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Addendum to 'On randomly weighted one-sided ergodic Hilbert transforms'

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I would like to mention here several improvements or additional results to the paper 'On randomly weighted one-sided ergodic Hilbert transforms'. We obtained there that iid random weights in $L \log L$ are universally good for the almost sure convergence of the weighted one-sided ergodic Hilbert transform for any Dunford–Schwartz operator in a probability space (X, Σ, μ) and any function g satisfying $\int_X |g| \log^+ |g| \, d\mu < +\infty$.

It is known [**1**] that in general, one cannot prove any randomly weighted ergodic theorem (i.e. Bourgain's return times theorem) in the duality $L^1 - L^1$. However, we obtain a result in that direction, with a slightly stronger normalization, answering in the course a question of Rosenblatt [**4**]. The method of proof does not apply to Dunford–Schwartz operators, but only to probability-preserving transformations.

We want to prove the following.

THEOREM 1. *Let (X_n) be a sequence of iid random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, symmetric and in $L^1(\mathbb{P})$. Let (u_n) be a strictly increasing sequence of integers satisfying $u_n = O(e^{n^\gamma})$ for some $\gamma \in (0, 1)$. Then there exists a set Ω' with $\mathbb{P}(\Omega') = 1$, such that for any dynamical system (X, Σ, μ, τ) and any g with $g \in L^1(\mu)$, the series*

$$\sum_{n \geq 1} \frac{X_n(\omega) g \circ \tau^{u_n}}{n \log n} \tag{1}$$

converges μ-almost surely for every $\omega \in \Omega'$.

We deduce the following corollaries.

COROLLARY 2. *Let (X_n) be a sequence of iid random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,* centered *and in $L^1(\mathbb{P})$. Then there exists a set Ω' with $\mathbb{P}(\Omega') = 1$, such that when $\omega \in \Omega'$, for any dynamical system (X, Σ, μ, τ) and any g with $g \in L^1(\mu)$, we have*

$$\lim_{n \to +\infty} \frac{1}{n \log n} \sum_{k=1}^{n} X_k(\omega) g \circ \tau^k = 0, \quad \mu\text{-almost surely.}$$

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COROLLARY 3. *There exists a sequence of complex (or real) numbers* (b_n) *with* $\sum_{n\geq 1}|b_n| = +\infty$ *and such that for any dynamical system* (X, Σ, μ, τ) *and any* g *with* $g \in L^1(\mu)$*, the series*

$$\sum_{n\geq 1} b_n g \circ \tau^n,$$

converges μ-almost surely.

Remarks. (1) Theorem 2 of [**2**] admits a corollary similar to Corollary 2; see the proof of Corollary 2 below.

(2) Theorem 1 allows one to obtain 'randomly' the sequence (b_n) in Corollary 3, and not any specific one. We will see that the Rudin–Shapiro sequence (see e.g. [**5**]) is convenient.

Corollary 3 answers a question of Rosenblatt [**4**].

Proof of Theorem 1. Theorem 1 follows easily from the two following lemmata.

LEMMA 4. *Let* (X_n) *be a sequence of iid centered random variables in* L^1*, defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. Let* (u_n) *be a strictly increasing sequence of integers satisfying* $u_n = O(e^{n^\gamma})$ *for some* $\gamma \in (0, 1)$*. Let* $\alpha \in (0, 1)$*. Define*

$$Y_n := X_n \mathbf{1}_{\{|X_n|\leq n^\alpha\}} - \mathbb{E}[X_n \mathbf{1}_{\{|X_n|\leq n^\alpha\}}], \quad \text{for all } n \geq 1.$$

We have:

(i) $\sum_{n\geq 1}(|X_n|/n\log n)\mathbf{1}_{\{|X_n|\geq n^\alpha\}} < +\infty$*,* $\mathbb{P}$*-almost surely;*

(ii) *for every* $\beta > 2/(1-\alpha)$*, we have* $\sup_{N\geq 1}(1/N^{\beta-1})\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}|Y_n| < +\infty$*,* $\mathbb{P}$*-almost surely;*

(iii) *if* $\alpha < 1-\gamma$*, then, for every* $\beta > (1+\alpha)/(1-\alpha-\gamma)$ *we have*

$$\sum_{N\geq 1} N^{\alpha/2} \sup_{t\in\mathbf{R}} \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{Y_n}{n\log n} e^{iu_n t} \right| < +\infty, \quad \mathbb{P}\text{-almost surely.}$$

LEMMA 5. *Let* (a_n) *be a sequence of complex numbers and* (u_n) *be a strictly increasing sequence of integers such that* $u_n = O(e^{n^\gamma})$ *for some* $\gamma \in (0, 1)$*. Assume there exist* $\beta > 0$ *and* $\alpha \in (0, 1)$ *such that:*

(i) $\sup_{N\geq 1}(1/N^{\beta-1})\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}|a_n| < +\infty$*;*

(ii) $\sum_{N\geq 1} N^{\alpha/2}\sup_{t\in\mathbf{R}}\left|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}(a_n/n\log n)e^{iu_n t}\right| < +\infty$.

Then, for every dynamical system (X, Σ, τ, μ) *and every* $g \in L^1$*, the series* $\sum_{n\geq 1}(a_n g \circ \tau^{u_n}/n\log n)$ *converges* μ*-almost surely.*

Proof of Lemma 4. We prove only point (i) since the proofs of points (ii) and (iii) are identical to the corresponding ones in Lemma 5 of [**2**].

The events $\{|X_n| \geq n\}$ take place $\mathbb{P}$-almost surely a finite number of times. Moreover, we have

$$\sum_{n\geq 1}\frac{\mathbb{E}[|X_n|\mathbf{1}_{\{n\geq|X_n|\geq n^\alpha\}}]}{n\log n} \leq \mathbb{E}\left[|X_1| \sum_{1\leq|X_1|\leq n\leq|X_1|^{1/\alpha}} \frac{1}{n\log n}\right]$$
$$\leq C\mathbb{E}[|X_1|(\log\log(|X_1|^{1/\alpha}) - \log\log|X_1|)] \leq C\mathbb{E}[|X_1|].$$

Hence point (i) is proved. □

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Proof of Lemma 5. We want to truncate the function g ($g \in L^1$), in order to use the spectral theorem. Since $g \in L^1$, the events $\{|g \circ \tau^n| \geq n\}$ take place a finite number of times μ-almost surely. (This is the argument which does not pass to Dunford–Schwartz operators, since the previous assertion does not make sense.)

On the other hand, by assumption (i), we have

$$\begin{aligned}
&\sum_{N\geq 1} \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \int_X |g\mathbf{1}_{\{n\geq |g|\geq N^\alpha\}}|(\tau^{u_n}x)\,\mu(dx)\frac{|a_n|}{n\log n} \\
&\qquad \leq \sum_{N\geq 1} \int_X |g\mathbf{1}_{\{(N+1)^\beta\geq |g|\geq N^\alpha\}}|(x)\,\mu(dx) \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{|a_n|}{n\log n} \\
&\qquad \leq C\sum_{N\geq 1} \int_X \frac{1}{N\log N}|g\mathbf{1}_{\{(N+1)^\beta|g|\geq N^\alpha\}}|(x)\,\mu(dx)\left(\frac{1}{N^{\beta-1}}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}|a_n|\right) \\
&\qquad \leq C\int_X |g| \sum_{1\leq |g|^{1/\beta}\leq N\leq |g|^{1/\alpha}} \frac{1}{n\log n}\,d\mu \\
&\qquad \leq C\int_X |g|(\log\log|g|^{1/\alpha} - \log\log|g|^{1/\beta})\mathbf{1}_{\{|g|\geq 1\}}\,d\mu \\
&\qquad \leq C\int_X |g|\,d\mu.
\end{aligned}$$

So $\sum_{N\geq 1}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}(|a_n|(|g|\mathbf{1}_{\{|g|\geq N^\alpha\}})\circ\tau^{u_n}/n) < +\infty$, μ-almost surely. Hence, it suffices to prove that the series

$$\sum_{n\geq 1}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{a_n(g\mathbf{1}_{\{|g|\leq N^\alpha\}})\circ\tau^{u_n}}{n\log n}$$

converges μ-almost surely and that

$$\sup_{[N^\beta]+1\leq n\leq[(N+1)^\beta]}\left|\sum_{k=[N^\beta]+1}^{n}\frac{a_k(g\mathbf{1}_{\{|g|\leq N^\alpha\}})\circ\tau^{u_k}}{k\log k}\right| \to 0, \quad \mu\text{-almost surely.}$$

By the spectral theorem, we obtain

$$\begin{aligned}
&\int_X \left|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{a_n(g\mathbf{1}_{\{|g|\leq N^\alpha\}})\circ\tau^{u_n}(x)}{n\log n}\right|\mu(dx) \\
&\qquad \leq \left(\int_X \left|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{a_n(g\mathbf{1}_{\{|g|\leq N^\alpha\}})\circ\tau^{u_n}}{n\log n}\right|^2\mu(dx)\right)^{1/2} \\
&\qquad \leq \sup_{t\in\mathbf{R}}\left|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\frac{a_n}{n\log n}e^{iu_nt}\right| \|g\mathbf{1}_{\{|g|\leq N^\alpha\}}\|_2 \\
&\qquad \leq \|g\|_1 N^{\alpha/2}\sup_{t\in\mathbf{R}}\left|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\frac{a_n}{n\log n}e^{iu_nt}\right|.
\end{aligned}$$

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By assumption (i), we have

$$\sup_{[N^\beta]+1\le n\le[(N+1)^\beta]}\left|\sum_{k=[N^\beta]+1}^{n}\frac{a_n(g\mathbf{1}_{\{|g|\le N^\alpha\}})\circ\tau^n}{n\log n}\right|\le N^\alpha\frac{1}{N^\beta}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}|a_n|$$
$$\le\frac{1}{N^{1-\alpha}}\sup_{N\ge1}\frac{1}{N^{\beta-1}}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}|a_n|.$$

This concludes the proof of the theorem. □

Proof of Corollary 2. By Kronecker's lemma, Corollary 2 is true for symmetric iid (X_n) in $L^1(\mathbb{P})$.

Now, let (X_n) be iid centered variables in $L^1(\mathbb{P})$. Let (Y_n) be an independent copy of (X_n) defined on a probability space $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$. By Corollary 2 applied to the iid symmetric sequence (X_n-Y_n) on $\Omega\times\Omega_1$, there exists a set $\Gamma\subset\Omega\times\Omega_1$, $\mathbb{P}\otimes\mathbb{P}_1(\Gamma)=1$, such that for every $(\omega,\omega_1)\in\Gamma$, any dynamical system (X,Σ,μ,τ), and any $g\in L^1(\mu)$, we have

$$\lim_{n\to+\infty}\frac{1}{n\log n}\sum_{k=1}^{n}(X_k(\omega)-Y_k(\omega_1))\,g\circ T^k=0,\quad\mu\text{-almost surely.}$$

By Fubini's theorem, there exists a set $\Omega'\subset\Omega$ with $\mathbb{P}(\Omega')=1$, such that, for every $\omega\in\Omega'$, the section $\Lambda_\omega:=\{\omega_1\in\Omega_1:(\omega,\omega_1)\in\Gamma\}$ satisfies $\mathbb{P}_1(\Lambda_\omega)=1$. Fix $\omega\in\Omega'$.

Take a dynamical system (X,Σ,μ,τ) and $g\in L^1(\mu)$. By the pointwise ergodic theorem applied to $\theta_1\times\tau$, where θ_1 is the shift of (Y_n), we have:

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}Y_k\,g\circ\tau^k=0,\quad\mathbb{P}_1\times\mu\text{-almost surely.}$$

In particular, there exists $\omega_1\in\Lambda_\omega$ such that

$$\lim_{n\to+\infty}\frac{1}{n\log n}\sum_{k=1}^{n}Y_k(\omega_1)\,g\circ\tau^k=0,\quad\mu\text{-almost surely.}$$

Hence for our $\omega\in\Omega'$ we have

$$\frac{1}{n\log n}\sum_{k=1}^{n}X_k(\omega)\,g\circ\tau^k=\frac{1}{n\log n}\sum_{k=1}^{n}(X_k(\omega)-Y_k(\omega_1))\,g\circ\tau^k$$
$$+\frac{1}{n\log n}\sum_{k=1}^{n}Y_k(\omega_1)\,g\circ\tau^k\longrightarrow0,$$

which yields Corollary 2. □

Proof of Corollary 3. By Theorem 1, applied, for instance, with $(X_n):=(\varepsilon_n)$ a Rademacher sequence (i.e. $\varepsilon_1=\pm1$ with probability $1/2$), one may take $(b_n):=(\varepsilon_n(\omega)/(n\log n))$ for some $\omega\in\Omega'$.

Now, if one wants a specific sequence, one can use the following.

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LEMMA 6. *Let (a_n) be a sequence of complex numbers such that there exists $\delta \in (0,1)$ such that*

$$\sup_{t\in\mathbf{R}} \left| \sum_{k=1}^{n} a_k e^{ikt} \right| = O(n^{\delta}). \tag{2}$$

Then, for every $\beta > 0$, we have

$$\sup_{t\in\mathbf{R}} \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{a_n}{n\log n} e^{int} \right| = O\left(\frac{1}{N^{\beta(1-\delta)}}\right).$$

Proof. Write $S_n(t) = \sum_{k=1}^{n} a_n e^{int}$ and $M_n = \sup_{t\in\mathbf{R}} |S_n(t)|$. By Abel summation by part we have

$$\begin{aligned}
&\left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{a_n}{n\log n} e^{int} \right| \\
&\quad = \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{S_n(t) - S_{n-1}(t)}{n\log n} \right| \\
&\quad \le C\left(\frac{|M_{[(N+1)^\beta]}|}{(N+1)^\beta \log(N+1)} + \frac{|M_{[N^\beta]}|}{N^\beta \log N} \right. \\
&\qquad \left. + \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]-1} S_n \left(\frac{1}{n\log n} - \frac{1}{(n+1)\log(n+1)} \right) \right| \right) \\
&\quad \le \frac{C}{N^{(1-\delta)\beta}}.
\end{aligned}$$

Taking the supremum over $t \in \mathbf{R}$, we obtain the announced result. □

Let (a_n) be either the Rudin–Shapiro sequence, or the sequence $(e^{in\log n})$. Then (a_n) satisfies the assumption of Lemma 6, see e.g. [**5**] for the Rudin–Shapiro sequence, and see e.g. Theorem V.4.7, Vol. I of [**6**] for the sequence $(e^{in\log n})$. For other sequences, satisfying (2), for some $\delta \in (0,1)$, one can refer to [**3**] and the references therein.

Hence $(b_n) := (a_n/(n\log n))$ satisfies $\sum_{n\ge 1} |b_n| = +\infty$; by Lemma 5, for any dynamical system (X, Σ, μ, τ) and any $g \in L^1(\mu)$, the series

$$\sum_{n\ge 1} b_n g \circ \tau^n$$

converges μ-almost surely.

The case where $(a_n) := (e^{in\log n})$ generalizes Theorem 9 of [**4**].

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