sigma(X_s,X_{\delta(s)})}{(X_s^n-X_s)}\xrightarrow{\text{a.s.}}I^2_{\{(X_s,X_{\delta(s)})\in D_\sigma\}}\frac{\partial\sigma(X_s,X_{\delta(s)})}{\partial x}.$$

In addition, since σ is Lipschitz continuous, there exists a constant $L_T>0$ such that

$$\left|\frac{\sigma(X_s^n,X_{\delta(s)})-\sigma(X_s,X_{\delta(s)})}{X_s^n-X_s}-\frac{\partial\sigma(X_s,X_{\delta(s)})}{\partial x}\right|I^2_{\{(X_s,X_{\delta(s)})\in D_\sigma\}}\le L_T.\tag{A19}$$

Then, combining with (A10), applying the dominated convergence theorem gives

$$G^{n,2,1}(2)\to 0.$$

This, together with (A18) and (A17), implies

$$\mathrm{E}\left[\sup_{0\le t\le T}|\tilde{G}_t^{n,2,1}|\right]\to 0.$$

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Similar to (A18), we have

$$G^{n,1,1}(1) \to 0.$$

In light of (A10) and (A19), to establish $G^{n,1,1}(2) \to 0$, it suffices to prove that

$$\mathrm{E}\int_0^T I^1_{\{(X_s, X_{\delta(s)})\in D_\sigma\}}\,\mathrm{d}s = 0. \tag{A20}$$

By the occupation time formula,

$$\begin{aligned}\mathrm{E}\int_0^T I^1_{\{(X_s, X_{\delta(s)})\in D_\sigma\}}\sigma(X_s, X_{\delta(s)})^2\,\mathrm{d}s &= \mathrm{E}\int_0^T I^1_{\{(X_s, X_{\delta(s)})\in D_\sigma\}}\,\mathrm{d}[X,X]_s\\ &\le \mathrm{E}\int_0^T I^1_{\{X_s\in D^1_\sigma\}}\,\mathrm{d}[X,X]_s\\ &= \mathrm{E}\int_{D^1_\sigma} L^x_T(X)\,\mathrm{d}x = 0.\end{aligned}$$

Then by assumption **(H4)**, we obtain (A20). That is, $G^{n,1,1}(2) \to 0$, which, in turn, implies that

$$\mathrm{E}\left[\sup_{0\le t\le T}|\tilde{G}^{n,1,1}_t|\right] \to 0.$$

By using a similar method with respect to the derivation of $\tilde{G}^{n,m,1}_t$, we arrive at

$$\mathrm{E}\left[\sup_{0\le t\le T}|\tilde{G}^{n,m,2}_t|\right] \to 0.$$

Combining (A11) with the Burkholder–Davis–Gundy inequality gives

$$\begin{aligned}\mathrm{E}\left[\sup_{0\le t\le T}|\tilde{G}^{n,m,3}_t|\right] &\le C\mathrm{E}\left[\int_0^T I^m_{\{(X_s, X_{\delta(s)})\in D_\sigma\}} b\big(X^n_{\lfloor s\rfloor_h}, X^n_{\lfloor\delta(s)\rfloor_h}\big)^2 n(s-\lfloor s\rfloor_h)^2\,\mathrm{d}s\right]^{1/2}\\ &\le \frac{C\tau}{\sqrt{n}}\mathrm{E}\left[\int_0^T I^m_{\{(X_s, X_{\delta(s)})\in D_\sigma\}} b\big(X^n_{\lfloor s\rfloor_h}, X^n_{\lfloor\delta(s)\rfloor_h}\big)^2\,\mathrm{d}s\right]^{1/2} \to 0,\end{aligned}$$

where we have used the fact that $|s-\lfloor s\rfloor_h| \le h = \tau/n$ in the last inequality.

As b is Lipschitz continuous and due to (A11), one obtains $A^1_t := b(X^n_{\lfloor\delta(s)\rfloor_h}, X^n_{\lfloor\delta(\lfloor\delta(s)\rfloor_h)\rfloor_h})$ is uniformly integrable. Note that $N^{n,3}_t \xrightarrow{\text{a.s.}} 0$. Then applying the similar techniques to $\tilde{G}^{n,m,3}_t$ yields

$$\mathrm{E}\left[\sup_{0\le t\le T}\tilde{G}^{n,m,5}_t\right] \to 0.$$

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Now let us estimate $\tilde{G}_t^{n,m,4}$, $m=1,2$. By the interpolation techniques, we have

$$\begin{aligned}|\tilde{G}_t^{n,m,4}| \le & \left| \int_0^t I^m_{\{(X_s,X_{\delta(s)})\in D_\sigma\}} \left[\left(\sigma\left(X_s^n, X_{\delta(s)}^n\right) - \sigma\left(X_{\lfloor s\rfloor_h}^n, X_{\delta(s)}^n\right)\right) \cdot \frac{\sigma\left(X_{\lfloor s\rfloor_h}^n, X_{\lfloor\delta(s)\rfloor_h}^n\right)}{\left(X_s^n - X_{\lfloor s\rfloor_h}^n\right)} \right.\right. \\ & \left.\left. - \frac{\partial\sigma(X_s, X_{\delta(s)})}{\partial x}\sigma(X_s, X_{\delta(s)}) \right] \mathrm{d}M_s^{n,4} \right| \\ & + \left| \int_0^t I^m_{\{(X_s,X_{\delta(s)})\in D_\sigma\}} \frac{\partial\sigma(X_s, X_{\delta(s)})}{\partial x}\sigma(X_s, X_{\delta(s)})\,\mathrm{d}M_s^{n,4} \right. \\ & \left. - \int_0^t \sqrt{\frac{\tau}{2}} I^m_{\{(X_s,X_{\delta(s)})\in D_\sigma\}} \frac{\partial\sigma(X_s, X_{\delta(s)})}{\partial x}\sigma(X_s, X_{\delta(s)})\,\mathrm{d}B_s^1 \right| \\ =: & \, G^{n,m,4}(1) + G^{n,m,4}(2), \quad m=1,2. \end{aligned} \tag{A21}$$

It follows from Lemma A.2 and (25) that $M_s^{n,4} \xrightarrow{\text{a.s.}} \sqrt{\tau/2}B_s^1$. Hence, by Lemma 2.2, we obtain

$$\begin{aligned} & \int_0^t I^m_{\{(X_s,X_{\delta(s)})\in D_\sigma\}} \frac{\partial\sigma(X_s, X_{\delta(s)})}{\partial x}\sigma(X_s, X_{\delta(s)})\,\mathrm{d}M_s^{n,4} \\ & \xrightarrow{P} \int_0^t \sqrt{\frac{\tau}{2}} I^m_{\{(X_s,X_{\delta(s)})\in D_\sigma\}} \frac{\partial\sigma(X_s, X_{\delta(s)})}{\partial x}\sigma(X_s, X_{\delta(s)})\,\mathrm{d}B_s^1. \end{aligned}$$

Owing to the Lipschitz continuousness of σ and (A11), applying the Burkholder–Davis–Gundy inequality gives

$$\begin{aligned} & \mathrm{E}\left| \int_0^T \frac{\partial\sigma(X_s, X_{\delta(s)})}{\partial x}\sigma(X_s, X_{\delta(s)})\,\mathrm{d}M_s^{n,4} \right|^2 \\ & \qquad \le C\int_0^T \mathrm{E}\left[\sup_{-\tau\le s\le T} |X_s|^2 n^2 (W_s - W_{\lfloor s\rfloor_h})^2 \right] \mathrm{d}s < \infty, \end{aligned}$$

which yields

$$\int_0^T \frac{\partial\sigma(X_s, X_{\delta(s)})}{\partial x}\sigma(X_s, X_{\delta(s)})\,\mathrm{d}M_s^{n,4}$$

is uniformly integrable. By the dominated convergence theorem, we have

$$\mathrm{E}\Big[\sup_{0\le t\le T} G^{n,m,4}(2) \Big] \to 0. \tag{A22}$$

By virtue of the Burkholder–Davis–Gundy inequality and the Hölder inequality, for $m=1,2$, we have

$$\begin{aligned} \mathrm{E}\Big[\sup_{0\le t\le T} G^{n,m,4}(1) \Big] \le C \Bigg[\int_0^T \Bigg(& \mathrm{E} I^m_{\{(X_s,X_{\delta(s)})\in D_\sigma\}} \left[\left(\sigma\left(X_s^n, X_{\delta(s)}^n\right) - \sigma\left(X_{\lfloor s\rfloor_h}^n, X_{\delta(s)}^n\right)\right) \right. \\ & \left. \cdot \frac{\sigma\left(X_{\lfloor s\rfloor_h}^n, X_{\lfloor\delta(\lfloor s\rfloor_h)\rfloor_h}^n\right)}{\left(X_s^n - X_{\lfloor s\rfloor_h}^n\right)} - \frac{\partial\sigma(X_s, X_{\delta(s)})}{\partial x}\sigma(X_s, X_{\delta(s)}) \right]^4 \Bigg)^{1/2} \\ & \left(\mathrm{E}(\sqrt{n}(W_s - W_{\lfloor s\rfloor_h}))^4\right)^{1/2} \mathrm{d}s \Bigg]^{1/2}. \end{aligned} \tag{A23}$$

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Recall that σ is Lipschitz continuous. Then it follows from the Hölder inequality, (A15), and (A23) that

$$\begin{aligned}
0 &\le \mathrm{E}\left[\sup_{0\le t\le T} G^{n,1,4}(1)\right] \\
&\le C\tau\left[\int_0^T \left(\mathrm{E}I^1_{\{(X_s,X_{\delta(s)})\in D_\sigma\}}\right)^{1/4}\left(\mathrm{E}\sup_{0\le s\le T}\left(1+|X^n_{\lfloor s\rfloor_h}|+|X_s|\right)^8\right)^{1/4}\mathrm{d}s\right]^{1/2} \\
&\le C\tau\left[\int_0^T \left(\mathrm{E}I^1_{\{(X_s,X_{\delta(s)})\in D_\sigma\}}\right)^{\frac{1}{7}4}\mathrm{d}s\right]^{1/2} \\
&\le C\tau T^{3/8}\left[\int_0^T \mathrm{E}I^1_{\{(X_s,X_{\delta(s)})\in D_\sigma\}}\,\mathrm{d}s\right]^{1/8} = 0,
\end{aligned} \tag{A24}$$

where we have used the fact that $\mathrm{E}\big(\sqrt{n}(W_s - W_{\lfloor s\rfloor_h})\big)^4 \le 3\tau^2$ in the first inequality. In addition, since σ is differential on $\{(X_s, X_{\delta(s)}) \notin D_\sigma\}$ and is Lipschitz continuous, we have

$$\begin{aligned}
&I^2_{\{(X_s,X_{\delta(s)})\in D_\sigma\}}\cdot\left(\sigma\left(X^n_s, X^n_{\delta(s)}\right)-\sigma(X^n_{\lfloor s\rfloor_h}, X^n_{\delta(s)})\right)\cdot\frac{\sigma\left(X^n_{\lfloor s\rfloor_h}, X^n_{\lfloor\delta(\lfloor s\rfloor_h)\rfloor_h}\right)}{\left(X^n_s - X^n_{\lfloor s\rfloor_h}\right)} \\
&\quad\to I^2_{\{(X_s,X_{\delta(s)})\in D_\sigma\}}\cdot\frac{\partial\sigma(X_s,X_{\delta(s)})}{\partial x}\sigma(X_s,X_{\delta(s)})
\end{aligned}$$

with probability 1. By the dominated convergence theorem,

$$\mathrm{E}\left[\sup_{0\le t\le T} G^{n,2,4}(1)\right]\to 0.$$

This, together with (A21), (A22), and (A24), implies

$$\mathrm{E}\left[\sup_{0\le t\le T}|\tilde{G}^{n,m,4}_t|\right]\to 0.$$

Similar techniques to $G^{n,m,4}_t$ gives

$$\mathrm{E}\left[\sup_{0\le t\le T}|\tilde{G}^{n,m,6}_t|\right]\to 0.$$

Summarizing all estimations above, it follows from (26) that

$$U^n_{[0,T]}\xrightarrow{\mathrm{P}}\hat{U}_{[0,T]}.$$

This completes this proof. □

Lemma A.4. *We have that $\alpha(n)$ converges in probability to 0.*

Proof. Under the conditions in Theorem 2.4, applying a standard approach (see, for instance, [32]) yields for any $p > 0$, there exists $C_p > 0$ such that

$$\mathrm{E}\left[\sup_{-\tau\le s\le T}|X^n_s|^p\right]\le C_p. \tag{A25}$$

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In addition, note that the Euler scheme defined by (13) coincides with the Milstein scheme for system (12). Hence, by using a standard argument, it is easy to see that

$$\mathrm{E}\left[\sup_{0\le t\le T}|X_t^n - X_t|^4\right] \le Ch^4. \tag{A26}$$

Then, in light of $b \in C_b^2(\mathbb{R}^2\times\mathbb{R};\mathbb{R})$ and (A26), we arrive at

$$\mathrm{E}|I_1| \le C\int_0^t n\mathrm{E}\big(X_s^n - X_s\big)^2\,\mathrm{d}s \le Ch,$$

which implies

$$\lim_{n\to\infty}\mathrm{E}|I_1| = 0.$$

Similar estimation to I_1, we have

$$\lim_{n\to\infty}\mathrm{E}|I_2| = \lim_{n\to\infty}\mathrm{E}|I_3| = \lim_{n\to\infty}\mathrm{E}|I_4| = 0.$$

Let us first consider I_7. It can be observed that

$$\begin{aligned} I_7 =& -\int_0^t \frac{n}{2}\frac{\partial^2 b_2\big(\theta_s^{n,3}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_1\partial x_1}\big(X_s^{n,1} - X^{n,1}_{\lfloor s\rfloor_h}\big)^2\,\mathrm{d}s \\ &-\int_0^t n\frac{\partial^2 b_2\big(\theta_s^{n,3}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_1\partial x_2}\big(X_s^{n,1} - X^{n,1}_{\lfloor s\rfloor_h}\big)\big(X_s^{n,2} - X^{n,2}_{\lfloor s\rfloor_h}\big)\,\mathrm{d}s \\ &-\left[\int_0^t \frac{n}{2}\frac{\partial^2 b_2\big(\theta_s^{n,3}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2\partial x_2}\big(X_s^{n,2} - X^{n,2}_{\lfloor s\rfloor_h}\big)^2\,\mathrm{d}s - \frac{\tau}{4}\int_0^t \frac{\partial^2 b_2\big(X_s, X^1_{s-\tau}\big)}{\partial x_2\partial x_2}\sigma^2\big(X_s^1, X^1_{s-\tau}\big)\,\mathrm{d}s\right] \\ =:& I_7^1 + I_7^2 + I_7^3. \end{aligned} \tag{A27}$$

Recalling that $b_1, b_2 \in C_b^2(\mathbb{R}^2\times\mathbb{R};\mathbb{R})$ and $\sigma \in C_b^2(\mathbb{R}\times\mathbb{R};\mathbb{R})$, by (13) and (A25),

$$\begin{aligned} \lim_{n\to\infty}\mathrm{E}|I_7^1| &\le \lim_{n\to\infty} C\mathrm{E}\int_0^t n\big(X_s^{n,1} - X^{n,1}_{\lfloor s\rfloor_h}\big)^2\,\mathrm{d}s \\ &= \lim_{n\to\infty} Cn\mathrm{E}\int_0^t b_1^2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)(s-\lfloor s\rfloor_h)^2\,\mathrm{d}s = 0 \end{aligned} \tag{A28}$$

and

$$\begin{aligned} \lim_{n\to\infty}\mathrm{E}|I_7^2| &\le \lim_{n\to\infty} C\mathrm{E}\int_0^t n\big|\big(X_s^{n,1} - X^{n,1}_{\lfloor s\rfloor_h}\big)\big(X_s^{n,2} - X^{n,2}_{\lfloor s\rfloor_h}\big)\big|\,\mathrm{d}s \\ &\le \lim_{n\to\infty} C\int_0^t n(s-\lfloor s\rfloor_h)\mathrm{E}|W_s - W_{\lfloor s\rfloor_h}|\,\mathrm{d}s = 0. \end{aligned} \tag{A29}$$

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Similar to I_7^1 and I_7^2, by (13), we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{E}|I_7^3|^2 &\le \lim_{n\to\infty} C\mathrm{E}\bigg|\frac{n}{2}\int_0^t \frac{\partial^2 b_2\big(\theta_s^{n,3}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2\partial x_2}\sigma^2\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)(W_s - W_{\lfloor s\rfloor_h})^2\,\mathrm{d}s \\
&\quad - \frac{\tau}{4}\int_0^t \frac{\partial^2 b_2\big(X_s, X^1_{s-\tau}\big)}{\partial x_2\partial x_2}\sigma^2\big(X^1_s, X^1_{s-\tau}\big)\,\mathrm{d}s\bigg|^2 \\
&\le \lim_{n\to\infty} C\mathrm{E}\bigg|\frac{n}{2}\int_0^t \bigg(\frac{\partial^2 b_2\big(\theta_s^{n,3}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2\partial x_2} - \frac{\partial^2 b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2\partial x_2}\bigg) \\
&\quad \times \sigma^2\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)(W_s - W_{\lfloor s\rfloor_h})^2\,\mathrm{d}s\bigg|^2 \\
&\quad + \lim_{n\to\infty} C\mathrm{E}\bigg|\frac{n}{2}\int_0^t \frac{\partial^2 b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2\partial x_2}\sigma^2\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)(W_s - W_{\lfloor s\rfloor_h})^2\,\mathrm{d}s \\
&\quad - \frac{n}{2}\int_0^t \frac{\partial^2 b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2\partial x_2}\sigma^2\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)(s - \lfloor s\rfloor_h)\,\mathrm{d}s\bigg|^2 \\
&\quad + \lim_{n\to\infty} C\mathrm{E}\bigg|\frac{n}{2}\int_0^t \frac{\partial^2 b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2\partial x_2}\sigma^2\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)(s - \lfloor s\rfloor_h)\,\mathrm{d}s \\
&\quad - \frac{\tau}{4}\int_0^t \frac{\partial^2 b_2\big(X_s, X^1_{s-\tau}\big)}{\partial x_2\partial x_2}\sigma^2\big(X^1_s, X^1_{s-\tau}\big)\,\mathrm{d}s\bigg|^2 \\
&=: I_7^{3,1} + I_7^{3,2} + I_7^{3,3}.
\end{aligned} \tag{A30}$$

Now, let us estimate these three terms one by one. Applying Hölder's inequality yields

$$\begin{aligned}
I_7^{3,1} &\le \lim_{n\to\infty} C\int_0^t \bigg[\mathrm{E}\bigg(\frac{\partial^2\big(b_2\big(\theta_s^{n,3}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big) - b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)\big)}{\partial x_2\partial x_2}\sigma^2\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)\bigg)^4\bigg]^{1/2} \\
&\quad \times \big(\mathrm{E}n^4\big(W_s - W_{\lfloor s\rfloor_h}\big)^8\big)^{1/2}\,\mathrm{d}s \\
&\le \lim_{n\to\infty} C\int_0^t \bigg[\mathrm{E}\bigg(\frac{\partial^2\big(b_2\big(\theta_s^{n,3}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big) - b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)\big)}{\partial x_2\partial x_2}\sigma^2\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)\bigg)^4\bigg]^{1/2}.
\end{aligned}$$

Note that $\theta_s^{n,3}$ can be represented as $X_s^n - \theta(X_s^n - X^n_{\lfloor s\rfloor_h})$ for some $\theta\in[0,1]$, and the fact that

$$\sup_{-\tau\le s\le T}|X_s^n - X^n_{\lfloor s\rfloor_h}| \xrightarrow{\text{a.s.}} 0$$

as $n\to\infty$. Then in light of $b_2\in C_b^2(\mathbb{R}^2\times\mathbb{R};\mathbb{R})$ and $\sigma\in C_b^2(\mathbb{R}\times\mathbb{R};\mathbb{R})$, the dominated convergence theorem gives

$$I_7^{3,1} = 0. \tag{A31}$$

Set $K = \lfloor t/h\rfloor$. Then we arrive at

$$\begin{aligned}
I_7^{3,2} &= \lim_{n\to\infty} C\mathrm{E}\bigg|\frac{n}{2}\sum_{l=0}^{K-1}\int_{lh}^{(l+1)h}\frac{\partial^2 b_2\big(X^n_{lh}, X^{n,1}_{lh-\tau}\big)}{\partial x_2\partial x_2}\sigma^2\big(X^{n,1}_{lh}, X^{n,1}_{lh-\tau}\big)\big((W_s - W_{lh})^2 - (s-lh)\big)\,\mathrm{d}s \\
&\quad + \frac{n}{2}\int_{Kh}^t \frac{\partial^2 b_2\big(X^n_{Kh}, X^{n,1}_{Kh-\tau}\big)}{\partial x_2\partial x_2}\sigma^2\big(X^{n,1}_{Kh}, X^{n,1}_{Kh-\tau}\big)\big((W_s - W_{Kh})^2 - (s-Kh)\big)\,\mathrm{d}s\bigg|^2.
\end{aligned}$$

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Define

$$f(l)=\frac{\partial^2 b_2\big(X_{lh}^n, X_{lh-\tau}^{n,1}\big)}{\partial x_2 \partial x_2}\sigma^2\big(X_{lh}^{n,1}, X_{lh-\tau}^{n,1}\big).$$

For any $i>j$, we have

$$\begin{aligned}
&\mathrm{E}\bigg[f(i)\int_{ih}^{(i+1)h}\big((W_s-W_{ih})^2-(s-ih)\big)\,\mathrm{d}s\bigg]\bigg[f(j)\int_{jh}^{(j+1)h}\big((W_u-W_{jh})^2-(u-jh)\big)\,\mathrm{d}u\bigg]\\
&\quad=\mathrm{E}\bigg[f(i)f(j)\int_{jh}^{(j+1)h}\big((W_s-W_{jh})^2-(s-jh)\big)\,\mathrm{d}s\\
&\qquad\times\mathrm{E}\bigg(\int_{ih}^{(i+1)h}\big((W_u-W_{ih})^2-(u-ih)\big)\,\mathrm{d}u\bigg|\mathcal{F}_{ih}\bigg)\bigg]\\
&\quad=0.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
I_7^{3,2}&\le\lim_{n\to\infty}C\mathrm{E}\bigg[\sum_{l=0}^{K-1}f^2(l)n^2\bigg(\int_{lh}^{(l+1)h}\big((W_s-W_{lh})^2-(s-lh)\big)\,\mathrm{d}s\bigg)^2\bigg]\\
&\quad+\lim_{n\to\infty}C\mathrm{E}\bigg[f^2(K)n^2\bigg(\int_{Kh}^{t}\big((W_s-W_{Kh})^2-(s-Kh)\big)\,\mathrm{d}s\bigg)^2\bigg]\\
&\le\lim_{n\to\infty}C\sum_{l=0}^{K-1}\mathrm{E}\bigg[f^2(l)\mathrm{E}\bigg(n\bigg(\int_{lh}^{(l+1)h}\big((W_s-W_{lh})^2-(s-lh)\big)\,\mathrm{d}s\bigg)^2\bigg|\mathcal{F}_{lh}\bigg)\bigg]\\
&\quad+\lim_{n\to\infty}C\mathrm{E}\bigg[f^2(K)\mathrm{E}\bigg(n^2\bigg(\int_{Kh}^{t}\big((W_s-W_{Kh})^2-(s-Kh)\big)\,\mathrm{d}s\bigg)^2\bigg|\mathcal{F}_{Kh}\bigg)\bigg]\\
&=\lim_{n\to\infty}C\sum_{l=0}^{K-1}\mathrm{E}\bigg[f^2(l)\mathrm{E}\bigg(n\int_{lh}^{(l+1)h}\big((W_s-W_{lh})^2-(s-lh)\big)\,\mathrm{d}s\bigg)^2\bigg]\\
&\quad+\lim_{n\to\infty}C\mathrm{E}\bigg[f^2(K)\mathrm{E}\bigg(n\int_{Kh}^{t}\big((W_s-W_{Kh})^2-(s-Kh)\big)\,\mathrm{d}s\bigg)^2\bigg].
\end{aligned}$$

Similar to (A5), one has

$$\begin{aligned}
&\mathrm{E}\bigg[n\int_{lh}^{(l+1)h}\big((W_s-W_{lh})^2-(s-lh)\big)\,\mathrm{d}s\bigg]^2\\
&\quad=\int_{lh}^{(l+l)h}\int_{lh}^{(l+l)h}n^2\mathrm{E}\big((W_s-W_{lh})^2-(s-lh)\big)\big((W_u-W_{lh})^2-(u-lh)\big)\,\mathrm{d}s\,\mathrm{d}u\\
&\quad=\int_{lh}^{(l+l)h}\int_{lh}^{(l+l)h}2n^2(u\wedge s-lh)^2\,\mathrm{d}s\,\mathrm{d}u\\
&\quad\le Ch^2,
\end{aligned}$$

which, together with (A25), implies

$$I_7^{3,2}\le\lim_{n\to\infty}C(K+1)h^2\mathrm{E}\bigg[\sup_{-\tau\le s\le T}f^2(s)\bigg]=0. \tag{A32}$$

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Note that

$$\begin{aligned}\frac{n}{2}\int_0^t f(l)(s-\lfloor s\rfloor_h)\,\mathrm{d}s &= \sum_{l=0}^{K-1} f(l)\int_{lh}^{(l+1)h}\frac{n}{2}(s-lh)\,\mathrm{d}s + f(K)\int_{Kh}^{t}\frac{n}{2}(s-Kh)\,\mathrm{d}s\\ &= \sum_{l=0}^{K-1} f(l)\frac{nh^2}{4} + f(K)\frac{n(t-Kh)^2}{4}\\ &= \frac{\tau}{4}\sum_{l=0}^{K-1} f(l)h + \frac{\tau}{4}f(K)(t-Kh) + f(K)\frac{n(t-Kh)(t-(K+1)h)}{4}\\ &\longrightarrow \frac{\tau}{4}\int_0^t \frac{\partial^2 b_2\big(X_s, X^1_{s-\tau}\big)}{\partial x_2\partial x_2}\sigma^2\big(X^1_s, X^1_{s-\tau}\big)\,\mathrm{d}s\end{aligned}$$

with probability one, where we have used the fact that

$$\left|f(K)\frac{n(t-Kh)(t-(K+1)h)}{4}\right| \le |f(K)|nh^2 \xrightarrow{\text{a.s.}} 0, \quad \text{as } n\to\infty.$$

Then with the help of (A25), it follows from the Lebesgue's dominated convergence theorem that

$$I_7^{3,3} = 0. \tag{A33}$$

Hence, it follows from (A30)–(A33) that $\lim_{n\to\infty}\mathrm{E}|I_7^3|^2 = 0$. This, together with (A27)–(A29), further implies

$$\lim_{n\to\infty}\mathrm{E}|I_7| = 0.$$

Similar to I_7^1, it is easy to see that

$$\lim_{n\to\infty}\mathrm{E}|I_5| = \lim_{n\to\infty}\mathrm{E}|I_8| = \lim_{n\to\infty}\mathrm{E}|I_9| = 0.$$

Now, it remains to prove $\lim_{n\to\infty}\mathrm{E}|I_6| = 0$. Note that

$$|I_6| = \left|\int_0^t \frac{\partial b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2}\sigma\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)n((W_s - W_{\lfloor s\rfloor_h})\,\mathrm{d}s - (\lfloor s\rfloor_h + h - s)\,\mathrm{d}W_s)\right|.$$

Let

$$g(l) = \frac{\partial b_2\big(X^n_{lh}, X^{n,1}_{lh-\tau}\big)}{\partial x_2}\sigma\big(X^{n,1}_{lh}, X^{n,1}_{lh-\tau}\big).$$

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Similar to (A1), by the formula of integration by parts, we arrive at

$$\begin{aligned}
|I_6| &= \left| \sum_{l=0}^{K-1} \int_{lh}^{(l+1)h} ng(l)(W_s - W_{lh})\,\mathrm{d}s - \sum_{l=1}^{K-1} \int_{lh}^{(l+1)h} ng(l)((l+1)h - s)\,\mathrm{d}W_s \right| \\
&\quad + \left| \int_{Kh}^{t} ng(Kh)(W_s - W_{Kh})\,\mathrm{d}s - \int_{Kh}^{t} ng(Kh)((K+1)h - s)\,\mathrm{d}W_s \right| \\
&= \left| ng(l) \left[\sum_{l=0}^{K-1} \int_{lh}^{(l+1)h} (W_s - W_{lh})\,\mathrm{d}s - \int_{lh}^{(l+1)h} ((l+1)h - s)\,\mathrm{d}W_s \right| \right. \\
&\quad + \left| ng(Kh) \left(\int_{Kh}^{t} (W_s - W_{Kh})\,\mathrm{d}s - \int_{Kh}^{t} ((K+1)h - s)\,\mathrm{d}W_s \right) \right| \\
&= \left| \sum_{l=1}^{K-1} ng(l) \left[\int_{lh}^{(l+1)h} W_s ds - ng(l) W_{lh}((l+1)h - lh) - ((l+1)h(W_{(l+1)h} - W_{lh}) \right. \right. \\
&\quad \left. \left. + (lh)W_{lh} + \int_{lh}^{(l+1)h} W_s\,\mathrm{d}s - (l+1)hW_{(l+1)h}) \right] \right| \\
&\quad + \left| ng(Kh) \left[\int_{Kh}^{t} W_s ds - W_{Kh}(t - Kh) - (K+1)h(W_t - W_{Kh}) \right. \right. \\
&\quad \left. \left. + tW_t - KhW_{Kh} - \int_{Kh}^{t} W_s\,\mathrm{d}s \right] \right| \\
&= |ng(Kh)((K+1)h - t)(W_t - W_{Kh})|.
\end{aligned}$$

Then it follows from (A25) that

$$\lim_{n\to\infty} \mathrm{E}|I_6| \le \lim_{n\to\infty} C\mathrm{E}|W_t - W_{Kh}| = 0.$$

Hence, the above estimations of I_i for $i = 1, \ldots, 9$ yield

$$\lim_{n\to\infty} \mathrm{E}|\alpha(n)| = 0, \tag{A34}$$

which implies that $\alpha(n)$ converges to zero in probability. This completes the proof. □

Remark A.2. This lemma reveals a key distinction between the proofs of Theorems 2.3 and 2.4 and shows that the proof of Theorem 2.4 is more complicated. Since (12) is a degenerate system, as noted in Remark 2.6, the proof of Theorem 2.4 needs a second-order expansion with respect to the coefficients, which leads to $\alpha(n)$ involving the second-order derivatives of the coefficients and convergence becomes more difficult.

On the other hand, in the first-order expansion, replacing $\sqrt{n}$ by n also leads to more complicated and difficult computations and proofs. For example, we need to examine the limit of B in Theorem 2.4 as follows:

$$B = -\int_0^t \frac{\partial b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2} \sigma\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big) n(W_s - W_{\lfloor s\rfloor_h})\,\mathrm{d}s$$

as $n \to \infty$. However, this computation is very difficult since B contains the non-martingale integral

$$\int_0^t n(W_s - W_{\lfloor s\rfloor_h})\,\mathrm{d}s.$$

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To overcome this difficulty, we introduce an auxiliary martingale process

$$A = -\int_0^t \frac{\partial b_2\big(X^n_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)}{\partial x_2}\sigma\big(X^{n,1}_{\lfloor s\rfloor_h}, X^{n,1}_{\lfloor s-\tau\rfloor_h}\big)\,\mathrm{d}\bar{M}^{n,2}_s,$$

and show that $B - A$ converges to zero. This implies that A and B share the same weak limiting distribution. To obtain the limit of A, the convergence results (b_2) and (b_3) in the following Lemma A.5 are necessary. Comparing with the proof of Theorem 2.3, the corresponding term is

$$M^{n,2}_t = \int_0^t \sqrt{n}(W_s - W_{\lfloor s\rfloor_h})\,\mathrm{d}s,$$

which converges to zero by Lemma A.2. Therefore, the proof of Theorem 2.3 does not require such complicated techniques and computations.

Lemma A.5. *Let $\bar{M}^n$ and $\bar{N}^n$ be given by (30). For any fixed $t \in [0, T]$,*

$$(b_1)\quad \bar{M}^{n,1}_t \longrightarrow \frac{t\tau}{2}, \qquad (b_2)\quad \big[\bar{M}^{n,2}, \bar{M}^{n,2}\big]_t \longrightarrow \frac{t\tau^2}{3},$$
$$(b_3)\quad \big[\bar{M}^{n,2}, W\big]_t \longrightarrow \frac{t\tau}{2}, \qquad (b_4)\quad \big[\bar{M}^{n,3}, \bar{M}^{n,3}\big]_t \longrightarrow \frac{t\tau^2}{3},$$
$$(b_5)\quad \big[\bar{M}^{n,3}, W\big]_t \longrightarrow \frac{t\tau}{2}, \qquad (b_6)\quad \big[\bar{M}^{n,2}, \bar{M}^{n,3}\big]_t \longrightarrow \frac{t\tau^2}{6},$$
$$(b_7)\quad \bar{N}^{n,1}_t \longrightarrow \frac{(t-\tau)\tau}{2}\cdot I_{\{t>\tau\}}, \qquad (b_8)\quad \big[\bar{N}^{n,2}, \bar{N}^{n,2}\big]_t \longrightarrow \frac{(t-\tau)\tau^2}{3}\cdot I_{\{t>\tau\}},$$
$$(b_9)\quad \big[\bar{N}^{n,2}, W\big]_t \longrightarrow \frac{(t-\tau)\tau}{2}\cdot I_{\{t>\tau\}}, \qquad (b_{10})\quad \big[\bar{M}^{n,2}, \bar{N}^{n,2}\big]_t \longrightarrow \frac{(t-\tau)\tau^2}{6}\cdot I_{\{t>\tau\}},$$
$$(b_{11})\quad \big[\bar{M}^{n,3}, \bar{N}^{n,2}\big]_t \longrightarrow \tfrac{(t-\tau)\tau^2}{3}\cdot I_{\{t>\tau\}}, \qquad (b_{12})\quad Z^n_t \longrightarrow \frac{\tau\big(\xi^1_{t\wedge\tau-\tau} - \xi^1_{-\tau}\big)}{2},$$
$$(b_{13})\quad \big[Z^{n,1}, Z^{n,2}\big]_t \longrightarrow 0,$$

where $Z^{n,1} = \bar{M}^{n,1}$ or $\bar{N}^{n,1}$ or Z^n_t, and $Z^{n,2} = \bar{M}^{n,2}$ or $\bar{M}^{n,3}$ or $\bar{N}^{n,2}$ or W.

Proof. In light of (A2), (b_1) holds.

Let $K = \lfloor t/h\rfloor$ and $h_1 = t - Kh$. It is easy to observe that

$$\begin{aligned}\big[\bar{M}^{n,2}, \bar{M}^{n,2}\big]_t &= \int_0^t n^2(\lfloor s\rfloor_h + h - s)^2\,\mathrm{d}s\\ &= \sum_{l=1}^{K-1}\int_{lh}^{(l+1)h} n^2((l+1)h - s)^2\,\mathrm{d}s + \int_{Kh}^t n^2((k+1)h - s)^2\,\mathrm{d}s\\ &= \frac{Kn^2h^3}{3} + \frac{n^2h^3 - n^2(h-h_1)^3}{3} \to \frac{t\tau^2}{3}\end{aligned} \tag{A35}$$

as $n \to \infty$, which implies that (b_2) holds.

Note that

$$\big[\bar{M}^{n,2}, W\big]_t = \int_0^t n(\lfloor s\rfloor_h + h - s)\,\mathrm{d}s = t\tau - \int_0^t n(\lfloor s\rfloor_h - s)\,\mathrm{d}s.$$

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Then (b_3) follows from (A2).

Now let us prove (b_4) and (b_5). Similar to (A2) and (A35), as $n \to \infty$,

$$\left[\bar{M}^{n,3}, \bar{M}^{n,3}\right]_t = \int_0^t n^2(s - \lfloor s \rfloor_h)^2 \, \mathrm{d}s \to \frac{t\tau^2}{3} \tag{A36}$$

and

$$\left[\bar{M}^{n,3}, W\right]_t = \int_0^t n(s - \lfloor s \rfloor_h) \, \mathrm{d}s \to \frac{t\tau}{2}. \tag{A37}$$

It follows from (A36) to (A37) that

$$\begin{aligned}\left[\bar{M}^{n,2}, \bar{M}^{n,3}\right]_t &= \int_0^t n^2(\lfloor s \rfloor_h + h - s)(s - \lfloor s \rfloor_h) \, \mathrm{d}s \\ &= \tau \int_0^t n(s - \lfloor s \rfloor_h) \, \mathrm{d}s - \int_0^t n^2(s - \lfloor s \rfloor_h)^2 \, \mathrm{d}s \to \frac{t\tau^2}{6}\end{aligned} \tag{A38}$$

as $n \to \infty$. Hence, (b_6) holds.

Noting that $s - \tau - \lfloor s - \tau \rfloor_h = s - \lfloor s \rfloor_h$, $h = \tau/n$, then we have

$$\bar{N}_t^{n,1} = \int_0^t n(s - \lfloor s \rfloor) I_{\{s-\tau>0\}} \, \mathrm{d}s.$$

Clearly, for $t \le \tau$, $\bar{N}_t^{n,1} = 0$. For $t > \tau$, from (A37) we see that

$$\bar{N}_t^{n,1} = \int_0^t n(s - \lfloor s \rfloor) I_{\{s-\tau>0\}} \, \mathrm{d}s = \int_0^t n(s - \lfloor s \rfloor) \, \mathrm{d}s - \int_0^\tau n(s - \lfloor s \rfloor) \, \mathrm{d}s \to \frac{t(t-\tau)}{2}$$

as $n \to \infty$. That is, (b_7) holds.

By (A36) and (A37), applying the similar techniques to (b_7) gives that as $n \to \infty$,

$$\left[\bar{N}^{n,2}, \bar{N}^{n,2}\right]_t = \int_0^t n^2(s - \lfloor s \rfloor)^2 I_{\{s-\tau>0\}} \, \mathrm{d}s \to \frac{(t-\tau)\tau^2}{3} I_{\{t>\tau\}}$$

and

$$\left[\bar{N}^{n,2}, W\right]_t = \int_0^t n^2(s - \lfloor s \rfloor) I_{\{s-\tau>0\}} \, \mathrm{d}s \to \frac{(t-\tau)\tau}{2} I_{\{t>\tau\}}.$$

That is, (b_8) and (b_9) hold.

Now let us turn to proving (b_{10}). Similar to (b_7), it follows from (A38) that

$$\left[\bar{M}^{n,2}, \bar{N}^{n,2}\right]_t = \int_0^t n^2(\lfloor s \rfloor_h + h - s)(s - \lfloor s \rfloor_h) \cdot I_{\{s-\tau>0\}} \, \mathrm{d}s \to \frac{(t-\tau)\tau^2}{6} I_{\{t>\tau\}} \quad \text{as } n \to \infty.$$

Hence, (b_{10}) holds.

For (b_{11}), according to the definition of $\bar{M}^{n,3}$ and $\bar{N}^{n,2}$, it is easy to see from (A36) that

$$\left[\bar{M}^{n,3}, \bar{N}^{n,2}\right]_t = \int_0^t n^2(s - \lfloor s \rfloor_h)^2 \cdot I_{\{s-\tau>0\}} \, \mathrm{d}s \to \frac{(t-\tau)\tau^2}{3} \cdot I_{\{t>\tau\}} \quad \text{as } n \to \infty.$$

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Now we are going to prove (b_{12}). Rewrite Z_t^n as

$$Z_t^n = \int_0^{t\wedge\tau} n\big(\xi^1_{s-\tau} - \xi^1_{\lfloor s\rfloor_h-\tau}\big)\,\mathrm{d}s.$$

For any $F \in C_b^1$, applying the mean value theorem gives

$$\begin{aligned}
\int_0^t n(F(s) - F(\lfloor s\rfloor_h))\,\mathrm{d}s &= n\sum_{l=0}^{K-1}\int_{lh}^{(l+1)h} F(s) - F(lh)\,\mathrm{d}s + n\int_{Kh}^{t} F(s) - F(Kh)\,\mathrm{d}s \\
&= n\sum_{l=0}^{K-1}\int_{lh}^{(l+1)h}\int_{lh}^{s} F'(u)\,\mathrm{d}u\,\mathrm{d}s + n\int_{Kh}^{t}\int_{Kh}^{s} F'(u)\,\mathrm{d}u\,\mathrm{d}s \\
&= n\sum_{l=0}^{K-1}\int_{lh}^{(l+1)h} F'(u)((l+1)h - u)\,\mathrm{d}u + n\int_{Kh}^{t} F'(u)(t-u)\,\mathrm{d}u \\
&= n\sum_{l=0}^{K-1} F'(\tilde{\eta}_l)\int_{lh}^{(l+1)h}((l+1)h - u)\,\mathrm{d}u + nF'(\tilde{\eta_K})\int_{Kh}^{t}(t-u)\,\mathrm{d}u \\
&= \frac{\tau}{2}\sum_{l=0}^{K-1} F'(\tilde{\eta}_l)h + nF'(\tilde{\eta_K})\frac{(t-Kh)^2}{2} \longrightarrow \frac{\tau}{2}\int_0^t F'(s)\,\mathrm{d}s \\
&= \frac{\tau}{2}(F(t) - F(0)), \quad \text{as } n\to\infty,
\end{aligned}$$

where $\tilde{\eta}_l \in [lh, (l+1)h]$ for $l = 0, 1, \ldots, K-1$ and $\tilde{\eta}_K \in [Kh, t)$. Then taking $F(s) = \xi^1_{s-\tau}$ yields

$$Z_t^n \longrightarrow \frac{\tau\big(\xi^1_{t\wedge\tau-\tau} - \xi^1_{-\tau}\big)}{2}, \quad \text{as } n\to\infty.$$

Since $\bar{M}^{n,1}$, $\bar{N}^{n,1}$, Z_t^n are functions of bounded variation, and $\bar{M}^{n,2}$, $\bar{M}^{n,3}$, $\bar{N}^{n,2}$, and W are square-integrable martingales, hence (b_{13}) holds. This proof is complete. □

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