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New aspects of Bargmann transform using Touchard polynomials and hypergeometric functions

Daniel Alpay, Antonino De Martino, and Kamal Diki

Abstract. In this paper, we study the ranges of the Schwartz space $\mathcal{S}$ and its dual $\mathcal{S}'$ (space of tempered distributions) under the Bargmann transform. The characterization of these two ranges leads to interesting reproducing kernel Hilbert spaces whose reproducing kernels can be expressed, respectively, in terms of the Touchard polynomials and the hypergeometric functions. We investigate the main properties of some associated operators and introduce two generalized Bargmann transforms in this framework. This can be considered as a continuation of an interesting research path that Neretin started earlier in his book on Gaussian integral operators.

1 Introduction and preliminary results

The Bargmann transform introduced in [8] is a fundamental mathematical model of quantum mechanics that maps the classical Hilbert space of square integrable functions on $\mathbb{R}^n$ onto the so-called Bargmann-Fock space. This space consists of entire functions on $\mathbb{C}^n$ that are square integrable with respect to the Gaussian measure. Some authors refer to this as the *boson* Fock space with n-degrees of freedom; see [21]. The book [31] presents a comprehensive introduction to the theory of Fock spaces and related topics in mathematical analysis. For applications in the study of coherent states and other topics in quantum mechanics, we refer to [2, 3, 18]. It is worth mentioning that the Bargmann transform was extended in [1] to the case of polyanalytic functions using time frequency analysis tools such as the short-time Fourier transform (STFT). A different approach using the true polyanalytic Bargmann and polyanalytic Fock spaces was introduced by Vasilevski in [30]. See also [5] for the case of polyanalytic functions of infinite order and [16] for the case of meta-analytic extensions. Recently, new connections with the theory of superoscillations are investigated in [6, 7]. For recent developments in quaternionic and Clifford analysis, one may consult [9–15, 22]. Before presenting our main results, we start by recalling some basic notions and properties of Fock spaces and Bargmann transform. For more details on this topic, we refer the reader to [31].

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Definition 1.1 An entire function $f : \mathbb{C} \to \mathbb{C}$ belongs to the Fock space $\mathcal{F}(\mathbb{C})$ if

$$\|f\|^2_{\mathcal{F}(\mathbb{C})} = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} d\lambda(z) < \infty,$$

where $d\lambda(z) = dxdy$ is the classical Lebesgue measure for $z = x + iy$.

The Fock space is a reproducing kernel Hilbert space, with reproducing kernel given by

$$K(z, w) = e^{z\bar{w}}, \qquad \forall z, w \in \mathbb{C}.$$

The Fock space has the following sequential characterization:

$$\mathcal{F}(\mathbb{C}) := \left\{ f(z) = \sum_{n=0}^{\infty} z^n a_n, \quad (a_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}, \quad \sum_{n=0}^{\infty} n!|a_n|^2 < \infty \right\}. \tag{1.1}$$

Another fundamental tool that we use in this paper is the Bargmann transform introduced in [8]. In the literature, there are several versions of this integral transform. In this paper, we use a modified version of the transform, which can be found in the book [31]. Given a function $\varphi \in L^2(\mathbb{R})$ the Bargmann transform is defined for any $z \in \mathbb{C}$ as

$$\mathcal{B}(\varphi)(z) = \int_{\mathbb{R}} A(z, x)\varphi(x)dx, \qquad A(z, x) = \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \exp(-\frac{x^2}{4} - \frac{z^2}{2} + zx). \tag{1.2}$$

The kernel of the Bargmann transform can be written in terms of generating function of the normalized Hermite functions. Precisely, the Hermite functions are defined by

$$h_n(x) = e^{-\frac{x^2}{4}} H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right), \tag{1.3}$$

where $H_n(x)$ are the probabilistic Hermite polynomials. Hence, the normalized Hermite functions are defined as

$$\xi_n(x) = \frac{h_n(x)}{\|h_n\|_{L^2}} = \frac{(-1)^n}{(2\pi)^{\frac{1}{4}}\sqrt{n!}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right).$$

Therefore, we can write the kernel of the Bargmann transform as

$$A(z, x) = \frac{1}{(2\pi)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{z^n h_n(x)}{n!}. \tag{1.4}$$

It is well-known that $\mathcal{B}$ is an unitary transform mapping $L^2(\mathbb{R})$ onto the Fock space $\mathcal{F}(\mathbb{C})$. The inverse of the Bargmann transform is denoted by $\mathcal{B}^{-1}$. Moreover, the Bargmann transform applied to the normalized Hermite functions gives back complex monomials:

$$\mathcal{B}(\xi_n)(z) = \frac{z^n}{\sqrt{n!}}, \qquad n \in \mathbb{N}_0. \tag{1.5}$$

The main purpose of this paper is to continue exploring this fascinating research topic and investigate new aspects of this transform involving relations with the Schwartz space and its dual space of tempered distributions. Indeed, inspired by the discussion in [21, Propositions 2.5 and 2.6], we aim to give a complete characterization

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of the range of the Schwartz space $\mathcal{S}$ and its dual $\mathcal{S}'$ under the Bargmann transform. We will develop a detailed study of these spaces. In literature, the Bargmann transform is studied on the Hilbert space L^2 that contains the Schwartz space $\mathcal{S}$ and which is contained in the space of tempered distributions $\mathcal{S}'$. Namely, we have

$$\mathcal{S} \subset L^2 \subset \mathcal{S}'. \tag{1.6}$$

It is well-known that the range of the space L^2 under the Bargmann transform is the Fock space. It is natural to wonder if there exists a characterization of the ranges of the Schwartz space $\mathcal{S}$ and the space of tempered distributions $\mathcal{S}'$ under the Bargmann transform. In this paper, we tackle this problem and we find out that the action of the Bargmann transform on the spaces $\mathcal{S}$ and $\mathcal{S}'$ lead to interesting reproducing kernel Hilbert spaces. Precisely, in this paper we introduce and study the following spaces:

$$\mathcal{H}_p(\mathbb{C}) := \left\{ f(z) = \sum_{n=0}^{\infty} z^n a_n, \quad (a_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}, \quad \sum_{n=0}^{\infty} \frac{|a_n|^2 n!}{(n+1)^{2p}} < \infty \right\},$$

$$\mathcal{F}_p(\mathbb{C}) := \left\{ f(z) = \sum_{n=0}^{\infty} z^n a_n, \quad (a_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}, \quad \sum_{n=0}^{\infty} |a_n|^2 n!(n+1)^{2p} < \infty \right\},$$

for $p > 0$. The space $\mathcal{H}_p(\mathbb{C})$ arises from the formal application of the Bargmann transform to the Schwartz space, while the space $\mathcal{F}_p(\mathbb{C})$ results from the formal application of the Bargmann transform to the space of tempered distributions.

The reproducing kernels of the spaces $\mathcal{H}_p(\mathbb{C})$ and $\mathcal{F}_p(\mathbb{C})$ can be expressed in terms of the Touchard polynomials and the hypergeometric functions, respectively. The Touchard polynomials were introduced first in [28] as polynomials having Stirling numbers of the second kind as coefficients. They share same interesting properties with other well-known families of orthogonal polynomials such as Rodrigues formula, generating functions, and orthogonality conditions. Associated with the spaces $\mathcal{H}_p(\mathbb{C})$ and $\mathcal{F}_p(\mathbb{C})$, we introduce and study two generalized Bargmann transforms that are constructed by taking the generating functions associated with some orthonormal basis against the Hermite functions. It is also possible to think about the Bargmann construction here by acting on the classical Bargmann transform with some specific operators combining powers of the momentum and annihilation operators. The technique that we use to achieve these objectives consists of computing higher derivatives of the classical Bargmann kernel and taking their linear combinations with suitable coefficients involving the Stirling numbers of the second kind. In particular, while developing the results of this paper we computed explicitly in Proposition 2.37 the higher derivatives of the classical Bargmann kernel $A(z, x)$, which is given by

$$\frac{d^k}{dz^k} A(z, x) = H_k(x - z)A(z, x),$$

where $k \geq 0$ and $H_k(z)$ denotes the holomorphic extension of the Hermite polynomials, see [26, 29].

Plan of the paper: in Section 2, we construct the new reproducing kernel Hilbert space $\mathcal{H}_p$ that are built upon the characterization of the range of the space of tempered distributions under the Bargmann transform. In particular, we obtain a new repro-

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ducing kernel function whose expression involves the Touchard polynomials. We also study some operators associated with the $\mathcal{H}_p$ space and investigate a generalized Bargmann transform based on this space. In Section 3, we study the reproducing kernel Hilbert space $\mathcal{F}_p$ that characterizes the range of the Schwartz space under the Bargmann transform. Finally, in Section 4, we present a connection between the reproducing kernel Hilbert space $\mathcal{H}_p$ and $\mathcal{F}_p$, through the Fock space.

2 The space $\mathcal{H}_p$ and associated operators

2.1 Construction and basic properties of the $\mathcal{H}_p$-space

It is well-known that the Schwartz space can be viewed as the intersection of weighted ℓ^2 spaces (see [21, Proposition 2.5]), while the union of weighted ℓ^2 spaces corresponds to the space of tempered distributions (see [21, Proposition 2.6]), see also [27]. Precisely we have

$$\bigcap_{p>0} \ell^2(\mathbb{N}, (n+1)^{2p}) = \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}) = \bigcup_{p>0} \ell^2(\mathbb{N}, (n+1)^{-2p}).$$

Lemma 2.1 *For a fixed $p > 0$, we have*

$$\mathcal{B}\left(\ell^2(\mathbb{N}, (n+1)^{-2p})\right) = \mathcal{H}_p(\mathbb{C}),$$

where

$$\mathcal{H}_p(\mathbb{C}) := \left\{ f(z) = \sum_{n=0}^{\infty} z^n a_n, \quad (a_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}, \quad \sum_{n=0}^{\infty} \frac{|a_n|^2 n!}{(n+1)^{2p}} < \infty \right\}. \tag{2.1}$$

Proof To show the result, we have to prove that

$$\mathcal{B}(S_p) = \mathcal{H}_p(\mathbb{C}),$$

where

$$S_p := \left\{ \sum_{n=1}^{\infty} \xi_n a_n, \quad (a_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}, \quad \sum_{n=1}^{\infty} (n+1)^{-2p} |a_n|^2 < \infty \right\}.$$

A function g belongs to $\mathcal{B}(S_p)$ if and only if there exists $f \in S_p$ such that $g(z) = \mathcal{B}(f)(z)$. From the action of the Bargmann transform applied to the nominalized Hermite functions, see (1.5), we get

$$\mathcal{B}(f)(z) = \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n!}} a_n.$$

We set $b_n := \frac{a_n}{\sqrt{n!}}$. Then we have

$$\sum_{n=1}^{\infty} |b_n|^2 \frac{n!}{(n+1)^{2p}} = \sum_{n=1}^{\infty} |a_n|^2 (n+1)^{-2p} < \infty.$$

Therefore, we get $\mathcal{B}(f)(z) \in \mathcal{H}_p(\mathbb{C})$ ∎

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Thanks to (1.5) the Bargmann transform can be defined on the whole of $\mathcal{S}'(\mathbb{R})$. This extension, still denoted by the letter $\mathcal{B}$, is defined by

$$\mathcal{B}((a_n)_{n\in\mathbb{N}_0}) = \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n!}} a_n,$$

or, formally, by

$$\mathcal{B}\left(\sum_{n=1}^{\infty} \xi_n a_n\right) = \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n!}} a_n.$$

We have the following result, characterizing the image of $\mathcal{S}'$ under $\mathcal{B}$.

Theorem 2.2 *$\mathcal{B}$ is bicontinuous from $\mathcal{S}'$ onto $\mathcal{B}(\mathcal{S}') = \bigcup_{p\geq 1} \mathcal{H}_p(\mathbb{C})$.*

Proof We denote by i_p the injection map from $\ell^2(\mathbb{N}, (n+1)^{-2p})$ into

$$\mathcal{S}' = \cup_{q=0}^{\infty} \ell^2(\mathbb{N}, (n+1)^{-2q}).$$

The inductive topology on the latter is by definition the finest topology for which all the injections i_p are continuous, see e.g. [17, Definition 1, p. 136]. A similar remark holds for the image of $\mathcal{S}'$ under $\mathcal{B}$.

By the characterization of continuous maps with domain a space, with an inductive topology and domain a locally convex space (see [25, Section 6.1, p. 54]), we have to show that for every p the map $\mathcal{B} \circ i_p$ is continuous from $\ell^2(\mathbb{N}, (n+1)^{-2p})$ into $\mathcal{B}(\mathcal{S}')$. But the restriction $\mathbf{B}_p$ of the Bargmann map $\mathcal{B}$ to $\ell^2(\mathbb{N}, (n+1)^{-2p})$ has range $\mathcal{B}(\ell^2(\mathbb{N}, (n+1)^{-2p}) = \mathcal{H}_p(\mathbb{C})$, see Lemma 2.1, and is continuous since it is isometric by construction. Furthermore, the inclusion map I_p from $\mathcal{B}(\ell^2(\mathbb{N}, (n+1)^{-2p}))$ into $\mathcal{B}(\mathcal{S}')$ is continuous by the definition of the inductive topology on $\mathcal{B}(\mathcal{S}')$. But for $a \in (\ell^2(\mathbb{N}, (n+1)^{-2p}))$ we have

$$i_p(a) = a \quad \text{and} \quad \mathbf{B}_p(a) = \mathcal{B}(a).$$

Hence,

$$\mathcal{B} \circ i_p(a) = \mathcal{B}(a) \quad \text{and} \quad I_p \circ \mathbf{B}_p(a) = \mathbf{B}_p(a) = \mathcal{B}(a).$$

Thus

$$\mathcal{B} \circ i_p = I_p \circ \mathbf{B}_p$$

so that $\mathcal{B} \circ i_p$ is continuous since $I_p \circ \mathbf{B}_p$ is continuous. Thus, $\mathcal{B}$ is continuous by definition of the inductive topology.

The proof for $\mathcal{B}^{-1}$ is similar. ∎

Now, we aim to discuss the main properties of the space $\mathcal{H}_p(\mathbb{C})$. This space takes an importance because it constitutes the building block of the range of the space $\mathcal{S}'$ under the Bargmann transform. First, we give a rigorous definition of this space.

Definition 2.3 Let $p \in \mathbb{N}$. We define the Hilbert space $\mathcal{H}_p(\mathbb{C})$ as the set of all entire functions endowed with the inner product

$$\langle f, g\rangle_{\mathcal{H}_p(\mathbb{C})} = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^{2p}} a_n \overline{b_n}, \tag{2.2}$$

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where $f(z) = \sum_{n=0}^{\infty} z^n a_n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, with $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$ such that $\sum_{n=0}^{\infty} \frac{|a_n|^2 n!}{(n+1)^{2p}} < \infty, \quad \sum_{n=0}^{\infty} \frac{|b_n|^2 n!}{(n+1)^{2p}} < \infty$.

The inner product defined in (2.2) induces the following norm:

$$\|f\|^2_{\mathcal{H}_p(\mathbb{C})} = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^{2p}} |a_n|^2, \qquad f \in \mathcal{H}_p(\mathbb{C}).$$

Remark 2.4 For $p = 0$, the space $\mathcal{H}_p(\mathbb{C})$ coincides with the classical Fock space, see (1.1).

As a consequence of (2.2), the inner product of two different monomials is given by the following formula:

$$\langle z^n, z^m \rangle_{\mathcal{H}_p(\mathbb{C})} := \frac{n!}{(n+1)^{2p}} \delta_{n,m}, \qquad n, m \geq 0. \tag{2.3}$$

Now, we consider the following function:

$$K_p(z, w) = \sum_{n=0}^{\infty} \frac{z^n \bar{w}^n (n+1)^{2p}}{n!}. \tag{2.4}$$

Proposition 2.5 *Let $w \in \mathbb{C}$ and $p \in \mathbb{N}$.*

i) *The function $K_p(z, w)$ belongs to $\mathcal{H}_p(\mathbb{C})$.*
ii) *The evaluation map $e_w(f) := f(w)$ is a continuous linear functional on $\mathcal{H}_p(\mathbb{C})$. Furthermore, we have*

$$|e_w(f)| \leq \|f\|_{\mathcal{H}_p(\mathbb{C})} \|K_p\|_{\mathcal{H}_p(\mathbb{C})}, \qquad \forall f \in \mathcal{H}_p(\mathbb{C}).$$

Proof

i) We set $\alpha(w) := \frac{\bar{w}^n (n+1)^{2p}}{n!}$ and thus we can write

$$K_p(z, w) = \sum_{n=0}^{\infty} z^n \alpha(w).$$

Hence, we have

$$\|K_p\|_{\mathcal{H}_p(\mathbb{C})} = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^{2p}} |\alpha(w)|^2 = \sum_{n=0}^{\infty} \frac{(n+1)^{2p}}{n!} |w|^n < \infty. \tag{2.5}$$

ii) Let $(a_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$. By the Schwartz inequality and (2.5) we have

$$\begin{aligned}
|e_w(f)| &\leq \sum_{n=0}^{\infty} |w|^n |a_n| \\
&= \sum_{n=0}^{\infty} \frac{\sqrt{n!}}{(n+1)^p} \frac{(n+1)^p}{\sqrt{n!}} |w|^n |a_n| \\
&\leq \left(\sum_{n=0}^{\infty} \frac{(n+1)^{2p}}{n!} |a_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{n!}{(n+1)^{2p}} |w|^{2n} \right)^{\frac{1}{2}} \\
&\leq \|f\|_{\mathcal{H}_p(\mathbb{C})} \|K_p\|_{\mathcal{H}_p(\mathbb{C})}.
\end{aligned}$$

■

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The previous result guarantees the existence of a reproducing kernel for the space $\mathcal{H}_p$.

Theorem 2.6 *Let $p \in \mathbb{N}$, the reproducing kernel of the space $\mathcal{H}_p(\mathbb{C})$ is given by $K_p(z,w)$.*

Proof Let $f(z) = \sum_{n=0}^{\infty} z^n a_n$, with $(a_n)_{n\geq 0} \subseteq \mathbb{C}$, being in the space $\mathcal{H}_p(\mathbb{C})$. We set $b_n := \frac{\bar{w}^n (n+1)^{2p}}{n!}$. By formula (2.2), we have

$$\langle f, K_p \rangle_{\mathcal{H}_p(\mathbb{C})} = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^{2p}} a_n \overline{b}_n = \sum_{n=0}^{\infty} a_n w^n = f(w).$$

This proves the reproducing kernel property. ∎

Now, we establish a relation between the reproducing kernel introduced in (2.4) and a special class of polynomials: the Touchard polynomials. These were first introduced in [28] and are defined as

$$T_n(x) := \sum_{k=0}^{n} S(n,k) x^k,$$

where $S(n,k)$ are the Stirling numbers of the second kind and can be calculated using the following explicit formula:

$$S(n,k) := \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n. \tag{2.6}$$

Examples of Touchard polynomials of degree up to five are given by

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= x,\\
T_2(x) &= x^2 + x,\\
T_3(x) &= x^3 + 3x^2 + x,\\
T_4(x) &= x^4 + 6x^3 + 7x^2 + x,\\
T_5(x) &= x^5 + 10x^4 + 25x^3 + 15x^2 + x.
\end{aligned}$$

The generating function for the polynomials $T_n(x)$ is given by

$$T_n(x) = e^{-x} \sum_{k=0}^{\infty} \frac{k^n x^k}{k!}. \tag{2.7}$$

Remark 2.7 It is clear that the Touchard polynomials can be easily extended in complex variables by extending x to $z \in \mathbb{C}$.

Proposition 2.8 *Let $p \in \mathbb{N}$, then the reproducing kernel of the space $\mathcal{H}_p(\mathbb{C})$ can be written as*

$$K_p(z\bar{w}) = \frac{e^{z\bar{w}}}{z\bar{w}} T_{2p+1}(z\bar{w}). \tag{2.8}$$

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Proof By formula (2.7) and Theorem 2.6, we have that

$$\begin{aligned}\frac{e^{z\bar{w}}}{z\bar{w}}T_{2p+1}(z\bar{w}) &= \frac{1}{z\bar{w}}\sum_{k=0}^{\infty}\frac{k^{2p+1}z^k\bar{w}^k}{k!}\\ &= \sum_{k=1}^{\infty}\frac{k^{2p}z^{k-1}\bar{w}^{k-1}}{(k-1)!}\\ &= \sum_{k=0}^{\infty}\frac{(k+1)^{2p}z^k\bar{w}^k}{k!}\\ &= K_p(z\bar{w}).\end{aligned}$$

■

Remark 2.9 The Touchard polynomials $T_{2p+1}(z\bar{w})$, for $p \geq 1$, can be always factorized in terms of $z\bar{w}$. This means that the right-hand side of formula (2.8) vanishes at the origin.

Remark 2.10 If we consider $p = 0$ in (2.8), since $T_1(z\bar{w}) = z\bar{w}$, we get back the reproducing kernel of the classical Fock space.

Now we tackle the problem to find the existence of a geometric description of the space $\mathcal{H}_p(\mathbb{C})$. This is related to study the one-dimensional (full)-moment problems: *Given a real sequence $s = (s_n)_{n\in\mathbb{N}_0}$ and a closed subset K of $\mathbb{R}$, the K-moment problems states: When does there exist a Radon measure μ on $\mathbb{R}$ supported on K such that $s_n = \int_{\mathbb{R}} x^n d\mu(x)$ for all $n \in \mathbb{N}_0$?*

For $K = [0, +\infty)$, the solution of the previous problem is given by the so-called Stieltjes theorem. In order to state this result, see [24, Theorem 3.12], we need to fix some notations. We denote by Es the shifted sequence given by

$$(Es)_n = s_{n+1}, \qquad n \in \mathbb{N}_0.$$

Furthermore, we denote the Hankel matrix $\mathsf{H}_n(x)$ by

$$\mathsf{H}_n(x) = \begin{pmatrix} s_0 & s_1 & s_2 & \cdots & s_n \\ s_1 & s_2 & s_3 & \cdots & s_{n+1} \\ s_2 & s_3 & s_4 & \cdots & s_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \\ s_n & s_{n+1} & s_{n+2} & \cdots & s_{2n} \end{pmatrix}.$$

Theorem 2.11 (Solution of the Stieltjes moment problem) *A real sequence $s = (s_n)_{n\in\mathbb{N}_0}$ is a Stieltjes sequence, that is, there exists a Radon measure μ on $[0, \infty)$ such that*

$$s_n = \int_0^{\infty} x^n d\mu(x), \quad \text{for} \quad n \in \mathbb{N}_0, \tag{2.9}$$

if and only if all the Hankel matrices $\mathsf{H}_n(s)$ and $\mathsf{H}_n(Es)$, $n \in \mathbb{N}_0$, are positive semidefinite.

In [24, Corollary 4.10], the author proved a condition that the Stieltjes moment problem has to satisfy in order to have a unique solution.

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Proposition 2.12 *Let $s = (s_n)_{n\in\mathbb{N}_0}$ being a Stieltjes sequence. Then if there exists a constant $M > 0$ such that*

$$s_n \le M^n(2n)! \qquad \text{for} \quad n \in \mathbb{N},$$

then the sequence s has only one representing measure on $[0, \infty)$.

To show the existence of a geometric description for the space $\mathcal{H}_p(\mathbb{C})$, we need also the following generic result of changing variables for an integral, see [23, Proposition 4.10].

Theorem 2.13 *If M is a continuous, nondecreasing function on the interval $[a, b]$, then for g being a nonnegative Borel function on $[M(a), M(b)]$ we have*

$$\int_{M(b)}^{M(a)} g(\rho)dN(\rho) = \int_a^b g(M(\rho))dN(M(\rho)).$$

Another fundamental tool that we use is the Hadamard (also called Schur) product between matrices. Given two matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times n}$, the Hadamard product is denoted by $A \odot B$ and its elements are given by

$$(A \odot B)_{ij} := (A)_{ij}(B)_{ij}.$$

The peculiarity of this product of matrices is that the product of two positive-semidefinite matrices is still positive-semidefinite. This result is known as Schur product theorem. Now, we have all the tools to prove the existence of a geometric description for the space $\mathcal{H}_p(\mathbb{C})$.

Theorem 2.14 *There exists a unique radial measure $d\sigma(x, y)$ such that*

$$\frac{1}{2\pi}\int_{\mathbb{C}} z^n \bar{z}^m d\sigma(x, y) = \frac{n!}{(n+1)^{2p}}\delta_{n,m}, \qquad z = x + iy.$$

Proof We start by passing in polar coordinates in the integral $\int_{\mathbb{C}} z^n \bar{z}^m d\sigma(x, y)$. Let $z = \rho e^{i\theta}$, where $\rho \in [0, \infty)$ and $\theta \in [0, 2\pi]$, then we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{C}} z^n \bar{z}^m d\sigma(x, y) &= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \rho^{n+m} e^{i(n-m)\theta} d\theta d\mu(\rho) \\ &= \delta_{n,m}\int_0^\infty \rho^{2n}\mu(\rho).\end{aligned}$$

This means that the problem is reduced to find a measure $\mu(\rho)$ such that

$$\int_0^\infty \rho^{2n} d\mu(\rho) = \frac{n!}{(n+1)^{2p}}. \tag{2.10}$$

Now, we use Theorem 2.13 with $M(\rho) = \rho^2$, $g(\rho) = \rho^n$, $dN = dv$, $a = 0$ and $b = k \in \mathbb{R}$, we get

$$\int_0^k \rho^{2n} dv(\rho^2) = \int_0^{k^2} \rho^n dv(\rho).$$

By sending k to infinity, we get

$$\int_0^\infty \rho^{2n} dv(\rho^2) = \int_0^\infty \rho^n dv(\rho).$$

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If we set $d\mu(\rho) := dv(\rho^2)$, we get

$$\int_0^\infty \rho^{2n} d\mu(\rho) = \int_0^\infty \rho^n dv(\rho).$$

By the expression (2.10), we have to find a measure $dv(\rho)$ such that

$$\int_0^\infty \rho^n dv(\rho) = \frac{n!}{(n+1)^{2p}}. \tag{2.11}$$

Now, the problem is reduced to the so-called Stieltjes-moment problem. We set $s_n := \frac{n!}{(n+1)^{2p}}$. In order to show that $(s_n)_{n\in\mathbb{N}_0}$ is a Stieltjes sequence, we have to use Theorem 2.11. Hence, we have to show that

$$T_N := \left(\frac{(n+m)!}{(n+m+1)^{2p}}\right)_{n,m=0}^N \quad \text{and} \quad S_N := \left(\frac{(n+m-1)!}{(n+m)^{2p}}\right)_{m,n=1}^N,$$

are positive semidefinite matrices. We observe that we can write the matrix T_N as the Hadamard product of two matrices

$$((n+m)!)_{n,m=0}^N \odot \left(\frac{1}{(m+m+1)^{2p}}\right)_{n,m=0}^N.$$

It is clear that the matrix

$$\left(\frac{1}{(m+m+1)^{2p}}\right)_{n,m=0}^N,$$

is positive semidefinite. Now, we observe that

$$((n+m)!)_{m,n=0}^N = \mathrm{Diag}(m!)\left(\frac{(n+m)!}{n!m!}\right)_{n,m=0}^N \mathrm{Diag}(n!),$$

where we denote by $\mathrm{Diag}(m!)$ and $\mathrm{Diag}(n!)$ the matrices that have entries $m!$ and $n!$ on the diagonal, respectively. By the Chu-Vandermonde formula

$$\binom{u+k}{k} = \sum_{\ell=0}^k \binom{k}{\ell}\binom{u}{\ell},$$

we get that

$$\left(\frac{(n+m)!}{n!m!}\right)_{n,m=0}^N,$$

is positive semidefinite. This implies that the matrix T_N is positive-semidefinite. To show that also the matrix S_N is positive definite, we observe that we can write

$$S_N = ((n+m-1)!)_{m,n=1}^N \odot \left(\frac{1}{(n+m)^{2p}}\right)_{n,m=1}^N.$$

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It is clear that $\left(\frac{1}{(n+m)^{2p}}\right)_{n,m=1}^{N}$ is positive-semidefinite. Now, we show that $((n+m-1)!)_{m,n=1}^{N}$ is also positive semidefinite. This follows from the fact that

$$(n+m-1)! = \Gamma(n+m-2) = \int_0^\infty t^{n+m-2}e^{-t}dt.$$

Therefore we get that also the matrix S_N is positive-semidefinite. Hence by Theorem 2.11 there exists a measure $dv(\rho)$ that admits a representation like the one in (2.11). The unicity of the solution follows by Corollary 2.12. Indeed by using the Stirling's approximation, it is not difficult to show that

$$s_n \leq M^n(2n!), \qquad \forall n \in \mathbb{N}_0.$$

This concludes the proof. ∎

2.2 Some operators on $\mathcal{H}_p$-space

The classical Fock space is the unique Hilbert space of power series defined in a neighborhood of the origin for which the annihilation operator ∂ and the momentum operator M_z, namely

$$M_z f(z) := zf(z), \qquad \partial f(z) := \frac{d}{dz}f(z).$$

are closely defined on the span of polynomials and are adjoints to each other, i.e.

$$\partial^* = M_z.$$

Moreover, the annihilation and the momentum operators satisfy the classical commutation rule

$$[M_z, \partial] = M_z\partial - \partial M_z = -\mathcal{I}, \tag{2.12}$$

where $\mathcal{I}$ is the identity operator. In the context of analytic functions in a neighborhood of the origin the backward-shift operator R_0 is defined as

$$R_0 = \begin{cases} \frac{f(z)-f(0)}{z}, & z \neq 0 \\ f'(0), & z = 0. \end{cases}$$

Another operator that plays a role in this context is the integration operator I, introduced in [4] and defined by

$$(If)(z) = \int_{[0,z]} f(s)ds.$$

In [4, Lemma 2.1], the authors proved that the integration operator is the adjoint of the backward shift operator, i.e.

$$R_0^* = I.$$

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The commutator of R_0 and I is equal on the monomials to

$$[R_0, I](z^n) = \begin{cases} 1, & n = 0 \\ -\frac{z^n}{n(n+1)}, & n = 1, 2, \dots \end{cases} \tag{2.13}$$

We denote by D_0 the (formal) diagonal operator defined by

$$D_0 := \operatorname{diag}\left(1, -\frac{1}{2}, -\frac{1}{6}, \dots, -\frac{1}{n(n+1)}, \dots\right). \tag{2.14}$$

Hence, we can write

$$[R_0, I] = D_0. \tag{2.15}$$

The goal of this section is to study the adjoint of the operators R_0, ∂, M_z and I for the space $\mathcal{H}_p(\mathbb{C})$.

Proposition 2.15 *Let $p \in \mathbb{N}$. The adjoint operator of the backward-shift operator in $\mathcal{H}_p(\mathbb{C})$ is given by*

$$R_0^* z^n = \frac{(n+2)^{2p}}{(n+1)^{2p+1}} z^{n+1}. \tag{2.16}$$

Proof We have to show that $\langle R_0 z^n, z^m \rangle_{\mathcal{H}_p(\mathbb{C})} = \langle z^n, R_0^* z^m \rangle_{\mathcal{H}_p(\mathbb{C})}$ for $m, n \in \mathbb{N}_0$. If we suppose $n \geq 1$, by formula (2.3), we have

$$\langle R_0 z^n, z^m \rangle_{\mathcal{H}_p(\mathbb{C})} = \langle z^{n-1}, z^m \rangle_{\mathcal{H}_p(\mathbb{C})} = \frac{(n-1)!}{n^{2p}} \delta_{n-1,m},$$

and

$$\begin{aligned} \langle z^n, R_0^* z^m \rangle_{\mathcal{H}_p(\mathbb{C})} &= \frac{(m+2)^{2p}}{(m+1)^{2p+1}} \langle z^n, z^{m+1} \rangle_{\mathcal{H}_p(\mathbb{C})} \\ &= \frac{(m+2)^{2p}}{(m+1)^{2p+1}} \frac{n!}{(n+1)^{2p}} \delta_{n,m+1} = \frac{(n-1)!}{n^{2p}} \delta_{n,m+1}. \end{aligned}$$

This proves the result for $n \geq 1$. For $n = 0$ the result is trivial, indeed we have

$$\langle R_0 1, z^m \rangle_{\mathcal{H}_p(\mathbb{C})} = 0 \qquad \text{and} \qquad \langle 1, R_0^* z^m \rangle_{\mathcal{H}_p(\mathbb{C})} = \frac{(m+2)^{2p}}{(m+1)^{2p}} \langle 1, z^{m+1} \rangle_{\mathcal{H}_p(\mathbb{C})} = 0.$$

■

Proposition 2.16 *Let $p \in \mathbb{N}$. The adjoint operator of the annihilation operator in $\mathcal{H}_p(\mathbb{C})$ is given by*

$$\partial^* z^n = \frac{(n+2)^{2p}}{(n+1)^{2p}} z^{n+1}. \tag{2.17}$$

Proof For $m, n \in \mathbb{N}_0$ we have to show that $\langle \partial z^n, z^m \rangle_{\mathcal{H}_p(\mathbb{C})} = \langle z^n, \partial^* z^m \rangle_{\mathcal{H}_p(\mathbb{C})}$. For $n \geq 1$ and by formula (2.3) we get

$$\langle \partial z^n, z^m \rangle_{\mathcal{H}_p(\mathbb{C})} = n \langle z^{n-1}, z^m \rangle_{\mathcal{H}_p(\mathbb{C})} = n \frac{(n-1)!}{n^{2p}} \delta_{n-1,m} = \frac{n!}{n^{2p}} \delta_{n-1,m}.$$

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By formula (2.17), we have

$$\begin{aligned}\langle z^n, \partial^* z^m\rangle_{\mathcal{H}_p(\mathbb{C})} &= \frac{(m+2)^{2p}}{(m+1)^{2p}}\langle z^n, z^{m+1}\rangle_{\mathcal{H}_p(\mathbb{C})}\\ &= \frac{(m+2)^{2p}}{(m+1)^{2p}}\frac{n!}{(n+1)^{2p}}\delta_{n,m+1} = \frac{n!}{n^{2p}}\delta_{n,m+1}.\end{aligned}$$

Hence, we have the result. For $n = 0$, the equality is trivial. Indeed, we have

$$\langle \partial 1, z^m\rangle_{\mathcal{H}_p(\mathbb{C})} = 0 \qquad \text{and} \qquad \langle 1, \partial^* z^m\rangle_{\mathcal{H}_p(\mathbb{C})} = \frac{(m+2)^{2p}}{(m+1)^{2p}}\langle 1, z^{m+1}\rangle_{\mathcal{H}_p(\mathbb{C})} = 0.$$

■

Proposition 2.17 *Let $p \in \mathbb{N}$. Then adjoint momentum operator in $\mathcal{H}_p(\mathbb{C})$ is given by*

$$M_z^* z^n = \frac{n^{2p+1}}{(n+1)^{2p}} z^{n-1}, \qquad M_z^*(1) = 0$$

Proof For $m, n \in \mathbb{N}_0$ we have to show $\langle M_z z^n, z^m\rangle_{\mathcal{H}_p(\mathbb{C})} = \langle z^n, M_z^* z^m\rangle_{\mathcal{H}_p(\mathbb{C})}$. For $m \geq 1$ and by formula (2.3) we have

$$\langle M_z z^n, z^m\rangle_{\mathcal{H}_p(\mathbb{C})} = \langle z^{n+1}, z^m\rangle_{\mathcal{H}_p(\mathbb{C})} = \frac{(n+1)!}{(n+2)^{2p}}\delta_{n+1,m},$$

while

$$\begin{aligned}\langle z^n, M_z^* z^m\rangle_{\mathcal{H}_p(\mathbb{C})} &= \frac{m^{2p+1}}{(m+1)^{2p}}\langle z^n, z^{m-1}\rangle_{\mathcal{H}_p(\mathbb{C})}\\ &= \frac{m^{2p+1}}{(m+1)^{2p}}\frac{n!}{(n+1)^{2p}}\delta_{n,m-1} = \frac{(n+1)!}{(n+2)^{2p}}\delta_{n,m-1}.\end{aligned}$$

This proves the result for $m \geq 1$. For $m = 0$, the result is trivial, indeed we have

$$\langle M_z z^n, 1\rangle_{\mathcal{H}_p(\mathbb{C})} = \langle z^{n+1}, 1\rangle_{\mathcal{H}_p(\mathbb{C})} = 0 \qquad \text{and} \qquad \langle z^n, M_z^* 1\rangle_{\mathcal{H}_p(\mathbb{C})} = 0.$$

■

Proposition 2.18 *Let $p \in \mathbb{N}$. The adjoint operator of the integration operator in $\mathcal{H}_p(\mathbb{C})$ is given by*

$$I^* z^n = \frac{n^{2p}}{(n+1)^{2p}} z^{n-1}, \qquad I^*(1) = 0. \tag{2.18}$$

Proof We have to prove that $\langle Iz^n, z^m\rangle_{\mathcal{H}_p(\mathbb{C})} = \langle z^n, I^* z^m\rangle_{\mathcal{H}_p(\mathbb{C})}$, for $m, n \in \mathbb{N}_0$. We show first the result for $m \geq 1$, by (2.3) we have

$$\langle Iz^n, z^m\rangle_{\mathcal{H}_p(\mathbb{C})} = \frac{1}{n+1}\langle z^{n+1}, z^m\rangle_{\mathcal{H}_p(\mathbb{C})} = \frac{1}{n+1}\frac{(n+1)!}{(n+2)^{2p}}\delta_{n+1,m} = \frac{n!}{(n+2)^{2p}}\delta_{n+1,m},$$

whereas

$$\langle z^n, I^* z^m\rangle_{\mathcal{H}_p(\mathbb{C})} = \frac{m^{2p}}{(m+1)^{2p}}\langle z^n, z^{m-1}\rangle_{\mathcal{H}_p(\mathbb{C})} = \frac{n!}{(n+2)^{2p}}\delta_{n,m-1}.$$

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This shows the result for $m \geq 1$. For $m = 0$ the result is trivial, indeed we have

$$\langle I1, z^m \rangle_{\mathcal{H}_p(\mathbb{C})} = 0 \qquad \text{and} \qquad \langle 1, I^* z^m \rangle_{\mathcal{H}_p(\mathbb{C})} = \frac{m^{2p}}{(m+1)^{2p}} \langle 1, z^{m-1} \rangle_{\mathcal{H}_p(\mathbb{C})} = 0.$$

■

We summarize the actions of all the adjoint operators previously studied in the following table:

Operator	$R_0 z^n$	∂z^n	$M_z z^n$	$I z^n$
Adjoint	$\frac{(n+2)^{2p}}{(n+1)^{2p+1}} z^{n+1}$	$\frac{(n+2)^{2p}}{(n+1)^{2p}} z^{n+1}$	$\frac{n^{2p+1}}{(n+1)^{2p}} z^{n-1}$	$\frac{n^{2p}}{(n+1)^{2p}} z^{n-1}$

In the following results, we show that the previous adjoint operators can be written in some specific forms. Precisely, let $\mathbf{A}^*$ be an adjoint operator in $\mathcal{H}_p(\mathbb{C})$. We are able to show that the operators R_0^*, ∂^*, M_z^*, and I^* can be written as the composition of the respective adjoint operators in the Fock space and a specific diagonal operator.

Theorem 2.19 *Let $p \in \mathbb{N}$. Then the adjoint operator of the backward-shift operator can be written as*

$$R_0^* = I(\mathcal{I} + R_0 I)^{2p},$$

where $\mathcal{I}$ is the identity operator.

Proof We show first the result for monomials z^n, with $n \in \mathbb{N}_0$. By formula (2.16) and the binomial theorem, we have

$$\begin{aligned} R_0^* z^n &= \frac{(n+2)^{2p}}{(n+1)^{2p+1}} z^{n+1} = \frac{1}{n+1} \left(1 + \frac{1}{n+1}\right)^{2p} z^{n+1} \\ &= \left(\sum_{k=0}^{2p} \binom{2p}{k} \frac{1}{(n+1)^{k+1}} \right) z^{n+1}. \end{aligned} \tag{2.19}$$

Now we observe that for $k \geq 0$, we have

$$(R_0 I)^k z^n = \frac{1}{(n+1)^k} z^n. \tag{2.20}$$

This implies that

$$I(R_0 I)^k z^n = \frac{1}{(n+1)^{k+1}} z^{n+1}. \tag{2.21}$$

By inserting (2.21) in (2.19) and applying another time the binomial theorem formula we get

$$R_0^* z^n = \left(\sum_{k=0}^{2p} \binom{2p}{k} I(R_0 I)^k \right) z^n = I \left(\sum_{k=0}^{2p} \binom{2p}{k} (R_0 I)^k \right) z^n = I(\mathcal{I} + R_0 I)^{2p} z^n.$$

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Now, let us consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $\mathcal{H}_p(\mathbb{C})$, with $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$. By (2.20) and the binomial theorem, we get

$$\begin{aligned}
\langle f, I(\mathcal{I} + R_0 I)^{2p} g\rangle_{\mathcal{H}_p(\mathbb{C})} &= \left\langle f, I(\mathcal{I} + R_0 I)^{2p} \sum_{n=0}^{\infty} b_n z^n \right\rangle_{\mathcal{H}_p(\mathbb{C})} \\
&= \left\langle f, \sum_{n=0}^{\infty} \frac{(n+2)^{2p}}{(n+1)^{2p+1}} z^{n+1} b_n \right\rangle_{\mathcal{H}_p(\mathbb{C})} \\
&= \left\langle f, \sum_{n=1}^{\infty} \frac{(n+1)^{2p}}{n^{2p+1}} z^n b_{n-1} \right\rangle_{\mathcal{H}_p(\mathbb{C})} \\
&= \sum_{n=1}^{\infty} a_n \overline{b_{n-1}} \frac{n!}{(n+1)^{2p}} \frac{(n+1)^{2p}}{n^{2p+1}} \\
&= \sum_{n=0}^{\infty} a_{n+1} \overline{b_n} \frac{n!}{(n+1)^{2p}} \\
&= \left\langle \sum_{n=0}^{\infty} a_{n+1} z^n, g \right\rangle_{\mathcal{H}_p(\mathbb{C})} \\
&= \langle R_0 f, g\rangle_{\mathcal{H}_p(\mathbb{C})}.
\end{aligned}$$

This shows that $R_0^* = I(\mathcal{I} + R_0 I)^{2p}$. ∎

Theorem 2.20 *Let $p \in \mathbb{N}$. Then we can write the adjoint operators of the annihilation operator in the following way:*

$$\partial^* = M_z(\mathcal{I} + R_0 I)^{2p},$$

where $\mathcal{I}$ is the identity operator.

Proof We show first the result for monomials z^n, with $n \in \mathbb{N}_0$. By formula (2.17) and the binomial theorem, we have

$$\partial^* z^n = \left(\frac{n+2}{n+1}\right)^{2p} z^{n+1} = \left(1 + \frac{1}{n+1}\right)^{2p} z^{n+1} = \left(\sum_{k=0}^{2p} \binom{2p}{k} \frac{1}{(n+1)^k}\right) z^{n+1}. \tag{2.22}$$

By (2.20) we have that

$$M_z(R_0 I)^k z^n = \frac{1}{(n+1)^k} z^{n+1}, \qquad k \geq 0. \tag{2.23}$$

By (2.22), (2.23) and the binomial theorem, we obtain

$$\partial^* z^n = M_z \left(\sum_{k=0}^{2p} \binom{2p}{k} (R_0 I)^k\right) z^n = M_z(\mathcal{I} + I R_0)^{2p}.$$

Let us consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $\mathcal{H}_p(\mathbb{C})$, with $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$. By (2.20) and the binomial theorem we get

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$$\begin{aligned}
\langle f, M_z(\mathfrak{I}+R_0I)^{2p}g\rangle_{\mathcal{H}_p(\mathbb{C})} &= \left\langle f, M_z(\mathfrak{I}+R_0I)^{2p}\sum_{n=0}^{\infty} b_n z^n\right\rangle_{\mathcal{H}_p(\mathbb{C})}\\
&= \left\langle f, \sum_{n=0}^{\infty}\left(\frac{n+2}{n+1}\right)^{2p} z^{n+1}b_n\right\rangle_{\mathcal{H}_p(\mathbb{C})}\\
&= \left\langle f, \sum_{n=1}^{\infty}\frac{(n+1)^{2p}}{n^{2p}} z^{n}b_{n-1}\right\rangle_{\mathcal{H}_p(\mathbb{C})}\\
&= \sum_{n=1}^{\infty} a_n\overline{b_{n-1}}\frac{n!}{n^{2p}}\\
&= \left\langle \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n, g\right\rangle_{\mathcal{H}_p(\mathbb{C})}\\
&= \langle \partial f, g\rangle_{\mathcal{H}_p(\mathbb{C})}.
\end{aligned}$$

This shows that $\partial^* = M_z(\mathfrak{I}+R_0I)^{2p}$. ∎

Theorem 2.21 *Let $p \in \mathbb{N}$, then we can write the adjoint operator of the momentum operator as*

$$M_z^* = \partial(\mathfrak{I}-R_0I)^{2p}, \tag{2.24}$$

where $\mathfrak{I}$ is the identity operator.

Proof We show first the result for monomials z^n, with $n \in \mathbb{N}$. By the binomial theorem and Proposition 2.17, we have

$$\begin{aligned}
M_z^* z^n &= \frac{n^{2p+1}}{(n+1)^{2p}}z^{n-1} = n\left(1-\frac{1}{n+1}\right)^{2p} z^{n-1}\\
&= \left(\sum_{k=0}^{2p}\binom{2p}{k}(-1)^k\frac{n}{(n+1)^k}\right)z^{n-1}.
\end{aligned} \tag{2.25}$$

By (2.20), we have that

$$\partial(R_0I)^k z^n = \frac{n}{(n+1)^k}z^{n-1}, \qquad k \geq 0. \tag{2.26}$$

By inserting (2.26) in (2.25), we obtain

$$M_z^* z^n = \partial\left(\sum_{k=0}^{2p}\binom{2p}{k}(-R_0I)^k\right)z^n = \partial(\mathfrak{I}-R_0I)^{2p}.$$

The result follows trivially for $n = 0$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be functions in $\mathcal{H}_p(\mathbb{C})$, with $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$. By (2.20) and the binomial theorem, we get

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$$\begin{aligned}\langle f, \partial(\mathcal{I} - R_0 I)^{2p} g\rangle_{\mathcal{H}_p(\mathbb{C})} &= \left\langle f, \partial(\mathcal{I} - R_0 I)^{2p} \sum_{n=0}^{\infty} b_n z^n \right\rangle_{\mathcal{H}_p(\mathbb{C})} \\ &= \left\langle f, \sum_{n=1}^{\infty} \frac{n^{2p+1}}{(n+1)^{2p}} z^{n-1} b_n \right\rangle_{\mathcal{H}_p(\mathbb{C})} \\ &= \left\langle f, \sum_{n=0}^{\infty} \frac{(n+1)^{2p+1}}{(n+2)^{2p}} z^n b_{n+1} \right\rangle_{\mathcal{H}_p(\mathbb{C})} \\ &= \sum_{n=0}^{\infty} a_n \overline{b_{n+1}} \frac{(n+1)!}{(n+2)^{2p}} \\ &= \left\langle \sum_{n=1}^{\infty} a_{n-1} z^n, g \right\rangle_{\mathcal{H}_p(\mathbb{C})} \\ &= \langle M_z f, g\rangle_{\mathcal{H}_p(\mathbb{C})}.\end{aligned}$$

This shows that $M_z^* = \partial(\mathcal{I} - R_0 I)^{2p}$. ∎

Theorem 2.22 *Let $p \in \mathbb{N}$. Then we can write the adjoint of the integration operator I as*

$$I^* = R_0(\mathcal{I} - R_0 I)^{2p}, \tag{2.27}$$

where $\mathcal{I}$ is the identity operator.

Proof We show first the result for monomials z^n, with $n \in \mathbb{N}$. By using the binomial theorem and formula (2.18), we get

$$I^* z^n = \frac{n^{2p}}{(n+1)^{2p}} z^{n-1} = \left(1 - \frac{1}{n+1}\right)^{2p} z^{n-1} = \left(\sum_{k=0}^{2p} \binom{2p}{k} (-1)^k \frac{1}{(n+1)^k}\right) z^{n-1}. \tag{2.28}$$

By (2.20), we have that

$$R_0(R_0 I)^k z^n = \frac{1}{(n+1)^k} z^{n-1}, \qquad k \geq 0. \tag{2.29}$$

By inserting (2.29) in (2.28) and using again the binomial theorem, we get

$$I^* z^n = R_0 \left(\sum_{k=0}^{2p} \binom{2p}{k} (-R_0 I)^k\right) z^n = R_0(\mathcal{I} - R_0 I)^{2p}.$$

The result follows trivially for $n = 0$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be functions in $\mathcal{H}_p(\mathbb{C})$, with $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$. By the binomial theorem we get

$$\begin{aligned}\langle f, R_0(\mathcal{I} - R_0 I)^{2p} g\rangle_{\mathcal{H}_p(\mathbb{C})} &= \left\langle f, R_0(\mathcal{I} - R_0 I)^{2p} \sum_{n=0}^{\infty} b_n z^n \right\rangle_{\mathcal{H}_p(\mathbb{C})} \\ &= \left\langle f, \sum_{n=1}^{\infty} \frac{n^{2p}}{(n+1)^{2p}} z^{n-1} b_n \right\rangle_{\mathcal{H}_p(\mathbb{C})} \\ &= \left\langle f, \sum_{n=0}^{\infty} \frac{(n+1)^{2p}}{(n+2)^{2p}} z^n b_{n+1} \right\rangle_{\mathcal{H}_p(\mathbb{C})}\end{aligned}$$

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$$= \sum_{n=0}^{\infty} a_n \overline{b_{n+1}} \frac{n!}{(n+2)^{2p}}$$
$$= \left\langle \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} z^n, g \right\rangle_{\mathcal{H}_p(\mathbb{C})}$$
$$= \langle If, g \rangle_{\mathcal{H}_p}.$$

This shows that $I^* = R_0(\mathcal{I} - R_0 I)^{2p}$. ∎

We sum up all the previous results in the following table:

Operator	R_0	∂	M_z	I
Adjoint	$I(\mathcal{I} + R_0 I)^{2p}$	$M_z(\mathcal{I} + R_0 I)^{2p}$	$\partial(\mathcal{I} - R_0 I)^{2p}$	$R_0(\mathcal{I} - R_0 I)^{2p}$

Remark 2.23 If we consider $p = 0$ in Theorems 2.19, 2.20, 2.21, and 2.22, we get the results that hold in classical theory of the Fock spaces.

Unlike to what happens in the classic case the operators M_z and M_z^* do not satisfy a commutation relation, as well as R_0 and R_0^*, see (2.12) and (2.15). We show that the commutators in the space $\mathcal{H}_p(\mathbb{C})$ are related to some specific diagonal operators. We denote a generic diagonal operator by

$$D = \mathrm{diag}(d_0, d_1, d_2, \ldots).$$

We define the forward and backward shifts of D as

$$D^{(1)} = \mathrm{diag}(0, d_0, d_1, \ldots), \tag{2.30}$$

$$D^{(-1)} = \mathrm{diag}(d_1, d_2, d_3, \ldots), \tag{2.31}$$

respectively.

Proposition 2.24 *The operators M_z and M_z^* satisfy the following formal relation:*

$$[M_z, M_z^*] = \left[\mathsf{A} \left(Id - [\mathsf{A}^{(-1)}]^{-1} \right)^{2p} - \mathsf{A}^{(-1)} \left(Id - [\mathsf{A}^{(-2)}]^{-1} \right)^{2p} \right], \tag{2.32}$$

where Id is the identity matrix, $\mathsf{A} := diag(0, 1, 2, 3, \ldots)$ and $[.]^{-1}$ is the inverse matrix.

Proof We prove first the result on the monomials. By Theorem 2.21, we have

$$[M_z, M_z^*](z^n) = M_z M_z^*(z^n) - M_z^* M_z(z^n)$$
$$= \left(\sum_{k=0}^{2p} \binom{2p}{k} (-1)^k M_z \partial (R_0 I)^k - \sum_{k=0}^{2p} \binom{2p}{k} (-1)^k \partial (R_0 I)^k M_z \right) z^n.$$

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By (2.20), we have

$$M_z\partial(IR_0)^k(z^n) = \frac{n}{(n+1)^k}z^n \qquad \text{and} \qquad \partial(IR_0)^k M_z(z^n) = \frac{(n+1)}{(n+2)^k}z^n.$$

Then by the binomial formula, we have

$$[M_z, M_z^*](z^n) = \left(n\sum_{k=0}^{2p}\binom{2p}{k}(-1)^k\frac{1}{(n+1)^k} - (n+1)\sum_{k=0}^{2p}\binom{2p}{k}(-1)^k\frac{1}{(n+2)^k}\right)z^n$$

$$= \left[n\left(1-\frac{1}{n+1}\right)^{2p} - (n+1)\left(1-\frac{1}{n+2}\right)^{2p}\right]z^n. \tag{2.33}$$

Now, by setting

$$\mathsf{A} := \operatorname{diag}(0,1,2,3,\ldots,n,\ldots), \qquad \mathsf{B} := (1,2,3,\ldots,n+1,\ldots), \tag{2.34}$$

$$\mathsf{C} := \operatorname{diag}\left(1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n+1},\ldots\right) \qquad \mathsf{D} := \operatorname{diag}\left(\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots,\frac{1}{n+2},\ldots\right), \tag{2.35}$$

we can formally write (2.33) in the following way:

$$[M_z, M_z^*] = \left[\mathsf{A}\,(Id-\mathsf{C})^{2p} - \mathsf{B}\,(Id-\mathsf{D})^{2p}\right].$$

Finally, we observe that $\mathsf{B} = \mathsf{A}^{(-1)}$, $\mathsf{C} = [\mathsf{A}^{(-1)}]^{-1}$ and $\mathsf{D} = [\mathsf{A}^{(-2)}]^{-1}$. These considerations imply (2.32). ∎

Remark 2.25 If we consider $p = 0$ in Proposition 2.24 we have that $M_z^* = \partial$ and

$$\mathsf{A} - \mathsf{A}^{(-1)} = \operatorname{diag}(-1,-1,-1,\ldots) = -id.$$

Hence, we get back to formula (2.12).

Proposition 2.26 *The operators R_0 and R_0^* satisfy the following formal relation:*

$$[R_0, R_0^*] = D_0\left[\mathsf{A}^{(-1)}\,(Id+\mathsf{E})^{2p} - \mathsf{A}\left(Id+\mathsf{E}^{(-1)}\right)^{2p}\right], \tag{2.36}$$

where Id is the identity matrix, $\mathsf{A} := diag(0,1,2,3,\ldots)$, $\mathsf{E} := diag\left(1,1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots\right)$ and D_0 is the diagonal matrix defined in (2.14).

Proof We prove the result on the monomials. By Theorem 2.19 and for $n \geq 1$, we have

$$[R_0, R_0^*](z^n) = R_0R_0^*(z^n) - R_0^*R_0(z^n)$$

$$= \sum_{k=0}^{2p}\binom{2p}{k}(R_0)^{k+1}(z^n) - \sum_{k=0}^{2p}\binom{2p}{k}I(R_0I)^kR_0(z^n). \tag{2.37}$$

By formula (2.20), we have

$$(R_0I)^{k+1}(z^n) = \frac{z^n}{(n+1)^{k+1}}, \qquad \text{and} \qquad [I(R_0I)^kR_0](z^n) = \frac{z^n}{n^{k+1}}.$$

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Hence, we get

$$[R_0, R_0^*](z^n) = \left(\sum_{k=0}^{2p}\binom{2p}{k}\frac{1}{(n+1)^{k+1}} - \sum_{k=0}^{2p}\binom{2p}{k}\frac{1}{n^{k+1}}\right)z^n$$
$$= -\frac{1}{n(n+1)}\left(\sum_{k=0}^{2p}\binom{2p}{k}\frac{(n+1)^{k+1}-n^{k+1}}{n^k(n+1)^k}\right)$$
$$= -\frac{1}{n(n+1)}\left((n+1)\sum_{k=0}^{2p}\binom{2p}{k}\frac{1}{n^k} - n\sum_{k=0}^{2p}\binom{2p}{k}\frac{1}{(n+1)^k}\right)z^n.$$

From the binomial theorem, we get

$$[R_0, R_0^*](z^n) = -\frac{1}{n(n+1)}\left((n+1)\left(1+\frac{1}{n}\right)^{2p} - n\left(1+\frac{1}{n+1}\right)^{2p}\right)z^n.$$

Now, for $n = 0$ we have that $R_0^*(1) = 4^p$ therefore by (2.37), we obtain

$$[R_0, R_0^*](1) = R_0R_0^*(1) - R_0^*R_0(1) = 4^p.$$

Therefore, we proved that

(2.38)

$$[R_0, R_0^*](z^n) = \begin{cases} 4^p, & n = 0 \\ -\frac{1}{n(n+1)}\left((n+1)\left(1+\frac{1}{n}\right)^{2p} - n\left(1+\frac{1}{n+1}\right)^{2p}\right)z^n, & n = 1, 2, 3, \ldots \end{cases}$$

Now by setting

$$\mathsf{E} := \operatorname{diag}\left(1, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right),$$

and by the notations introduced in (2.34) and (2.35) we can write (2.38) formally as

$$[R_0, R_0^*] = D_0\left[\mathsf{B}\,(Id + \mathsf{E})^{2p} - \mathsf{A}\,(Id + \mathsf{C})^{2p}\right].$$

Finally, by observing that $\mathsf{B} = \mathsf{A}^{(-1)}$ and $\mathsf{C} = \mathsf{E}^{(-1)}$ we get (2.36). ∎

Remark 2.27 If we take $p = 0$ in Proposition 2.26 we have that $R_0^* = I$ and

$$\mathsf{A}^{(-1)} - \mathsf{A} = \operatorname{diag}(1, 1, 1, \ldots) = Id.$$

Since $D_0Id = D_0$ we get back the identity (2.13).

2.3 The $\mathcal{H}_p$-Bargmann transform

In this section, we show another characterization of the space $\mathcal{H}_p(\mathbb{C})$. This is obtained by means of a specific map from $L^2(\mathbb{R})$ onto $\mathcal{H}_p(\mathbb{C})$.

Proposition 2.28 *Let $p \in \mathbb{N}$. For every $x \in \mathbb{R}$ and $z \in \mathbb{C}$ we define the function*

(2.39)
$$A_p(z, x) = \sum_{n=0}^{\infty}\frac{(n+1)^p z^n}{\sqrt{n!}}\xi_n(x),$$

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where ξ_n are the normalized Hermite functions. Then, we have:

1) *the function $A_p(.,x)$ is entire for every $x \in \mathbb{R}$.*
2) *a function $f \in \mathcal{H}_p(\mathbb{C})$ if and only if there exists $\varphi \in L^2(\mathbb{R})$ such that*

$$f(z) = \int_{\mathbb{R}} A_p(z,x)\varphi(x)dx = \langle \varphi, \overline{A_p(z,.)} \rangle_{L^2}. \tag{2.40}$$

Proof We know by [19] that the normalized Hermite functions are uniformly bounded by some constants, i.e.

$$\exists C > 0 \quad \text{such that} \quad |\xi_n(x)| < C \quad \text{for every} \quad n \in \mathbb{N} \quad \text{and} \quad x \in \mathbb{R}.$$

This implies that the series in (2.28) is convergent, and so $A_p(.,x)$ is entire. Now, we show the second point of the statement. Let us suppose that $f(z) = \langle g, \overline{A_p(z,.)} \rangle_{L^2}$, for $\varphi \in L^2(\mathbb{R})$. We have to prove that $f \in \mathcal{H}_p(\mathbb{C})$. Then, we have

$$\begin{aligned} f(z) &= \int_{\mathbb{R}} \left(\sum_{n=0}^{\infty} \frac{(n+1)^p z^n}{\sqrt{n!}} \xi_n(x)\varphi(x) \right) dx \\ &= \sum_{n=0}^{\infty} \frac{(n+1)^p z^n}{\sqrt{n!}} \int_{\mathbb{R}} \xi_n(x)\varphi(x)dx \\ &= \sum_{n=0}^{\infty} z^n \alpha_n, \end{aligned}$$

where $\alpha_n := \frac{(n+1)^p}{\sqrt{n!}} \int_{\mathbb{R}} \xi_n(x)\varphi(x)dx$. Since $\{\xi_n\}_{n=0}^{\infty}$ is an orthonormal basis of $L^2(\mathbb{R})$ and by the Parseval's equality we have that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{n!}{(n+1)^{2p}} |\alpha_n|^2 &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \left| \int_{\mathbb{R}} \xi_n(x)\varphi(x)dx \right|^2 \\ &= \int_{\mathbb{R}} |\varphi(x)|^2 dx \\ &= \|\varphi\|_{L^2}^2 < \infty. \end{aligned}$$

This proves that $f \in \mathcal{H}_p(\mathbb{C})$.

Now, we show the converse. From the definition of the space $\mathcal{H}_p(\mathbb{C})$ we have $f(z) = \sum_{n=0}^{\infty} z^n a_n$ and $\sum_{n=0}^{\infty} \frac{n!}{(n+1)^{2p}} |a_n|^2 < \infty$, with $(a_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{C}$. We set

$$\varphi(x) := \sum_{n=0}^{\infty} \frac{\sqrt{n!}}{(n+1)^p} a_n \xi_n(x).$$

This function belongs to $L^2(\mathbb{R})$. Indeed we have

$$\|\varphi\|_{L^2}^2 = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^{2p}} |a_n|^2 < \infty.$$

Since the normalized Hermite functions are a basis of the space $L^2(\mathbb{R})$ we have

$$\langle \varphi, \overline{A_p(z,.)} \rangle_{L^2} = \sum_{n=0}^{\infty} \frac{z^n \sqrt{n!}}{(n+1)^p} \frac{(n+1)^p}{\sqrt{n!}} a_n = \sum_{n=0}^{\infty} z^n a_n = f(z).$$

This concludes the proof. ∎

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The previous result gives the motivation to consider a Bargmann transform defined in the space $\mathcal{H}_p(\mathbb{C})$.

Definition 2.29 Let $p \in \mathbb{N}$. For any $\varphi \in L^2(\mathbb{R})$ we define the $\mathcal{H}_p$-Bargmann transform as

$$\mathcal{B}_p(\varphi(z)) := \int_{\mathbb{R}} A_p(z,x)\varphi(x)dx, \qquad A_p(z,x) := \frac{1}{(2\pi)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{(n+1)^p z^n}{n!} h_n(x),$$

where h_n are the Hermite functions defined in (1.3).

Remark 2.30 If we consider $p = 0$ in Definition 2.29 we get back to the classical Bargmann transform, see (1.2) and (1.4).

To show the next result, we need the following notation:

$$A_p^z(x) := A_p(z,x), \tag{2.41}$$

that it means that the variable z is fixed.

Proposition 2.31 *Let $p \in \mathbb{N}$. For any $z, w \in \mathbb{C}$ we have that*

$$\langle A_p^z, A_p^w \rangle_{L^2} = \frac{e^{z\bar{w}}}{z\bar{w}} T_{2p+1}(z\bar{w}). \tag{2.42}$$

Proof Let $z, w \in \mathbb{C}$, by the definition of the kernel A_p we have

$$\begin{aligned} \langle A_p^z, A_p^w \rangle_{L^2} &= \int_{\mathbb{R}} A_p^z(x)\overline{A_p^w(x)}dx \\ &= \int_{\mathbb{R}} \left(\sum_{j=0}^{\infty} \frac{(j+1)^p z^j}{\sqrt{j!}} \xi_j(x) \right) \left(\sum_{k=0}^{\infty} \frac{(k+1)^p \bar{w}^k}{\sqrt{k!}} \xi_k(x) \right) dx. \end{aligned} \tag{2.43}$$

The Hermite functions form an orthogonal basis of the space $L^2(\mathbb{R})$, hence we have

$$\int_{\mathbb{R}} \xi_j(x)\xi_k(x)dx = \delta_{j,k}, \qquad j,k \in \mathbb{N}, \tag{2.44}$$

where $\delta_{j,k}$ is the Kronecker symbol. By Proposition 2.28 we can exchange the double series in (2.43) and so by formula (2.44) we have

$$\begin{aligned} \langle A_p^z, A_p^w \rangle_{L^2} &= \sum_{j,k=0}^{\infty} \frac{(j+1)^p (k+1)^p z^j \bar{w}^k}{\sqrt{j!k!}} \int_{\mathbb{R}} \xi_j(x)\xi_k(x)dx \\ &= \sum_{k=0}^{\infty} \frac{(k+1)^{2p} z^k \bar{w}^k}{k!} \\ &= K_p(z,w). \end{aligned}$$

The result follows by Proposition 2.8. ∎

Remark 2.32 Formula (2.42) gives another representation of the Touchard polynomials in terms of the L^2-inner product of the kernel of the $\mathcal{H}_p$-Bargmann transform.

Now, our goal is to find a closed expression of of the $\mathcal{H}_p$-Bargmann transform. We recall that the number operator is defined by means of the combination of the

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momentum and annihilation operators, i.e. $M_z\partial$. It acts on the monomial z^n, with $n \geq 0$, in the following way:

$$(M_z\partial)\, z^n = nz^n.$$

By iterating the action of the number operator to z^n we get

$$(M_z\partial)^j\, z^n = n^j z^n, \qquad j \geq 0. \tag{2.45}$$

In [20], the author proved that the action of the number operator can be written in terms of the Stirling numbers of the second type, see Stirling. Precisely, it holds that

$$(M_z\partial)^j = \sum_{k=0}^{j} S(j,k) z^k \frac{d^k}{dz^k}. \tag{2.46}$$

Theorem 2.33 *For $p \in \mathbb{N}$ we can write the kernel of the $\mathcal{H}_p$-Bargmann transform as*

$$A_p(z,x) = (\mathcal{I} + M_z\partial)^p\, A(z,x), \qquad \forall (z,x) \in \mathbb{C} \times \mathbb{R}. \tag{2.47}$$

Moreover, we can write

$$A_p(z,x) = (\partial M_z)^p\, A(z,x), \tag{2.48}$$

where $A(z,x)$ is the kernel of the classical Bargmann transform, see (1.2).

Proof By the binomial formula, we have

$$(n+1)^p = \sum_{j=0}^{p} \binom{p}{j} n^j.$$

This implies that we can write the kernel of the $\mathcal{H}_p$-Bargmann transform as

$$\begin{aligned} A_p(z,x) &= \frac{1}{(2\pi)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{(n+1)^p}{n!} h_n(x) z^n \\ &= \frac{1}{(2\pi)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{j=0}^{p} \binom{p}{j} n^j \right) h_n(x) z^n \\ &= \sum_{j=0}^{p} \binom{p}{j} \left[\frac{1}{(2\pi)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{h_n(x)}{n!} n^j z^n \right]. \end{aligned} \tag{2.49}$$

By (2.45) and using another time the binomial formula, we have

$$\begin{aligned} A_p(z,x) &= \sum_{j=0}^{p} \binom{p}{j} \left[\frac{1}{(2\pi)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{h_n(x)}{n!} (M_z\partial)^j z^n \right] \\ &= \sum_{j=0}^{p} \binom{p}{j} (M_z\partial)^j \left[\frac{1}{(2\pi)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{h_n(x)}{n!} z^n \right] \\ &= \sum_{j=0}^{p} \binom{p}{j} (M_z\partial)^j\, A(z,x) \\ &= (\mathcal{I} + M_z\partial)^p\, A(z,x). \end{aligned}$$

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Finally, the equality (2.48) follows from (2.12). ∎

A direct application of the previous result is given by the following generating formula.

Corollary 2.34 *For $p \in \mathbb{N}$ we have that*

$$\sum_{p=0}^{\infty} \frac{A_p(z,x)}{p!} = exp\left[\left(\mathcal{I} + M_z\partial\right)\right] A(z,x).$$

Now, we show a recurrence-like relation for the kernel $A_p(z,x)$.

Proposition 2.35 *Let $p \in \mathbb{N}$. Then, we have*

$$A_{p+1}(z,x) - A_p(z,x) = M_z\left(\mathcal{I} + \partial M_z\right)^p \partial A(z,x), \qquad \forall (z,x) \in \mathbb{C} \times \mathbb{R}.$$

Proof By Theorem 2.33 we have

$$\begin{aligned} A_{p+1}(z,x) &= \left(\mathcal{I} + M_z\partial\right)^{p+1} A(z,x) \\ &= \left(\mathcal{I} + M_z\partial\right) A_p(z,x) \\ &= A_p(z,x) + M_z\partial A_p(z,x). \end{aligned}$$

By using another time Theorem 2.33, we have

$$A_{p+1}(z,x) - A_p(z,x) = M_z\partial\left(\mathcal{I} + M_z\partial\right)^p A(z,x). \tag{2.50}$$

Now, for $j \geq 0$ we have

$$\partial\left(M_z\partial\right)^j = \partial\left(M_z\partial\right)^{j-1} M_z\partial = \left(\partial M_z\right)^j \partial. \tag{2.51}$$

By the binomial formula and (2.51), we have

$$\begin{aligned} \partial\left(\mathcal{I} + M_z\partial\right)^p A(z,x) &= \sum_{j=0}^{p} \binom{p}{j} \partial\left(M_z\partial\right)^j A(z,x) \\ &= \sum_{j=0}^{p} \binom{p}{j} \left(\partial M_z\right)^j \partial A(z,x) \\ &= \left(\mathcal{I} + \partial M_z\right)^p \partial A(z,x). \end{aligned} \tag{2.52}$$

By plugging (2.52) into (2.50) we get the final result. ∎

Now, we prove that the kernel of $\mathcal{B}_p$ can be written also in terms of the Stirling numbers of the second kind.

Proposition 2.36 *Let $p \in \mathbb{N}$. Then we can write the kernel of the $\mathcal{H}_p$-Bargmann transform as*

$$A_p(z,x) = \sum_{j=0}^{p} \sum_{k=0}^{j} \binom{p}{j} S(j,k) z^k \frac{d^k}{dz^k} A(z,x), \tag{2.53}$$

where $S(n,k)$ are the Stirling number of the second type.

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Proof By Theorem 2.33 and formula (2.46), we have

$$A_p(z,x) = \sum_{j=0}^{p} \binom{p}{j} (z\partial)^j A(z,x) = \sum_{j=0}^{p} \sum_{k=0}^{j} \binom{p}{j} S(j,k) z^k \frac{d^k}{dz^k} A(z,x).$$

This proves the result. ∎

To show the next result we recall the extension of the Hermite polynomials in the complex variable z, see [26, 29]. Precisely, these are given by

$$H_n(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!(-1)^k}{k!(n-2k)!} \frac{(\sqrt{2}z)^{n-2k}}{2^{\frac{n}{2}}}.$$

These polynomials satisfied the following recurrence relation:

$$H_{k+1}(z) = zH_k(z) - \frac{d}{dz} H_k(z). \tag{2.54}$$

Proposition 2.37 *For $k \geq 0$ we have*

$$\frac{d^k}{dz^k} A(z,x) = H_k(x-z) A(z,x),$$

where $A(z,x)$ is the kernel of the classical Bargmann transform.

Proof We prove the result by induction on k. For $k = 0$, the result is trivial because $H_0(x-z) = 1$. Now we suppose that it is true for k and we prove it for $k+1$. We start by observing that

$$\frac{d}{dz} A(z,x) = (x-z) A(z,x).$$

Hence by (2.54) and the inductive hypothesis we get

$$\begin{aligned}
\frac{d^{k+1}}{dz^{k+1}} A(z,x) &= \frac{d}{dz} \left(\frac{d^k}{dz^k} A(z,x) \right) \\
&= \frac{d}{dz} \left(H_k(x-z) A(z,x) \right) \\
&= -\frac{d}{dz} H_k(x-z) A(z,x) + H_k(x-z)(x-z) A(z,x) \\
&= \left[H_k(x-z)(x-z) - \frac{d}{dz} H_k(x-z) \right] A(z,x) \\
&= H_{k+1}(x-z) A(z,x).
\end{aligned}$$

∎

The above result paved the way to get a closed formula for the $\mathcal{H}_p$-Bargmann transform.

Proposition 2.38 *Let $p \in \mathbb{N}$. Then we can write the $\mathcal{H}_p$-Bargmann transform in terms of the classical Bargmann transform. Precisely, for any $\varphi \in L^2(\mathbb{R})$ it holds that*

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$$\begin{aligned}\mathcal{B}_p(\varphi(z)) &= (\mathcal{I} + M_z\partial)^p\,(\mathcal{B}\varphi)(z)\\ &= (\partial M_z)^p\,(\mathcal{B}\varphi)(z)\\ &= \sum_{j=0}^{p}\sum_{k=0}^{j}\binom{p}{j}S(j,k)z^k\frac{d^k}{dz^k}(\mathcal{B}\varphi)(z).\end{aligned} \tag{2.55}$$

Moreover, we can write the $\mathcal{H}_p$-Bargmann transform as

(2.56)

$$\mathcal{B}_p(\varphi(z)) = \left(\frac{1}{2\pi}\right)^{\frac{1}{4}}\sum_{j=0}^{p}\sum_{k=0}^{j}\binom{p}{j}S(j,k)z^k\int_{\mathbb{R}}H_k(x-z)\exp\left(-\frac{x^2}{4}+xz-\frac{z^2}{2}\right)\varphi(x)dx.$$

Proof The result follows from Theorem 2.33 and Proposition 2.53. Finally by (2.55) and Proposition 2.37 we get (2.56). ∎

2.4 The Fock space connected to the $\mathcal{H}_p$ space

In this section, we study a connection between the space $\mathcal{H}_p(\mathbb{C})$ and the Fock space. This is achieved by a diagonal specific operator. A similar operator has been also considered in [21, Proposition 2.6], but in this paper we follow a different approach. At the end of this section, by using the connection between the spaces $\mathcal{F}(\mathbb{C})$ and $\mathcal{H}_p(\mathbb{C})$, we show also some basic properties of the $\mathcal{H}_p$-Bargmann transform.

Theorem 2.39 *Let $p \in \mathbb{N}$. The the operator*

$$\mathcal{E}_p := (\mathcal{I} + M_z\partial)^p\,. \tag{2.57}$$

realizes a surjective isometry from the space $\mathcal{F}(\mathbb{C})$ onto $\mathcal{H}_p(\mathbb{C})$.

Proof We show first that the operator $\mathcal{E}_p$ is injective. To do this we prove that $\mathcal{E}_p$ realizes an isometry. Let us consider $f \in \mathcal{F}(\mathbb{C})$. We apply the operator $\mathcal{E}_p$ to a function $f = \sum_{n=0}^{\infty} z^n a_n$, with $(a_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$. Since $\mathcal{E}_p(z^n) = (n+1)^p z^n$ we get

$$\begin{aligned}\mathcal{E}_p(f) &= \mathcal{E}_p\left(\sum_{n=0}^{\infty} z^n a_n\right)\\ &= \sum_{n=0}^{\infty}(n+1)^p z^n a_n\\ &= \sum_{n=0}^{\infty} z^n \alpha_n,\end{aligned} \tag{2.58}$$

where $\alpha_n := (n+1)^p a_n$. Therefore, we have

$$\|\mathcal{E}_p(f)\|^2_{\mathcal{H}_p(\mathbb{C})} = \sum_{n=0}^{\infty}\frac{n!}{(n+1)^{2p}}|\alpha_n|^2 = \sum_{n=0}^{\infty} n!|a_n|^2 = \|f\|^2_{\mathcal{F}(\mathbb{C})}.$$

This proves that the operator $\mathcal{E}_p$ is an isometry between the spaces $\mathcal{F}(\mathbb{C})$ and $\mathcal{H}_p(\mathbb{C})$. Now, we show that the operator $\mathcal{E}_p$ is surjective. Let $g \in \mathcal{H}_p(\mathbb{C})$. Hence, we can write

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$g(z) = \sum_{n=0}^{\infty} z^n a_n$, with $(a_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$. To achieve our aim we have to find a function $f \in \mathcal{F}(\mathbb{C})$, such that

$$g(z) = \mathcal{E}_p(f)(z).$$

Precisely, we have to pick a sequence $(b_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} z^n b_n$ and $\|f\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} n!|b_n|^2 < \infty$. Let us consider $b_n = \frac{a_n}{(n+1)^p}$. Then we have

$$\begin{aligned}\mathcal{E}_p(f)(z) &= \sum_{n=0}^{\infty} (n+1)^p z^n b_n \\ &= \sum_{n=0}^{\infty} z^n a_n \\ &= g(z).\end{aligned}$$

Moreover since $g \in \mathcal{H}_p(\mathbb{C})$, we have

$$\|f\|_{\mathcal{F}(\mathbb{C})}^2 = \sum_{n=0}^{\infty} n!|b_n|^2 = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^{2p}}|a_n|^2 < \infty.$$

Therefore, with the above choice of $(b_n)_{n\in\mathbb{N}_0}$ we have that $f \in \mathcal{F}(\mathbb{C})$. ∎

Remark 2.40 The operator $\mathcal{E}_p$ is the identity if we take $p = 0$.

Proposition 2.41 *For $p \in \mathbb{N}$ the adjoint of the operator $\mathcal{E}_p$ is given by $(R_0 I)^p$. Moreover, the inverse of the operator $\mathcal{E}_p$ is given by its adjoint.*

Proof Let us consider $g \in \mathcal{H}_p(\mathbb{C})$ and $f \in \mathcal{F}(\mathbb{C})$. To prove the result, we have to show that

$$\langle \mathcal{E}_p(f), g\rangle_{\mathcal{H}_p(\mathbb{C})} = \langle f, (R_0 I)^p(g)\rangle_{\mathcal{F}(\mathbb{C})}.$$

We can write $f(z) = \sum_{n=0}^{\infty} z^n a_n$, $g(z) = \sum_{n=0}^{\infty} z^n b_n$, with $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$. From the definition of the operator $\mathcal{E}_p$, we have that

$$\langle \mathcal{E}_p(f), g\rangle_{\mathcal{H}_p(\mathbb{C})} = \left\langle \sum_{n=0}^{\infty} (n+1)^p z^n a_n, g \right\rangle_{\mathcal{F}(\mathbb{C})} = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^p} a_n \overline{b_n}. \tag{2.59}$$

On the other side since

$$(R_0 I)^p z^n = \frac{z^n}{(n+1)^p},$$

we get

$$\langle f, (R_0 I)^p g\rangle_{\mathcal{F}(\mathbb{C})} = \left\langle f, \sum_{n=0}^{\infty} \frac{z^n a_n}{(n+1)^p} \right\rangle_{\mathcal{H}_p(\mathbb{C})} = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^p} a_n \overline{b_n}. \tag{2.60}$$

Since (2.59) and (2.60) are equal we get that $\mathcal{E}_p^* = (R_0 I)^p$. Finally, since the operator $\mathcal{E}_p$ is unitary its adjoint coincides with its inverse. ∎

Now we show that we can write the reproducing kernel of the space $\mathcal{H}_p(\mathbb{C})$ in terms of the operator $\mathcal{E}_p$.

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Lemma 2.42 *Let $p \in \mathbb{N}$. Then we have*

$$\mathcal{E}_p^{\bar{w}} \mathcal{E}_p^{z}(e^{z\bar{w}}) := \frac{e^{z\bar{w}}}{z\bar{w}} T_{2p+1}(z\bar{w}),$$

where $\mathcal{E}_p^{\bar{w}}$, $\mathcal{E}_p^{z}$ denote the operator $\mathcal{E}_p$ with respect to the variables z and $\bar{w}$, respectively.

Proof Since $\mathcal{E}_p^{z}(z^n) = (n+1)^p z^n$ and $\mathcal{E}_p^{\bar{w}}(\bar{w}^n) = (n+1)^p \bar{w}^n$ we have

$$\begin{aligned}
\mathcal{E}_p^{\bar{w}} \mathcal{E}_p^{z}(e^{z\bar{w}}) &= \mathcal{E}_p^{\bar{w}} \mathcal{E}_p^{z} \left(\sum_{n=0}^{\infty} \frac{z^n \bar{w}^n}{n!} \right) \\
&= \mathcal{E}_p^{\bar{w}} \left(\sum_{n=0}^{\infty} \frac{(n+1)^p z^n \bar{w}^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \frac{(n+1)^{2p} z^n \bar{w}^n}{n!} \\
&= K_p(z, w),
\end{aligned}$$

where $K_p(z, w)$ is the reproducing kernel of the space $\mathcal{H}_p(\mathbb{C})$. By Proposition 2.8, we get the result. ∎

By combining the properties of the operator $\mathcal{E}_p$ and the ones of the classical Bargmann transform, we can deduce the main properties of the $\mathcal{H}_p$-Bargmann transform.

Proposition 2.43 *Let $p \in \mathbb{N}$. Then the $\mathcal{H}_p$-Bargmann transform is a unitary operator and satisfies*

$$\|\mathcal{B}_p(\varphi)\|_{\mathcal{H}_p(\mathbb{C})} = \|\varphi\|_{L^2(\mathbb{R})}, \qquad \text{and} \qquad \mathcal{B}_p(\xi_n)(z) = \frac{(n+1)^p z^n}{\sqrt{n!}}.$$

Proof First we show that $\mathcal{B}_p$ is an unitary operator. By Proposition 2.38, Theorem 2.39 and the fact that the classical Bargmann transform is unitary we get

$$\|\mathcal{B}_p(\varphi)\|_{\mathcal{H}_p(\mathbb{C})} = \|\mathcal{E}_p(\mathcal{B})(\varphi)\|_{\mathcal{H}_p(\mathbb{C})} = \|\mathcal{B}(\varphi)\|_{\mathcal{F}(\mathbb{C})} = \|\varphi\|_{L^2(\mathbb{R})}.$$

By using another time Proposition 2.38 and formula (1.5), we get

$$\mathcal{B}_p(\xi_n) = \mathcal{E}_p(\mathcal{B}(\xi_n)(z)) = \frac{\mathcal{E}_p(z^n)}{\sqrt{n!}} = \frac{(n+1)^p}{\sqrt{n!}} z^n.$$

∎

3 The space $\mathcal{F}_p$ and associated operators

3.1 Construction and basic properties of the $\mathcal{F}_p$-space

In this section, we study the range of the Schwartz space $\mathcal{S}(\mathbb{R})$ under the Bargmann transform. The goal of this section is to show the main properties of the space that come out from this application.

Lemma 3.1 *For a fixed $p \in \mathbb{N}$ we have*

$$\mathcal{B}\left(\ell^2(\mathbb{N}, (n+1)^{2p})\right) = \mathcal{F}_p(\mathbb{C}),$$

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where

$$\mathcal{F}_p(\mathbb{C}) := \left\{ f(z) = \sum_{n=0}^{\infty} z^n a_n; \quad (a_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}, \quad \sum_{n=0}^{\infty} |a_n|^2 n!(n+1)^{2p} < \infty \right\}. \tag{3.1}$$

Proof The result follows from similar computations performed in Lemma 2.1. ∎

Theorem 3.2 *Let $p \in \mathbb{N}$. Then we have*

$$\mathcal{B}\left(\bigcap_{p>0} \ell^2(\mathbb{N}, (n+1)^{2p}) \right) = \bigcap_{p>0} \mathcal{B}\left(\ell^2(\mathbb{N}, (n+1)^{2p}) \right).$$

Proof We show the equality by a double inclusion. Let $f \in \bigcap_{p>0} \ell^2(\mathbb{N}, (n+1)^{2p})$. Then for any $p \in \mathbb{N}$ we have $f \in \ell^2(\mathbb{N}, (n+1)^{2p})$. Since the Bargmann transform is unitary we have

$$\mathcal{B}f \in \mathcal{B}\left(\ell^2(\mathbb{N}, (n+1)^{2p}) \right).$$

This implies that $\mathcal{B}f \in \bigcap_{p>0} \mathcal{B}\left(\ell^2(\mathbb{N}, (n+1)^{2p}) \right)$. Now we show the other inclusion. Let $g \in \bigcap_{p>0} \mathcal{B}\left(\ell^2(\mathbb{N}, (n+1)^{2p}) \right)$. Then for any $p \in \mathbb{N}$ we have $g \in \mathcal{B}\left(\ell^2(\mathbb{N}, (n+1)^{2p}) \right)$. Then there exist $\alpha_p \in \ell^2(\mathbb{N}, (n+1)^{2p})$ such that

$$g = \mathcal{B}(\alpha_p).$$

Similarly, we can take α_q such that we have

$$g = \mathcal{B}(\alpha_q).$$

Therefore, we have

$$\mathcal{B}(\alpha_p) = \mathcal{B}(\alpha_q).$$

Since the Bargmann transform is unitary, for any $p > 0$ we have

$$\alpha_q = \alpha_p := \alpha \in \ell^2(\mathbb{N}, (n+1)^{2p}).$$

Therefore we have

$$g = \mathcal{B}(\alpha) \in \mathcal{B}\left(\bigcap_{p>0} \ell^2(\mathbb{N}, (n+1)^{2p}) \right).$$

This concludes the proof. ∎

Now we give a rigorous definition of the space $\mathcal{F}_p(\mathbb{C})$.

Definition 3.3 Let $p \in \mathbb{N}$. We define the Hilbert space $\mathcal{F}_p(\mathbb{C})$ as the set of all entire functions endowed with the inner product given by

$$\langle f, g \rangle_{\mathcal{F}_p(\mathbb{C})} = \sum_{n=0}^{\infty} n!(n+1)^{2p} a_n \overline{b_n}, \tag{3.2}$$

where $f(z) = \sum_{n=0}^{\infty} z^n a_n$ and $g(z) = \sum_{n=0}^{\infty} z^n b_n$ with $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$ such that $\sum_{n=0}^{\infty} n!(n+1)^{2p}|a_n|^2 < \infty$, $\sum_{n=0}^{\infty} n!(n+1)^{2p}|b_n|^2 < \infty$.

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The inner product defined in (3.2) induces the following norm:

$$\|f\|^2_{\mathcal{F}_p(\mathbb{C})} = \sum_{n=0}^{\infty} n!(n+1)^{2p}|a_n|^2.$$

As an obvious consequence of (3.2) we have the following:

$$\langle z^n, z^m\rangle_{\mathcal{F}_p(\mathbb{C})} = n!(n+1)^{2p}\delta_{n,m}. \tag{3.3}$$

Remark 3.4 For $p = 0$ the space $\mathcal{F}_p(\mathbb{C})$ coincides with the classic Fock space, see (1.1).

Let $z, w \in \mathbb{C}$. We consider the following function

$$\mathcal{K}_p(z,w) = \sum_{n=0}^{\infty} \frac{z^n \bar{w}^n}{n!(n+1)^{2p}}.$$

By using similar arguments of Proposition 2.5 we have the following result.

Proposition 3.5 *Let $w \in \mathbb{C}$ and $p \in \mathbb{N}$.*

i) *The function $\mathcal{K}_p(z,w)$ belongs to $\mathcal{F}_p(\mathbb{C})$.*
ii) *The evaluation map $e_w(f) := f(w)$ is a continuous linear functional on $\mathcal{F}_p(\mathbb{C})$. Furthermore, we have*

$$|e_w(f)| \le \|f\|_{\mathcal{F}_p(\mathbb{C})}\|\mathcal{K}_p\|_{\mathcal{H}_p(\mathbb{C})}, \qquad \forall f \in \mathcal{F}_p(\mathbb{C}).$$

The above result guarantees the existence of the reproducing kernel of the space $\mathcal{F}_p(\mathbb{C})$.

Theorem 3.6 *Let $p \in \mathbb{N}$. For any $z, w \in \mathbb{C}$ the reproducing kernel of the space $\mathcal{F}_p(\mathbb{C})$ is given by $\mathcal{K}_p(z,w)$*

Proof The result follows from similar arguments used to prove Theorem 2.6. ∎

The reproducing kernel of the space $\mathcal{F}_p(\mathbb{C})$ can be written in terms of a specific generalized hypergeometric series that are defined as

$${}_qF_q(x_1,\dots,x_q;y_1,\dots,y_q;z) = \sum_{n=0}^{\infty} \frac{(x_1)_n \cdot \ldots \cdot (x_q)_n}{(y_1)_n \cdot \ldots \cdot (y_q)_n}\frac{z^n}{n!}, \qquad z \in \mathbb{C}, \tag{3.4}$$

where $(a)_n$ is the Pochhammer symbol defined for $a > 0$ as $(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}$.

Proposition 3.7 *Let $z, w \in \mathbb{C}$. Then for $p \in \mathbb{N}$ the reproducing kernel $\mathcal{K}_p$ of the space $\mathcal{F}_p(\mathbb{C})$ can be written as*

$$\mathcal{K}_p(z,w) = {}_{2p}F_{2p}(1,\dots,1;2,\dots,2;z\bar{w}).$$

Proof By Theorem 3.6 and the expression of the generalized hypergeometric series we have

$$\begin{aligned}\mathcal{K}_p(z,w) &= \sum_{n=0}^{\infty} \frac{(z\bar{w})^n}{(n+1)^{2p}n!} \\ &= \sum_{n=0}^{\infty} \frac{(n!)^{2p}}{[(n+1)!]^{2p}}\frac{(z\bar{w})^n}{n!}\end{aligned}$$

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$$= \sum_{n=0}^{\infty} \frac{(\Gamma(n+1))^{2p}}{(\Gamma(n+2))^{2p}} \frac{(z\bar{w})^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{[(1)_n]^{2p}}{[(2)_n]^{2p}} \frac{(z\bar{w})^n}{n!}$$
$$= {}_{2p}F_{2p}(1, \dots, 1; 2, \dots, 2; z\bar{w}).$$

■

Unlike to what happens for the space $\mathcal{H}_p(\mathbb{C})$ studied in Section 3 the space $\mathcal{F}_p(\mathbb{C})$ do not satisfy any geometric description. This is proved in the following result.

Proposition 3.8 *Let $p \in \mathbb{N}$. There is no a radial measure $\sigma(x, y)$ such that*

$$\frac{1}{2\pi} \int_{\mathbb{C}} z^n \bar{z}^m d\sigma(x, y) = n!(n+1)^{2p} \delta_{n,m}. \tag{3.5}$$

Proof To show this result we use Theorem 2.11. We set $s_n := n!(n+1)^{2p}$. The matrix

$$S_N := \left((n+m)!(n+m+1)^{2p}\right)_{m,n\geq 0}^{N},$$

is not positive semidefinite. Indeed, if we pick $n, m = 0, 1$ and $N = 1$ we can write the following 2×2 matrix

$$\begin{pmatrix} 1 & 4^p \\ 4^p & 2 \cdot 9^p \end{pmatrix}.$$

This matrix is not positive semidefinite. Hence by Theorem 2.11 we cannot find a radial measure $\sigma(x, y)$ as in (3.5). ■

3.2 Some operators on $\mathcal{F}_p$-space

The goal of this section is to study the adjoint of the operators R_0, ∂, M_z and I in the space $\mathcal{F}_p(\mathbb{C})$. First we show how the adjoint operators acts on the monomials.

Theorem 3.9 *Let $p \in \mathbb{N}$. Then for $n \in \mathbb{N}$ we have that*

- *the adjoint of the backward-shift operator is given by*

$$R_0^* z^n = \frac{(n+1)^{2p-1}}{(n+2)^{2p}} z^{n+1}, \tag{3.6}$$

- *the adjoint of the annihilation operator is given by*

$$\partial^* z^n = \frac{(n+1)^{2p}}{(n+2)^{2p}} z^{n+1}, \tag{3.7}$$

- *the adjoint of the momentum operator is given by*

$$M_z^* z^n = \frac{(n+1)^{2p}}{n^{2p-1}} z^{n-1}, \qquad M_z^*(1) = 0, \tag{3.8}$$

- *the adjoint of the integration operator is given by*

$$I^* z^n = \frac{(n+1)^{2p}}{n^{2p}} z^{n-1}, \qquad I^*(1) = 0. \tag{3.9}$$

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Proof It follows by similar arguments used to prove Proposition 2.15, Proposition 2.16, Proposition 2.17, and Proposition 2.18 and by using formula (3.3). ■

In Section 4, we proved that in the space $\mathcal{H}_p(\mathbb{C})$ the operators R_0^*, ∂_z^*, M_z^*, and I^* can be written as a composition of the respective adjoint operators in the Fock space and a diagonal operator. In the next results, we show that there is a similar result for the operators R_0^*, ∂_z^*, M_z^*, and I^* in the space $\mathcal{F}_p(\mathbb{C})$.

Theorem 3.10 *Let $p \in \mathbb{N}$. Then we can write the adjoint of the backward-shift operator in the space $\mathcal{F}_p(\mathbb{C})$ as*

$$R_0^* = I(\mathcal{I} - R_0^2 I M_z)^{2p}. \tag{3.10}$$

Proof We show first the result for monomials z^n, with $n \in \mathbb{N}_0$. By the binomial theorem, we have

$$I\left(\mathcal{I} - R_0^2 I M_z\right)^{2p} z^n = I\left(\sum_{k=0}^{2p} \binom{2p}{k} (-1)^k (R_0^2 I M_z)^k\right) z^n. \tag{3.11}$$

Now we observe that for $k > 0$ we have

$$(R_0^2 I M_z)^k z^n = \frac{1}{(n+2)^k} z^n. \tag{3.12}$$

By plugging (3.12) into (3.11) we get

$$\begin{aligned} I\left(\mathcal{I} - R_0^2 I M_z\right)^{2p} z^n &= I\left(\sum_{k=0}^{2p} \binom{2p}{k} \frac{(-1)^k}{(n+2)^k}\right) z^n \\ &= \frac{1}{n+1}\left(\sum_{k=0}^{2p} \binom{2p}{k} \frac{(-1)^k}{(n+2)^k}\right) z^{n+1} \\ &= \frac{1}{n+1}\left(\sum_{k=0}^{2p-1} \binom{2p}{k} \frac{(-1)^k}{(n+2)^k} + \frac{1}{(n+2)^{2p}}\right) z^{n+1} \\ &= \frac{1}{n+1}\left(\frac{\sum_{k=0}^{2p-1} \binom{2p}{k} (-1)^k (n+2)^{2p-k} + 1}{(n+2)^{2p}}\right) z^{n+1}. \end{aligned} \tag{3.13}$$

Now, we change the index of the above sum with $\ell = 2p - k$ and we get

$$\begin{aligned} \sum_{k=0}^{2p-1} \binom{2p}{k} (-1)^k (n+2)^{2p-k} &= \sum_{\ell=1}^{2p} \binom{2p}{2p-\ell} (n+2)^{\ell} (-1)^{2p-\ell} \\ &= \sum_{\ell=1}^{2p} \binom{2p}{\ell} (n+2)^{\ell} (-1)^{\ell}. \end{aligned} \tag{3.14}$$

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By substituting (3.14) in (3.13), by the binomial theorem and by (3.6), we get

$$
\begin{aligned}
I\left(\mathcal{I}-R_0^2IM_z\right)^{2p}z^n &= \frac{1}{n+1}\left(\frac{\sum_{k=1}^{2p}\binom{2p}{k}(n+2)^k(-1)^k+1}{(n+2)^{2p}}\right)z^{n+1} \\
&= \frac{1}{n+1}\left(\frac{\sum_{k=0}^{2p}\binom{2p}{k}(-1)^k(n+2)^k}{(n+2)^{2p}}\right)z^{n+1} \\
&= \frac{1}{n+1}\frac{(n+1)^{2p}}{(n+2)^{2p}}z^{n+1} \\
&= \frac{(n+1)^{2p-1}}{(n+2)^{2p}}z^{n+1} \\
&= I\left(\mathcal{I}-R_0^2IM_z\right)^{2p}z^n.
\end{aligned} \tag{3.15}
$$

Now we show the result for a generic function f in $\mathcal{F}_p(\mathbb{C})$. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$ be another function in $\mathcal{F}_p(\mathbb{C})$, with $(a_n)_{n\in\mathbb{N}_0},(b_n)_{n\in\mathbb{N}_0}\subseteq\mathbb{C}$. By the binomial theorem, we get

$$
\begin{aligned}
\langle f, I\left(\mathcal{I}-R_0^2IM_z\right)^{2p}g\rangle_{\mathcal{F}_p(\mathbb{C})} &= \left\langle f, I\left(\mathcal{I}-R_0^2IM_z\right)^{2p}\sum_{n=0}^{\infty}b_nz^n\right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \left\langle f, \sum_{n=0}^{\infty}\frac{(n+1)^{2p-1}}{(n+2)^{2p}}z^{n+1}b_n\right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \left\langle f, \sum_{n=1}^{\infty}\frac{n^{2p}}{n(n+1)^{2p}}z^nb_{n-1}\right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \sum_{n=1}^{\infty}a_n\overline{b_{n-1}}(n-1)!n^{2p} \\
&= \left\langle\sum_{n=0}^{\infty}a_{n+1}z^n, g\right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \langle R_0f, g\rangle_{\mathcal{F}_p(\mathbb{C})}.
\end{aligned}
$$

This shows that $R_0^*=I\left(\mathcal{I}-R_0^2IM_z\right)^{2p}$. ∎

Theorem 3.11 *Let $p\in\mathbb{N}$. Then we can write the adjoint annihilation operator for the space $\mathcal{F}_p(\mathbb{C})$ as*

$$
\partial^* = M_z(\mathcal{I}-R_0^2IM_z)^{2p}, \tag{3.16}
$$

Proof We show first the result for monomials z^n, with $n\in\mathbb{N}_0$. By similar arguments used to prove (3.15), we deduce that

$$
(\mathcal{I}-R_0^2IM_z)^{2p}z^n = \frac{(n+1)^{2p}}{(n+2)^{2p}}z^n.
$$

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By (3.7), we get

$$M_z(\mathcal{I} - R_0^2 I M_z)^{2p} z^n = \frac{(n+1)^{2p}}{(n+2)^{2p}} z^{n+1} = \partial^* z^n. \tag{3.17}$$

Now we show the result for a generic function f in $\mathcal{F}_p(\mathbb{C})$. Let us consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $\mathcal{F}_p(\mathbb{C})$, with $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$. By the binomial theorem we get

$$\begin{aligned}
\langle f, M_z(\mathcal{I} - R_0^2 I M_z)^{2p} g\rangle_{\mathcal{F}_p(\mathbb{C})} &= \left\langle f, M_z(\mathcal{I} - R_0^2 I M_z)^{2p} \sum_{n=0}^{\infty} b_n z^n \right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \left\langle f, \sum_{n=0}^{\infty} \left(\frac{n+1}{n+2}\right)^{2p} z^{n+1} b_n \right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \left\langle f, \sum_{n=1}^{\infty} \frac{n^{2p}}{(n+1)^{2p}} z^n b_{n-1} \right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \sum_{n=1}^{\infty} a_n \overline{b_{n-1}} n! n^{2p} \\
&= \left\langle \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n, g \right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \langle \partial f, g\rangle_{\mathcal{F}_p(\mathbb{C})}.
\end{aligned}$$

This shows that $\partial^* = M_z(\mathcal{I} - R_0^2 I M_z)^{2p}$. ∎

Theorem 3.12 *Let $p \in \mathbb{N}$. Then we can write the adjoint of the momentum operator for the space $\mathcal{F}_p(\mathbb{C})$ as*

$$M_z^* = \partial(\mathcal{I} + I R_0)^{2p}. \tag{3.18}$$

Proof We show first the result for monomials z^n, with $n \in \mathbb{N}$. By the binomial theorem formula we have

$$\partial(\mathcal{I} + I R_0)^{2p} z^n = \partial\left(\sum_{k=0}^{2p} \binom{2p}{k} (I R_0)^k\right) z^n. \tag{3.19}$$

We observe that for $k \geq 0$ we have

$$(I R_0)^k z^n = \frac{z^n}{n^k}. \tag{3.20}$$

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By inserting (3.20) into (3.19) we get

$$\begin{aligned}\partial(\mathcal{I}+IR_0)^{2p}z^n &= \partial\left(\sum_{k=0}^{2p}\binom{2p}{k}\frac{1}{n^k}\right)z^n\\ &= n\left(\sum_{k=0}^{2p-1}\binom{2p}{k}\frac{1}{n^k}+\frac{1}{n^{2p}}\right)z^n\\ &= n\left(\frac{\sum_{k=0}^{2p-1}\binom{2p}{k}n^{2p-k}+1}{n^{2p}}\right)z^n. \qquad (3.21)\end{aligned}$$

Now, we change the index of the sum above with $\ell = 2p-k$ and we get

$$\sum_{k=0}^{2p-1}\binom{2p}{k}n^{2p-k} = \sum_{\ell=1}^{2p}\binom{2p}{2p-\ell}n^{\ell}. \qquad (3.22)$$

We observe that $\binom{2p}{2p-\ell} = \binom{2p}{\ell}$. This implies that

$$\sum_{k=0}^{2p-1}\binom{2p}{k}\frac{n^{2p}}{n^k}+1 = \sum_{\ell=1}^{2p}\binom{2p}{2p-\ell}n^{\ell}+1 = \sum_{\ell=0}^{2p}\binom{2p}{\ell}n^{\ell}. \qquad (3.23)$$

Finally by inserting (3.23) in (3.21) we get

$$\partial(\mathcal{I}+IR_0)^{2p}z^n = \partial\left(\frac{\sum_{\ell=0}^{2p}\binom{2p}{\ell}n^{\ell}}{n^{2p}}\right)z^n = \frac{(n+1)^{2p}}{n^{2p-1}}z^{n-1}. \qquad (3.24)$$

Finally by (3.8) we have

$$\partial(\mathcal{I}+IR_0)^{2p}z^n = M_z^*z^n. \qquad (3.25)$$

Formula (3.25) is still valid for $n=0$. Since

$$M_z^*(1) = 0, \qquad \partial(\mathcal{I}+IR_0)^{2p}(1) = 0.$$

Now, we prove the result for a generic f in $\mathcal{F}_p(\mathbb{C})$. Let $f(z) = \sum_{n=0}^{\infty}a_nz^n$ and $g(z) = \sum_{n=0}^{\infty}b_nz^n$ be another function in $\mathcal{F}_p(\mathbb{C})$, with $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}\subseteq\mathbb{C}$, then we have

$$\begin{aligned}\langle f, \partial(\mathcal{I}+IR_0)^{2p}g\rangle_{\mathcal{F}_p(\mathbb{C})} &= \left\langle f, \partial(\mathcal{I}+IR_0)^{2p}\sum_{n=0}^{\infty}b_nz^n\right\rangle_{\mathcal{F}_p(\mathbb{C})}\\ &= \left\langle f, \sum_{n=1}^{\infty}\frac{(n+1)^{2p}}{n^{2p-1}}z^{n-1}b_n\right\rangle_{\mathcal{F}_p(\mathbb{C})}\\ &= \left\langle f, \sum_{n=0}^{\infty}\frac{(n+2)^{2p}}{(n+1)^{2p-1}}z^nb_{n+1}\right\rangle_{\mathcal{F}_p(\mathbb{C})}\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{n=0}^{\infty} a_n \overline{b_{n+1}} (n+1)! (n+2)^{2p} \\
&= \left\langle \sum_{n=1}^{\infty} a_{n-1} z^n, g \right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \langle M_z f, g \rangle_{\mathcal{F}_p(\mathbb{C})}.
\end{aligned}
$$

This shows that $M_z^* = \partial(\mathcal{I} + IR_0)^{2p}$. ∎

Theorem 3.13 *Let $p \in \mathbb{N}$. Then we can write the adjoint of the integration operator for the space $\mathcal{F}_p(\mathbb{C})$ as*

$$
I^* = R_0(\mathcal{I} + IR_0)^{2p}. \tag{3.26}
$$

Proof We show first the result for monomials z^n with $n \in \mathbb{N}$. By similar arguments used to prove (3.24) we have

$$
(\mathcal{I} + IR_0)^{2p} z^n = \frac{(n+1)^{2p}}{n^{2p}} z^n.
$$

Hence by (3.9) we have

$$
R_0(\mathcal{I} + IR_0)^{2p} z^n = \frac{(n+1)^{2p}}{n^{2p}} z^{n-1} = I^* z^n.
$$

If we consider $n = 0$ we have

$$
I^*(1) = 0, \qquad R_0(\mathcal{I} + IR_0)^{2p}(1) = 0.
$$

Now, we prove the result for a generic $f \in \mathcal{F}_p(\mathbb{C})$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $\mathcal{F}_p(\mathbb{C})$, with $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$. By the binomial theorem we get

$$
\begin{aligned}
\langle f, R_0(\mathcal{I} + IR_0)^{2p} g \rangle_{\mathcal{F}_p(\mathbb{C})} &= \left\langle f, \partial(\mathcal{I} + IR_0)^{2p} \sum_{n=0}^{\infty} b_n z^n \right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \left\langle f, \sum_{n=1}^{\infty} \frac{(n+1)^{2p}}{n^{2p}} z^{n-1} b_n \right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \left\langle f, \sum_{n=0}^{\infty} \frac{(n+2)^{2p}}{(n+1)^{2p}} z^n b_{n+1} \right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \sum_{n=0}^{\infty} a_n \overline{b_{n+1}} n! (n+2)^{2p} \\
&= \left\langle \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} z^n, g \right\rangle_{\mathcal{F}_p(\mathbb{C})} \\
&= \langle If, g \rangle_{\mathcal{F}_p(\mathbb{C})}.
\end{aligned}
$$

This shows that $I^* = R_0(\mathcal{I} + IR_0)^{2p}$. ∎

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We sum up all the previous results in the following table

Operator	R_0	∂	M_z	I
Adjoint	$I(\mathcal{I} - R_0^2 I M_z)^{2p}$	$M_z(\mathcal{I} - R_0^2 I M_z)^{2p}$	$\partial(\mathcal{I} + IR_0)^{2p}$	$R_0(\mathcal{I} + IR_0)^{2p}$

Now we investigate the commutators of M_z and M_z^* as well as R_0 and R_0^*. Similarly to what happens for the space $\mathcal{H}_p(\mathbb{C})$ we do not have any commutation relations, like in the classic case. In the next result, we use the notation of backward of a diagonal operator, see (2.31).

Proposition 3.14 *Let $p \in \mathbb{N}$. The operators M_z and M_z^* satisfy the following formal relation:*

$$[M_z, M_z^*] = \mathsf{A}(Id + \mathsf{E})^{2p} - \mathsf{A}^{(-1)}(Id + [\mathsf{E}^{(-1)}])^{2p}, \tag{3.27}$$

where Id is the identity matrix, $\mathsf{A} := diag(0, 1, 2, 3, \ldots)$ and $\mathsf{E} := diag\left(1, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right)$.

Proof We show the theorem only for monomials z^n, with $n \in \mathbb{N}$. By Theorem 3.12 we have

$$\begin{aligned}
[M_z, M_z^*](z^n) &= M_z M_z^*(z^n) - M_z^* M_z(z^n) \\
&= \sum_{k=0}^{2p} \binom{2p}{k} M_z \partial (IR_0)^k (z^n) - \sum_{k=0}^{2p} \binom{2p}{k} \partial (IR_0)^k (z^{n+1}) \\
&= \left[n \sum_{k=0}^{2p} \binom{2p}{k} \frac{1}{n^k} - (n+1) \sum_{k=0}^{2p} \binom{2p}{k} \frac{1}{(n+1)^k} \right] z^n.
\end{aligned}$$

By the binomial formula we have

$$[M_z, M_z^*](z^n) = \left[n \left(1 + \frac{1}{n}\right)^{2p} - (n+1) \left(1 + \frac{1}{n+1}\right)^{2p} \right] z^n.$$

If we take $n = 0$ we have

$$[M_z, M_z^*](1) = M_z M_z^*(1) - M_z^* M_z(1) = -4^p.$$

Therefore,

$$[M_z, M_z^*](z^n) = \begin{cases} -4^p, & n = 0 \\ \left[n \left(1 + \frac{1}{n}\right)^{2p} - (n+1) \left(1 + \frac{1}{n+1}\right)^{2p} \right] z^n, & n = 1, 2, 3, \ldots \end{cases}$$

Finally by the notion of the backward of a diagonal operator, see (2.31), we have

$$[M_z, M_z^*] = \left[\mathsf{A}(Id + \mathsf{E})^{2p} - \mathsf{A}^{(-1)}(Id + [\mathsf{E}^{(-1)}])^{2p} \right].$$

■

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Proposition 3.15 *Let $p \in \mathbb{N}$. The operators R_0 and R_0^* satisfy the following formal relation:*

$$[R_0, R_0^*] = D_0 \left[\mathsf{A}^{(-1)} \left(Id - [\mathsf{A}^{(-1)}]^{-1} \right)^{2p} - \mathsf{A} \left(Id - [\mathsf{A}^{(-2)}]^{-1} \right)^{2p} \right], \tag{3.28}$$

where Id is the identity matrix, $\mathsf{A} := diag(0, 1, 2, 3, \ldots)$, $[.]^{-1}$ is the inverse matrix and D_0 is a diagonal matrix introduced in in (2.14).

Proof We prove the result for monomials z^n, for $n \in \mathbb{N}$. By Theorem 3.10, we have

$$\begin{aligned}
[R_0, R_0^*](z^n) &= R_0 R_0^*(z^n) - R_0^* R_0(z^n) \\
&= \sum_{k=0}^{2p} \binom{2p}{k} (-1)^k R_0 I(R_0^2 I M_z)(z^n) - \sum_{k=0}^{2p} \binom{2p}{k} (-1)^k I(R_0^2 I M_z)(z^{n-1}) \\
&= \left(\sum_{k=0}^{2p} \binom{2p}{k} (-1)^k \frac{1}{(n+2)^k (n+1)} - \sum_{k=0}^{2p} \binom{2p}{k} (-1)^k \frac{1}{(n+1)^k n} \right) z^n.
\end{aligned}$$

By the binomial formula, we have

$$\begin{aligned}
[R_0, R_0^*](z^n) &= \left[\frac{1}{n+1} \left(1 - \frac{1}{n+2} \right)^{2p} - \frac{1}{n} \left(1 - \frac{1}{n+1} \right)^{2p} \right] z^n \\
&= -\frac{1}{n(n+1)} \left[(n+1) \left(1 - \frac{1}{n+1} \right)^{2p} - n \left(1 - \frac{1}{n+2} \right)^{2p} \right] z^n.
\end{aligned}$$

For $n = 0$ we have

$$[R_0, R_0^*](1) = R_0 R_0^*(1) - R_0^* R_0(1) = \frac{1}{4^p}.$$

Therefore,

$$[R_0, R_0^*](z^n) = \begin{cases} \frac{1}{4^p}, & n = 0 \\ -\frac{1}{n(n+1)} \left[(n+1) \left(1 - \frac{1}{n+1} \right)^{2p} - n \left(1 - \frac{1}{n+2} \right)^{2p} \right] z^n, & n = 1, 2, \ldots \end{cases}$$

Finally by the notion of the backward of a diagonal operator, see (2.31), and the notation (2.14) we have

$$[R_0, R_0^*] = D_0 \left[\mathsf{A}^{(-1)} \left(Id - [\mathsf{A}^{(-1)}]^{-1} \right)^{2p} - \mathsf{A} \left(Id - [\mathsf{A}^{(-2)}]^{-1} \right)^{2p} \right].$$

■

Remark 3.16 By taking $p = 0$ in (3.27) and (3.28) we get back the relations (2.12) and (2.15), respectively. Indeed

$$D_0 \left[\mathsf{A}^{(-1)} - \mathsf{A} \right] = D_0 \left[\mathrm{diag}\,(1, 2, 3, \ldots) - \mathrm{diag}\,(0, 1, 2, 3, \ldots) \right] = D_0 Id = D_0.$$

3.3 The $\mathcal{F}_p$-Bargmann transform

The goal of this section is to further develop a characterization of the space $\mathcal{F}_p(\mathbb{C})$ through a unitary operator from $L^2(\mathbb{R})$ onto $\mathcal{F}_p(\mathbb{C})$. We find out that this operator can be considered a generalization of the Bargmann transform in the space $\mathcal{F}_p(\mathbb{C})$.

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Proposition 3.17 *Let $p \in \mathbb{N}$. For every $x \in \mathbb{R}$ and $z \in \mathbb{C}$ we define the function*

$$\mathcal{A}_p(z,x) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)^p \sqrt{n!}} \xi_n(x), \tag{3.29}$$

where ξ_n are the normalized Hermite functions. Then we have

1) *The function $\mathcal{A}_p(.,x)$ is entire for every $x \in \mathbb{R}$.*
2) *A function $f \in \mathcal{F}_p(\mathbb{C})$ if and only if there exists $\varphi \in L^2(\mathbb{R})$ such that*

$$f(z) = \int_{\mathbb{R}} \mathcal{A}_p(z,x)\varphi(x)dx = \langle \varphi, \overline{\mathcal{A}_p(z,.)} \rangle_{L^2}. \tag{3.30}$$

Proof The result follows from similar arguments used to prove Proposition 2.28. ∎

The above result motivates the following

Definition 3.18 For $p \in \mathbb{N}$ and any $\varphi \in L^2(\mathbb{R})$ we define the $\mathcal{F}_p$-Bargmann transform as

$$\mathcal{SB}_p(\varphi(z)) := \int_{\mathbb{R}} \mathcal{A}_p(z,x)\varphi(x)dx, \qquad \mathcal{A}_p(z,x) := \frac{1}{(2\pi)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{z^n h_n(x)}{(n+1)^p n!}.$$

By using the notations fixed in (2.41) we have

Proposition 3.19 *Let $p \in \mathbb{N}$. For any $z, w \in \mathbb{C}$ we have*

$$\langle \mathcal{A}_p^z, \mathcal{A}_p^w \rangle_{L^2} = {}_{2p}F_{2p}(1,\ldots,1;2,\ldots,2;z\bar{w}),$$

where ${}_{2p}F_{2p}$ is the generalized hypergeometric series.

Proof By the definition of the kernel $\mathcal{A}_p$ and Proposition 3.7 we have

$$\begin{aligned}
\langle \mathcal{A}_p^z, \mathcal{A}_p^w \rangle_{L^2} &= \int_{\mathbb{R}} \left(\sum_{j=0}^{\infty} \frac{z^j \xi_j(x)}{(j+1)^p \sqrt{j!}} \right) \left(\sum_{k=0}^{\infty} \frac{\bar{w}^k \xi_k(x)}{(k+1)^p \sqrt{k!}} \right) dx \\
&= \sum_{j,k=0}^{\infty} \frac{z^j \bar{w}^k}{(j+1)^p (k+1)^p \sqrt{j!}\sqrt{k!}} \int_{\mathbb{R}} \xi_j(x)\xi_k(x)dx \\
&= \sum_{j=0}^{\infty} \frac{z^j \bar{w}^j}{j!(j+1)^{2p}} \\
&= \mathcal{K}_p(z,w) \\
&= {}_{2p}F_{2p}(1,\ldots,1;2,\ldots,2;z\bar{w}).
\end{aligned}$$

∎

The kernel of the $\mathcal{F}_p$-Bargmann transform can be written in a specific way.

Theorem 3.20 *For $p \in \mathbb{N}$ we have*

$$\mathcal{A}_p(z,x) = (R_0 I)^p A(z,x), \qquad \forall (z,x) \in \mathbb{C} \times \mathbb{R}, \tag{3.31}$$

where $A(z,x)$ is the kernel of the $\mathcal{F}_p$-Bargmann transform.

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Proof We observe that $(R_0 I)^p = \frac{z^n}{(n+1)^p}$ then by the definition of the kernel of the classical Bargmann we have

$$\begin{aligned}\mathcal{A}_p(z,x) &= \frac{1}{(2\pi)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{z^n h_n(x)}{(n+1)^p n! 2^{\frac{n}{2}}} \\ &= (R_0 I)^p \left(\frac{1}{(2\pi)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{z^n h_n(x)}{n! 2^{\frac{n}{2}}} \right) \\ &= (R_0 I)^p A(z,x).\end{aligned}$$

■

The following result, see [4, Theorem 4.6], will be useful to write the kernel $\mathcal{A}_p(z,x)$ in another way.

Proposition 3.21 *Given D_0 as (2.14) we have*

$$(IR_0)^n = \sum_{k=1}^{n} \Lambda_{k,n} I^k R_0^k, \tag{3.32}$$

where $\Lambda_{k,n}$ is a diagonal operator defined by means of the following recurrence relation

$$\Lambda_{k,n} := \Lambda_{k-1,n-1} + \sum_{\ell=1}^{k} D_0^{(\ell)} \Lambda_{k,n-1}, \tag{3.33}$$

where $D_0^{(\ell)}$ is the forward diagonal of D_0, see (2.30). The initial values are given by

$$\Lambda_{n,n} = Id \qquad and \qquad \Lambda_{0,n} = diag(0,0,0,\ldots), \quad n \in \mathbb{N},$$

and for the remaining values k, p we have

$$\Lambda_{k,n} = diag(0,0,0,\ldots).$$

Proposition 3.22 *Let $p \in \mathbb{N}$ then for any $z, w \in \mathbb{C}$ we have*

$$\mathcal{A}_p(z,w) = (D_0 + IR_0)^p A(z,w) \tag{3.34}$$

$$= \sum_{j=0}^{p} \sum_{k=1}^{j} D_0^{p-j} \binom{p}{j} \Lambda_{k,j} I^k R_0^k A(z,w), \tag{3.35}$$

where $\Lambda_{k,j}$ are defined in (3.33).

Proof The equality (3.34) follows by Theorem 3.20 and the relation (2.15). Now we show (3.35). By the binomial formula and Proposition 3.21 we have

$$\begin{aligned}\mathcal{A}_p(z,w) &= (R_0 I)^p A(z,w) \\ &= \sum_{j=0}^{p} \binom{p}{j} D_0^{p-j} (IR_0)^j A(z,w) \\ &= \sum_{j=0}^{p} \sum_{k=1}^{j} D_0^{p-j} \binom{p}{j} \Lambda_{k,j} I^k R_0^k A(z,w).\end{aligned}$$

■

The previous result paved the way to write the $\mathcal{F}_p$-Bargmann transform in three different ways.

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Proposition 3.23 *For $p \in \mathbb{N}$ and any $\varphi \in L^2(\mathbb{R})$ we can write the $\mathcal{F}_p$-Bargmann transform as*

$$\begin{aligned}\mathcal{SB}_p(\varphi(z)) &= (R_0 I)^p(\mathcal{B}\varphi(z)) \\ &= (D_0 + IR_0)^p(\mathcal{B}\varphi(z)) \\ &= \sum_{j=0}^{p}\sum_{k=1}^{j} D_0^{p-j}\binom{p}{j}\Lambda_{k,j} I^k R_0^k(\mathcal{B}\varphi(z)),\end{aligned}$$

where D_0 is the diagonal matrix defined in (2.14).

Proof The result follows by Definition 3.18, Theorem 3.20, and Proposition 3.22. ∎

3.4 The Fock space connected to the $\mathcal{F}_p$-space

In this section, we show a connection between the Fock space and $\mathcal{F}_p(\mathbb{C})$. Similarly to what happens for the space $\mathcal{H}_p(\mathbb{C})$ in Section 2.4 the connection is provided by a diagonal operator. Moreover, in this section we further investigate the properties of the $\mathcal{F}_p$-Bargmann transform.

$$\mathcal{V}_p := (R_0 I)^p. \tag{3.36}$$

Theorem 3.24 *Let $p \in \mathbb{N}$. Then the operator $\mathcal{V}_p$ is a surjective isometry from the space $\mathcal{F}(\mathbb{C})$ onto $\mathcal{F}_p(\mathbb{C})$. Moreover the adjoint of $\mathcal{V}_p$ is given by the operator $\mathcal{E}_p$ defined in (2.57).*

Proof Firstly we show that the operator $\mathcal{V}_p$ is an isometric operator. We observe that $\mathcal{V}_p(z^n) = \frac{z^n}{(n+1)^p}$. Therefore for $f \in \mathcal{F}(\mathbb{C})$, such that $f(z) = \sum_{n=0}^{\infty} z^n a_n$, with $(a_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$, we have

$$\mathcal{V}_p(f) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)^p} a_n = \sum_{n=0}^{\infty} z^n \beta_n,$$

where $\beta_n := \frac{a_n}{(n+1)^p}$. Hence we get

$$\|\mathcal{V}_p(f)\|^2_{\mathcal{F}_p(\mathbb{C})} = \sum_{n=0}^{\infty} n!(n+1)^{2p}|\beta_n|^2 = \|f\|^2_{\mathcal{F}(\mathbb{C})}.$$

Now, we show that the operator $\mathcal{V}_p$ is surjective. Let $g \in \mathcal{F}_p(\mathbb{C})$. Hence we have $g(z) = \sum_{n=0}^{\infty} z^n a_n$ with $(a_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$. Our goal is to find a function $g \in \mathcal{F}(\mathbb{C})$ such that

$$g(z) = \mathcal{V}_p(f)(z). \tag{3.37}$$

We have to choose a sequence $(b_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$ such that $g(z) = \sum_{n=0}^{\infty} z^n b_n$ and $\|g\|^2_{\mathcal{F}(\mathbb{C})} < \infty$. If we take $b_n := (n+1)^p a_n$, we have

$$\mathcal{V}_p(f) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)^p} b_n = g(z).$$

Now, since $f \in \mathcal{F}_p(\mathbb{C})$ we have

$$\|g\|^2_{\mathcal{F}(\mathbb{C})} = \sum_{n=0}^{\infty} n!|b_n|^2 = \sum_{n=0}^{\infty} n!(n+1)^{2p}|a_n|^2 < \infty.$$

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Therefore $g \in \mathcal{F}(\mathbb{C})$. Now we show that the adjoint of $\mathcal{V}_p$ is $\mathcal{E}_p$. We have to show that

$$\langle \mathcal{V}_p(f), g\rangle_{\mathcal{F}_p(\mathbb{C})} = \langle f, \mathcal{E}_p(g)\rangle_{\mathcal{F}(\mathbb{C})},$$

for $f \in \mathcal{F}(\mathbb{C})$ and $g \in \mathcal{F}_p(\mathbb{C})$. Let $f(z) = \sum_{n=0}^{\infty} z^n a_n$, $g(z) = \sum_{n=0}^{\infty} z^n b_n$, with $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$ then we have

$$\langle \mathcal{V}_p(f), g\rangle_{\mathcal{F}_p(\mathbb{C})} = \sum_{n=0}^{\infty} n!(n+1)^p a_n \overline{b_n}. \tag{3.38}$$

We observe that $\mathcal{E}_p(z^n) = (n+1)^p z^n$. This implies that

$$\langle f, \mathcal{E}_p(g)\rangle_{\mathcal{F}(\mathbb{C})} = \sum_{n=0}^{\infty} n!(n+1)^p a_n \overline{b_n}. \tag{3.39}$$

Since (3.38) and (3.39) are equal we get the result. ∎

Lemma 3.25 *Let $p \in \mathbb{N}$. Then for any $zw \in \mathbb{C}$ we have*

$$\mathcal{V}_p^{\bar{w}}\mathcal{V}_p^{z}(e^{z\bar{w}}) = {}_2F_{2p}(1, \ldots, 1; 2, \ldots, 2; z\bar{w}),$$

where $\mathcal{V}_p^{\bar{w}}\mathcal{V}_p^{z}$ denote the application of the operator $\mathcal{V}_p$ with respect to the variables $\bar{w}$ and z, respectively.

Proof We start by observing that $\mathcal{V}_p^{\bar{w}}(\bar{w}^n) = \frac{\bar{w}^n}{(n+1)^p}$ and $\mathcal{V}_p^{z}(z^n) = \frac{z^n}{(n+1)^p}$. By Proposition 3.7 we have

$$\begin{aligned}\mathcal{V}_p^{\bar{w}}\mathcal{V}_p^{z}(e^{z\bar{w}}) &= \sum_{n=0}^{\infty} \frac{z^n \bar{w}^n}{n!(n+1)^{2p}} \\ &= \mathcal{K}_p(z, w) \\ &= {}_2F_{2p}(1, \ldots, 1; 2, \ldots, 2; z\bar{w}).\end{aligned}$$

∎

Proposition 3.26 *Let $p \in \mathbb{N}$. Then the $\mathcal{F}_p$-Bargmann transform is a unitary operator and holds that*

$$\|\mathcal{SB}_p(\varphi)\|_{\mathcal{F}_p(\mathbb{C})} = \|\varphi\|_{L^2(\mathbb{R})}, \qquad \textit{and} \qquad \mathcal{B}_p(\xi_n)(z) = \frac{z^n}{(n+1)^p\sqrt{n!}}.$$

Proof Since the Bargmann transform $\mathcal{SB}_p$ is a composition of two unitary operators: the classical Bargmann transform and the operator $\mathcal{V}_p$, see Theorem 3.24, we get

$$\|\mathcal{SB}_p(\varphi)\|_{\mathcal{F}_p(\mathbb{C})} = \|\varphi\|_{L^2(\mathbb{R})}.$$

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Finally by formula (1.5), the fact that $\mathcal{V}_p(z^n) = \frac{z^n}{(n+1)^p}$ and by using another time Proposition 3.23 we get that

$$\mathcal{SB}_p(\xi_n)(z) = \frac{z^n}{(n+1)^p\sqrt{n!}}.$$

■

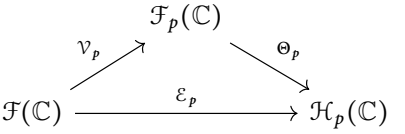
4 A relation between the $\mathcal{H}_p$ and $\mathcal{F}_p$ spaces

In this section we show how the two spaces $\mathcal{H}_p(\mathbb{C})$ and $\mathcal{F}_p(\mathbb{C})$ are connected each other through the Fock space.

Theorem 4.1 *Let $p \in \mathbb{N}$. The operator that maps the space $\mathcal{F}_p(\mathbb{C})$ onto $\mathcal{H}_p(\mathbb{C})$ is given by*

$$\Theta_p = (\mathcal{I} + M_z\partial)^{2p}.$$

Proof By combining Theorems 2.39 and 3.24 we have the following scheme:



Then we can write the operator Θ_p as

$$\Theta_p := \mathcal{E}_p \circ (\mathcal{V}_p)^{-1}.$$

By Theorem 3.24 we know that the operator $\mathcal{V}_p$ is unitary, hence its adjoint coincide with its inverse. So we have

$$\Theta_p = \mathcal{E}_p \circ (\mathcal{V}_p)^{-1} = \mathcal{E}_p \circ \mathcal{E}_p = (\mathcal{I} + M_z\partial)^{2p}.$$

■

Theorem 4.2 *Let $p \in \mathbb{N}$. Then the inverse of the operator Θ_p is given by*

$$\Lambda_p = (\Theta_p)^{-1} = (R_0 I)^{2p}.$$

Therefore, the operator Λ_p maps the space $\mathcal{H}_p(\mathbb{C})$ to $\mathcal{F}_p(\mathbb{C})$.

Proof By Theorem 4.1 we have that

$$\Lambda_p = \left(\mathcal{E}_p \circ (\mathcal{V}_p)^{-1}\right)^{-1} = \mathcal{V}_p \circ (\mathcal{E}_p)^{-1}.$$

By Proposition 2.41 we get

$$\Lambda_p = \mathcal{V}_p \circ \mathcal{V}_p = (R_0 I)^{2p}.$$

■

By means of the operators Θ_p and Λ_p we can relate the reproducing kernel of the spaces $\mathcal{F}_p(\mathbb{C})$ and $\mathcal{H}_p(\mathbb{C})$

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Theorem 4.3 *Let $p \in \mathbb{N}$. Then for every $z, w \in \mathbb{C}$ we have*

$$\Theta_p^{\bar{w}} \Theta_p^z (\mathcal{K}_p(z,w)) = K_p(z,w), \tag{4.1}$$

and

$$\Lambda_p^{\bar{w}} \Lambda_p^z (K_p(z,w)) = \mathcal{K}_p(z,w), \tag{4.2}$$

where we denote by $\Theta_p^{\bar{w}}$, Θ_p^z, $\Lambda_p^{\bar{w}}$, Λ_p^z the operators Θ_p and Λ_p with respect the variables z and $\bar{w}$, respectively.

Proof We start by observing that $\Theta_p^z(z^n) = (n+1)^{2p} z^n$ and $\Theta_p^{\bar{w}}(\bar{w}^n) = (n+1)^{2p} \bar{w}^n$. Now by the definition of the kernel $\mathcal{K}_p(z,w)$ we have

$$\begin{aligned}
\Theta_p^{\bar{w}} \Theta_p^z (\mathcal{K}_p(z,w)) &= \Theta_p^{\bar{w}} \Theta_p^z \left(\sum_{n=0}^{\infty} \frac{z^n \bar{w}^n}{(n+1)^{2p} n!} \right) \\
&= \sum_{n=0}^{\infty} \frac{(n+1)^{2p} z^n \bar{w}^n}{n!} \\
&= K_p(z,w).
\end{aligned}$$

The equality (4.2) follows from the fact that $\Lambda_p^z(z^n) = \frac{z^n}{(n+1)^{2p}}$, $\Lambda_p^{\bar{w}}(\bar{w}^n) = \frac{\bar{w}^n}{(n+1)^{2p}}$ and similar computations performed to get formula (4.1). ∎

The above result implies the following connection between the generalized hypergeometric series and the Touchard polynomials.

Corollary 4.4 *Let $p \in \mathbb{N}$. Then for every $z, w \in \mathbb{C}$ we have*

$$\Theta_p^{\bar{w}} \Theta_p^z ({}_{2p}F_{2p}(1, \ldots, 1; 2, \ldots, 2; z\bar{w})) = \frac{e^{z\bar{w}}}{z\bar{w}} T_{2p+1}(z\bar{w}),$$

and

$$\Lambda_p^{\bar{w}} \Lambda_p^z \left(\frac{e^{z\bar{w}}}{z\bar{w}} T_{2p+1}(z\bar{w}) \right) = {}_{2p}F_{2p}(1, \ldots, 1; 2, \ldots, 2; z\bar{w}),$$

Proof The result follows by Theorem 4.3, Propositions 2.8 and 3.7. ∎

The operators Θ_p and Λ_p are also useful to relate the two type of Bargmann transforms introduced in this paper.

Proposition 4.5 *Let $p \in \mathbb{N}$. Then we have*

$$\Theta_p(\mathcal{SB}_p) = \mathcal{B}_p,$$

and

$$\Lambda_p(\mathcal{B}_p) = \mathcal{SB}_p.$$

Proof By Proposition 3.23 we have

$$\Theta_p(\mathcal{SB}_p) = \Theta_p \mathcal{V}_p(\mathcal{B}).$$

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Since $\Theta_p = \mathcal{E}_{2p}$ and by Theorem 3.24 we have $\mathcal{V}_p = (\mathcal{E}_p)^{-1}$, by using Proposition 2.38 we get

$$\Theta_p(\mathcal{SB}_p) = \mathcal{E}_p(\mathcal{B}) = \mathcal{B}_p.$$

Now by Proposition 2.38 we have

$$\Lambda_p(\mathcal{B}_p) = \Lambda_p\mathcal{E}_p(\mathcal{B}).$$

We observe that $\Lambda_p = \mathcal{V}_{2p}$ and by Proposition 2.41 we have $\mathcal{E}_p = (\mathcal{V}_p)^{-1}$. Therefore by Proposition 3.23 we get

$$\Lambda_p(\mathcal{B}_p) = \mathcal{V}_p(\mathcal{B}) = \mathcal{SB}_p.$$

■

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Schmid College of Science and Technology, Chapman University, California 92866, USA
e-mail: alpay@chapman.edu

Politecnico di Milano, Dipartimento di Matematica, 20133 Milano, Italy
e-mail: antonino.demartino@polimi.it

Clifford research group, Department of Electronics and Information Systems, Faculty of Engineering and Architecture, Ghent University, 9000 Ghent, Belgium
e-mail: Kamal.Diki@UGent.be

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