There are several further properties available with this scenario.

Note finally that the formula $OL \times OM = 2R \times LM$ (circumradius R), may be obtained from a routine calculation in areal coordinates.

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100.26 An example of Aitken acceleration

Introduction

At the beginning of the 13th century Fibonacci posed what is now an extremely well-known problem concerning the growth of a rabbit population under idealised circumstances [1]. In a recent *Gazette* article [2] I consider a logical extension of this problem in which two neighbouring rabbit populations are, in addition to growing, competing for units of grassland. The probabilistic process that results from this scenario gives rise potentially to two random walks involving the Fibonacci numbers. It is with the second of these that we concern ourselves here. This one-dimensional, non-simple random walk on the integers [3] is described in the following two paragraphs, recalling that the kth Fibonacci number F_k is defined by way of the recurrence relation $F_k = F_{k-1} + F_{k-2}$ for $k \geqslant 3$, where $F_1 = F_2 = 1$. The reason for considering the second process rather than the first is that the former leads to a convergent infinite series, whereas the latter does not.

In order that our extended problem be as concise as possible, we may, rather that state it with respect to competing rabbit populations, couch it in terms of an equivalent process involving black and white discs and an urn. To this end, we have an urn initially containing two discs of identical shape and size. One disc is black and the other is white. A marker is placed at position 0, and its first step is determined by picking out a disc at random. If the disc is black then the marker moves one unit in the positive direction to arrive at position 1, while if the disc is white it moves one unit in the negative direction to arrive at position -1. The disc is then put back in the urn. In order to decide the direction of the second step, we add a black disc to the urn, so that it now contains one white disc and two black discs, and withdraw one of these discs at random. This time the marker moves one unit in the negative direction if a black disc is drawn from the urn, and one unit in the positive direction if a white disc is drawn.

More generally, for the kth step, where $k \geqslant 2$, we add a further F_{k-2} white and F_{k-1} black discs to the urn. The direction is subsequently chosen by picking a disc at random from the urn, which now contains F_k white and F_{k+1} black counters. If $k = 2n - 1$ for some $n \in \mathbb{N}$ then the marker

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moves one unit in the positive direction if a black disc is drawn from the urn, and one unit in the negative direction if a white disc is drawn. However, if $k = 2n$ then the marker moves one unit in the negative direction if a black disc is drawn, and one unit in the positive direction if a white disc appears.

Let Y_k denote the random variable representing the position of the marker after the kth step. Using results obtained in [2] it can be shown that the expectation of Y_{2n} is given by

$$\mathrm{E}(Y_{2n}) = -2\left(\frac{1}{F_3F_4} + \frac{1}{F_5F_6} + \ \ldots + \frac{1}{F_{2n+1}F_{2n+2}}\right)$$

for $n \geqslant 1$. For the sake of convenience, however, we consider here the related series

$$S_n = \tfrac{1}{2}\left|\mathrm{E}(Y_{2n})\right| = \frac{1}{F_3F_4} + \frac{1}{F_5F_6} + \ \ldots + \frac{1}{F_{2n+1}F_{2n+2}}. \tag{1}$$

rather than $\mathrm{E}(Y_{2n})$ itself. In particular, we are going to take a brief look at the rate of convergence of the sequence $(S_n)_{n \geqslant 1}$ to

$$S = \sum_{k=1}^{\infty} \frac{1}{F_{2k+1}F_{2k+2}},$$

noting, as a consequence of the exponential-type behaviour of the Fibonacci numbers, that S is a convergent series (with slight abuse of notation, we use S here to denote both the infinite series and its sum).

The question we ask ourselves here is: Can we obtain, with very little in the way of extra effort, a sequence $(\hat{S}_n)_{n \geqslant 1}$ that leads to a dramatic increase in the speed of convergence to S? As will be described in the following sections, we are able to answer in the affirmative by way of a nice application of a process termed *Aitken acceleration* or, alternatively, *Aitken's delta-squared process*.

A candidate for acceleration

The first point to be made here is that, because of the aforementioned behaviour of the Fibonacci numbers, $(S_n)_{n \geqslant 1}$ converges to S relatively quickly without any additional numerical tricks. For example, using a well-known computer algebra package, we obtained the relative error in using S_{10} for S as

$$\frac{S - S_{10}}{S} = 4.4966 \times 10^{-9},$$

correct to five significant figures. As will be seen, however, it turns out that our series $(S_n)_{n \geqslant 1}$ is in fact a candidate for Aitken's delta-squared process, which is a method of accelerating the convergence of series exhibiting *linear convergence*. (By 'accelerate' here we mean increase the rate of convergence.) A sequence $(a_n)_{n \geqslant 1}$ is said to converge linearly to l if there exists some real number $\mu \in (0, 1)$ such that

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$$\lim_{n\to\infty}\frac{l-a_{n+1}}{l-a_n}=\mu.$$

In order to write S_n in a form that is slightly more amenable to analysis than (1), we may employ *Binet's formula* for the nth Fibonacci number [4, 5, 6], which is as follows:

$$F_n=\frac{1}{\sqrt{5}}\left(\alpha^n-\beta^n\right), \tag{2}$$

where

$$\alpha=\frac{1+\sqrt{5}}{2}$$

is known as the *golden ratio* [4, 5, 6] and

$$\beta=\frac{1-\sqrt{5}}{2}.$$

Noting that $\alpha+\beta=1$ and $\alpha\beta=-1$, we obtain, in conjunction with (2),

$$\begin{aligned}F_{2n+3}F_{2n+4}&=\frac{1}{5}\left(\alpha^{2n+3}-\beta^{2n+3}\right)\left(\alpha^{2n+4}-\beta^{2n+4}\right)\\&=\frac{1}{5}\left(\alpha^{4n+7}+1+\beta^{4n+7}\right).\end{aligned}$$

Then, via the formula for the sum to infinity of a geometric series, the fact that $|\beta|<1$ and some rather messy algebra, it may be shown that

$$\lim_{n\to\infty}\frac{S-S_{n+1}}{S-S_n}=\frac{1}{\alpha^4}. \tag{3}$$

It follows from this that $(S_n)_{n\geqslant 1}$ converges linearly to S, and hence that it is suitable for Aitken acceleration.

The acceleration

From (3) we know that

$$\frac{S-S_n}{S-S_{n-1}}\approx\frac{S-S_{n-1}}{S-S_{n-2}}, \tag{4}$$

where the symbol '≈' denotes the fact that, by choosing n sufficiently large, we may make each of these fractions as close to $1/\alpha^4$ as we like. In a sense that may also be made rigorous, a rearrangement of (4) then gives

$$S\approx\frac{S_{n-2}S_n-S_{n-1}^2}{S_{n-2}-2S_{n-1}+S_n}.$$

Finally, the nth term of the accelerated sequence is, for $n\geqslant 3$, given by

$$\hat{S}_n\approx\frac{S_{n-2}S_n-S_{n-1}^2}{S_{n-2}-2S_{n-1}+S_n}.$$

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n	5	10	20	40
$\dfrac{S - S_n}{S}$	6.8020×10^{-5}	4.4966×10^{-9}	1.9651×10^{-17}	3.7529×10^{-34}
$\dfrac{\lvert S - \hat{S}_n \rvert}{S}$	6.3483×10^{-9}	2.7747×10^{-17}	5.2992×10^{-34}	1.9328×10^{-67}

TABLE 1: A comparison of the rates of convergence of $(S_n)_{n \geqslant 1}$ and $(\hat{S}_n)_{n \geqslant 1}$

The numerical entries in Table 1 were obtained using a well-known computer algebra package. They are all correct to five significant figures, and serve to highlight the acceleration that may be achieved using Aitken's delta-squared process. Furthermore, if the accelerated sequence $(\hat{S}_n)_{n \geqslant 1}$ converges linearly, then we can apply the Aitken delta-squared process to it, and so on.

Speaking more generally, it is often the case that a closed-form expression for the sum of an infinite series either does not exist or is not known. In such a situation we would be interested in accelerating the convergence of the series since this would both help reduce the computation time taken to achieve a given accuracy and minimise the effect of compounding errors in calculations involving approximations to various quantities.

Accelerating the convergence of infinite series is just one of the many topics that arise in the vast mathematical field of numerical analysis, and Aitken acceleration is itself just one of several ways in which series may be accelerated. Some methods of acceleration are particularly well suited to certain types of convergent infinite series. We already know, for example, that a series exhibiting linear convergence will be a candidate for Aitken's delta-squared process.

As well as leading to speedier convergence, Aitken acceleration can be used to give good predictions of the error term for a truncated infinite series. Techniques for accelerating the convergence of series sometimes allow us to improve the speed at which numerical integration may be carried out. Finally, series acceleration techniques may also be used, for example, to obtain a variety of identities associated with special functions [7].

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100.27 Proof without words: $\arcsin x + 2\arcsin\sqrt{\dfrac{1-x}{2}} = \dfrac{\pi}{2}$

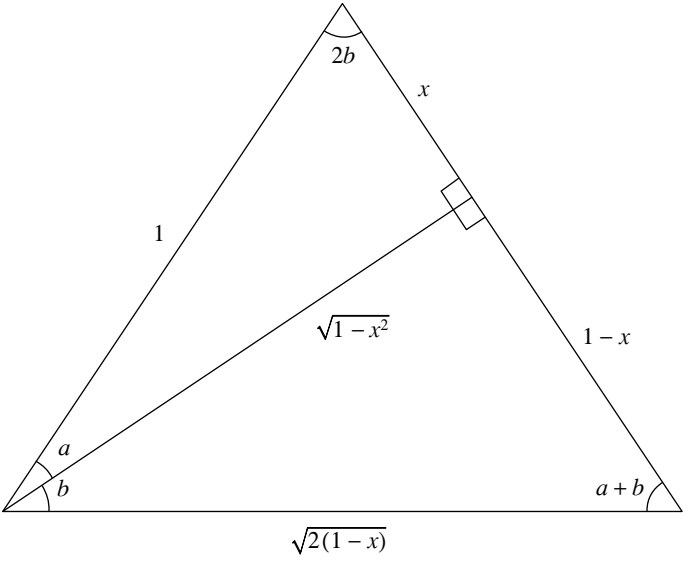


FIGURE 1

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