Ergod. Th. & Dynam. Sys. (2007), **27**, 1851–1863
doi: 10.1017/S0143385707000181
© *2007 Cambridge University Press*
Printed in the United Kingdom

On the ergodicity of Weyl sum cocycles

GERNOT GRESCHONIG†, MAHESH NERURKAR‡ and DALIBOR VOLNÝ§

† *Faculty of Mathematics, University of Vienna, Nordbergstraße 15, A-1090 Vienna, Austria*
(e-mail: gernot.greschonig@univie.ac.at)

‡ *Department of Mathematical Sciences, Rutgers University – Camden, Armitage Hall, 311 North 5th Street, Camden, NJ 08102, USA*
(e-mail: nmahesh@crab.rutgers.edu)

§ *Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS, Avenue de l'Université, F-76801 Saint Etienne du Rouvray, France*
(e-mail: dalibor.volny@univ-rouen.fr)

(*Received* 5 *September* 2005 *and accepted in revised form* 7 *December* 2006)

Abstract. We present the quadratic Weyl sums $\sum_{k=0}^{n-1} e^{2\pi i(k^2\theta+2kx)}$ with $\theta, x \in [0, 1)$ as cocycles over a measure-preserving transformation on the two-dimensional torus. We show then that these cocycles are not coboundaries for every irrational $\theta \in [0, 1)$, and that for a dense G_δ set of $\theta \in [0, 1)$ the corresponding skew product is ergodic. For each of those θ, there exists a dense G_δ set of full measure of $x \in [0, 1)$ for which the sequence $\sum_{k=0}^{n-1} e^{2\pi i(k^2\theta+2kx)}$, $n = 1, 2, \ldots$, is dense in $\mathbb{C}$.

1. *Introduction*

In the paper [**HL**], Hardy and Littlewood studied the growth of the series

$$\sum_{k=0}^{n-1} e^{2\pi i(k^2\theta+2kx)}$$

as $n \to \infty$. They showed that, if θ has bounded partial quotients, then there exists a constant $0 < C < \infty$ such that

$$\left|\sum_{k=0}^{n-1} e^{2\pi i(k^2\theta+2kx)}\right| < C\sqrt{n}$$

for all x and all $n \geq 1$ (cf. [**HL**, Theorem 2.25]). In the paper [**Fo1**], Forrest defined these sums as a topological dynamical system and used dynamical methods to prove the following result.

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

THEOREM A. *If $\theta \in [0, 1)$ is irrational and $\liminf_{q\to\infty} q^{3/2}\|q\theta\| < \infty$, then there exists a dense G_δ set of $x \in [0, 1)$ such that the sequence of partial sums*

$$\sum_{k=0}^{n-1} e^{2\pi i(k^2\theta+2kx)}, \quad n = 1, 2, \ldots,$$

is dense in $\mathbb{C}$.

In [**Fo2**], Forrest proved the following theorem.

THEOREM B. *If $\theta \in [0, 1)$ has a continued fraction representation $[0; a_1, a_2, \ldots]$ such that $\sum_{j=1}^{\infty} 1/a_j < \infty$ and $\liminf_{q\to\infty} q^{3+\epsilon}\|q\theta\| = 0$ for some $\epsilon > 0$, then there exists a set of full measure of $x \in [0, 1)$ such that the sequence of partial sums*

$$\sum_{k=0}^{n-1} e^{2\pi i(k^2\theta+2kx)}, \quad n = 1, 2, \ldots,$$

is dense in $\mathbb{C}$.

In proving both these theorems, Forrest considered the partial sums as a cocycle of a compact dynamical system and studied the topological dynamics of the corresponding skew product. In both of his papers, he posed as an open problem the existence of an irrational θ for which the system is ergodic in the measure theoretic sense. The measure theoretic ergodicity (as we show in Theorem 3) is of course a stronger property than its topological analogue proved by Forrest and it implies the density of the sequence of partial sums for a set of $x \in [0, 1)$ of full measure. In the present paper, we show that there exists a dense G_δ set of $\theta \in [0, 1)$ such that the system is ergodic. This set is different from the set in [**Fo2**], as the condition that $\sum_{j=1}^{\infty} 1/a_j < \infty$ in Forrest's result from [**Fo2**] is not necessary here.

The present paper is based on [**VN**]. In [**VN**], the authors used a minoration of $\left|\sum_{k=0}^{n-1} e^{2\pi i(k^2\theta+2kx)}\right|$; this minoration was mistakenly announced by Forrest in [**Fo1**] and attributed to Hardy and Littlewood [**HL**].

2. *Cocycles and ergodicity of skew products*

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space equipped with an ergodic, bijective, bimeasurable, and measure-preserving transformation $S : \Omega \to \Omega$. Let G be an Abelian locally compact second countable group with identity element $\mathbf{0}$ and Haar measure λ_G. A measurable function $f : \Omega \to G$ as well as the mapping $f : \Omega \times \mathbb{Z} \to G$ defined by

$$f(\omega, n) = \begin{cases} \sum_{i=0}^{n-1} f(S^i\omega) & \text{if } n > 0, \\ \mathbf{0} & \text{if } n = 0, \\ -\sum_{i=-n}^{-1} f(S^i\omega) & \text{if } n < 0 \end{cases}$$

will be called a cocycle. The transformation $\mathbf{S}_f : \Omega \times G \to \Omega \times G$ defined by

$$\mathbf{S}_f(\omega, g) = (S\omega, g + f(\omega))$$

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

preserves the product measure $\mu \times \lambda_G$ and is called a skew product (see [**S**]). This transformation is related to the map $f(\omega, n)$ by the equality

$$\mathbf{S}_f^n(\omega, g) = (S^n\omega,\ f(\omega, n) + g)$$

for every $\omega \in \Omega$, $g \in G$, and every integer n.

Definition. We say that $g \in G$ is an *essential value* of a cocycle $f(\omega, n)$ if, for every $B \in \mathcal{A}$ with $\mu(B) > 0$ and for every neighbourhood $\mathcal{U}(g)$ of g, there exists a *non-zero* integer n such that

$$\mu(B \cap S^{-n}B \cap \{x:\ f(n, x) \in \mathcal{U}(g)\}) > 0.$$

A cocycle is called *recurrent* if and only if the identity element $\mathbf{0}$ is an essential value. The set of all essential values of f is called the *essential range*, denoted by $E(f)$.

For a recurrent cocycle, the essential range defined above coincides with the essential range defined in [**S**]. Thus it follows from [**S**, Lemma 3.3 and Corollary 5.4] that for a recurrent cocycle $f(\omega, n)$ the set $E(f) \neq \emptyset$ is a closed subgroup of G, and that the skew product $\mathbf{S}_f$ is ergodic if and only if the cocycle $f(\omega, n)$ has full essential range $E(f) = G$.

Definition. A cocycle $f(\omega, n)$ is called a *coboundary* if there exists a measurable function $b : \Omega \to G$ such that $f(\omega) = b(S\omega) - b(\omega)$ for μ-a.e. $\omega \in \Omega$. It follows then for every integer n that

$$f(\omega, n) = b(S^n\omega) - b(\omega) \quad \text{for } \mu\text{-a.e. } \omega \in \Omega.$$

The essential range of a coboundary is always $\{\mathbf{0}\}$, and a recurrent cocycle $f(\omega, n)$ with $E(f) = \{\mathbf{0}\}$ which is not a coboundary is called a type III_0 cocycle. We can also define essential values in the one-point compactification $\bar{G}$ of the *topological space* G. Then $f(\omega, n)$ is a coboundary if and only if it has only $\mathbf{0}$ as an essential value in $\bar{G}$, and it is of type III_0 if and only if its essential values in $\bar{G}$ are $\mathbf{0}$ and ∞.

In the proof of our main theorem the following result will be necessary.

PROPOSITION 1. [**S**, Proposition 3.8] *Let $f(\omega, n)$ be a cocycle and suppose that K is a compact subset of G such that $K \cap E(f) = \emptyset$. Then for every $B \in \mathcal{A}$ with $\mu(B) > 0$ there exists a set $C \in \mathcal{A}$ with $C \subset B$ and $\mu(C) > 0$ such that*

$$C \cap S^{-n}C \cap \{x:\ f(n, x) \in K\} = \emptyset$$

for every integer $n \neq 0$.

We shall also need a generalization of [**S**, Theorem 11.8], which says that an $\mathbb{R}^d$-valued cocycle is a coboundary if and only if the distributions of $f(\cdot, n)$ are uniformly tight.

LEMMA 1. *Suppose that $f(\omega, n)$ is an $\mathbb{R}^d$-valued cocycle which is not a coboundary, and let $\|\cdot\|$ denote a norm on $\mathbb{R}^d$. Then for every $\epsilon > 0$, $K > 0$, and positive integer N there exists an integer $n \geq N$ such that $\mu(\{\omega \in \Omega : \|f(\omega, n)\| < K\}) < \epsilon$.*

Proof. Assume that the converse is true, i.e. that there exist $\epsilon > 0$, $K > 0$, and a positive integer N with $\mu(\{\omega \in \Omega : \|f(\omega, n)\| < K\}) \geq \epsilon$ for all integers $n \geq N$. If we replace K by a larger number K' and use that

$$\begin{aligned}\mu(\{\omega \in \Omega : \|f(\omega, -n)\| < K'\}) &= \mu(\{\omega \in \Omega : \|f(S^{-n}\omega, n)\| < K'\}) \\ &= \mu(\{\omega \in \Omega : \|f(\omega, n)\| < K'\}),\end{aligned}$$

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

then we can obtain that $\mu(\{\omega \in \Omega : \|f(\omega, n)\| < K'\}) \geq \epsilon$ for all integers n. We fix an integer $1 \leq i \leq d$ and observe that the coordinate $f_i(\omega, n)$ of the cocycle fulfils the condition that $\mu(\{\omega \in \Omega : |f_i(\omega, n)| < K'\}) \geq \epsilon$ for all integers n. Thus the set $B = \Omega \times \{t \in \mathbb{R} : |t| \leq (K'+1)/2\}$ with $(\mu \times \lambda_{\mathbb{R}})(B) = K'+1$ has the property that $(\mu \times \lambda_{\mathbb{R}})(B \cap \mathbf{S}^n_{f_i} B) > \epsilon$ for every integer n. We can conclude that

$$\int \frac{1}{n}\left(\sum_{k=0}^{n-1} \chi_B(\mathbf{S}^k_{f_i}(\omega, t)) \cdot \chi_B(\omega, t)\right) d(\mu \times \lambda_{\mathbb{R}})(\omega, t) > \epsilon$$

for all positive integers n, and thus that

$$(\mu \times \lambda_{\mathbb{R}})\left(\left\{(\omega, t) : \limsup_{n\to\infty} \left|\frac{1}{n}\sum_{k=0}^{n-1} \chi_B(\mathbf{S}^k_{f_i}(\omega, t))\right| > 0\right\}\right) > 0.$$

Let $(Y, \mathcal{S}, \nu)$ be a measure space and $\{q_y : y \in Y\}$ denote an ergodic decomposition of $\mu \times \lambda_{\mathbb{R}}$ with respect to $\mathbf{S}_{f_i}$. Then [**S**, Lemma 11.6] implies that there exists a set $C \in \mathcal{S}$ with $\nu(C) > 0$ such that $q_y(\Omega \times \mathbb{R}) < \infty$ for every $y \in C$. From [**S**, Proposition 11.5] it follows then that $q_y(\Omega \times \mathbb{R}) < \infty$ for every $y \in X$ and that $f_i(\omega, n)$ is a coboundary. As $1 \leq i \leq d$ was arbitrary, every coordinate of the cocycle $f(\omega, n)$ is a coboundary. A contradiction occurs, as the cocycle $f(\omega, n)$ is then a coboundary. □

3. *Ergodicity of Weyl sum cocycles*

Let θ be an irrational number and let $\tau_\theta : \mathbb{T}^2 \to \mathbb{T}^2$ be a transformation defined by

$$\tau_\theta(x, y) = (x + \theta, y + 2x + \theta),$$

the sums being understood modulo 1. The transformation τ_θ is bijective, bimeasurable, and preserves the Lebesgue measure $\lambda_{\mathbb{T}^2}$ on the Borel σ-algebra $\mathcal{B}(\mathbb{T}^2)$. It is the skew product extension of the rotation by θ on $\mathbb{T}$ defined by the cocycle $f : \mathbb{T} \to \mathbb{T}$, $f(x) = 2x + \theta$, and in [**Fu**] it is proved that τ_θ is uniquely ergodic. Notice that

$$\tau^n_\theta(x, y) = (x + n\theta, y + 2nx + n^2\theta),$$
$$\tau^{-n}_\theta(x, y) = (x - n\theta, y - 2nx + n^2\theta),$$

for all integers n. We study the skew product extension $\mathbf{T}_\theta = \tau_{\theta,a} : \mathbb{T}^2 \times \mathbb{C} \to \mathbb{T}^2 \times \mathbb{C}$ of τ_θ defined by the cocycle $a((x, y)) = a((x, y), 1) = e^{2\pi i y}$:

$$\mathbf{T}_\theta((x, y), z) = (\tau_\theta(x, y), z + e^{2\pi i y}).$$

For every integer n, it holds that

$$a((x, y), n) = \begin{cases} \displaystyle\sum_{k=0}^{n-1} e^{2\pi i(k^2\theta + 2kx + y)} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ \displaystyle -\sum_{k=-n}^{-1} e^{2\pi i(k^2\theta + 2kx + y)} & \text{if } n < 0. \end{cases}$$

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

The action of $\mathbb{T}$ on $\mathbb{T}^2$ by the translations $R_v : \mathbb{T}^2 \to \mathbb{T}^2$, $R_v(x, y) = (x, y + v)$ with $v \in \mathbb{T}$ is in the centralizer of τ_θ and preserves $\lambda_{\mathbb{T}^2}$, and for every $v \in \mathbb{T}$ it holds that

$$a((x, y + v), n) = a(R_v(x, y), n) = e^{2\pi i v} a((x, y), n) \tag{1}$$

and hence

$$|a((x, y), n)| = |a((x, 0), n)|. \tag{2}$$

Owing to these properties of the cocycle a, the existence of a non-zero essential value implies that a whole circle is included in $E(a)$. Moreover, we shall use the idea of Aaronson's essential value condition in the next lemma (cf. [**AaLV, Aa, V**]).

LEMMA 2. *Suppose that for each Borel set $B \in \mathcal{B}(\mathbb{T}^2)$ with $\lambda_{\mathbb{T}^2}(B) > 0$ and for each $\epsilon > 0$ there exist $R \in \mathcal{B}(\mathbb{T}^2)$ with $\lambda_{\mathbb{T}^2}(R) > 0$ and a positive integer N such that $\lambda_{\mathbb{T}^2}(B|R) > 1 - \epsilon$, $\lambda_{\mathbb{T}^2}(\tau_\theta^{-N} R|R) > 1 - \epsilon$, and $||a(\cdot, N)| - 1| < \epsilon$ on R. Then the transformation $\mathbf{T}_\theta$ is ergodic.*

Proof. Suppose that the cocycle a has no non-zero finite essential value. Let K be the annulus $K = \{z \in \mathbb{C} : 3/4 \le |z| \le 5/4\}$. By Proposition 1 the compactness of K implies that there exists a Borel set B of positive measure such that

$$B \cap \tau_\theta^{-n} B \cap \{a(\cdot, n) \in K\} = \emptyset \tag{3}$$

holds for every integer n. For this set B and $\epsilon = 1/4$ then choose a set $R \in \mathcal{B}(\mathbb{T}^2)$ of positive measure and a positive integer N as in the assumptions of this lemma. From $\lambda_{\mathbb{T}^2}(B \cap R) > (3/4)\lambda_{\mathbb{T}^2}(R)$ it follows that $\lambda_{\mathbb{T}^2}(\tau_\theta^{-N} B \cap \tau_\theta^{-N} R) > (3/4)\lambda_{\mathbb{T}^2}(R)$, and $\lambda_{\mathbb{T}^2}(\tau_\theta^{-N} R \cap R) > (3/4)\lambda_{\mathbb{T}^2}(R)$ implies then that $\lambda_{\mathbb{T}^2}(\tau_\theta^{-N} B \cap R) > (1/2)\lambda_{\mathbb{T}^2}(R)$. The condition $\lambda_{\mathbb{T}^2}(B \cap R) > (3/4)\lambda_{\mathbb{T}^2}(R)$ now shows that $\lambda_{\mathbb{T}^2}(\tau_\theta^{-N} B \cap B \cap R) > (1/4)\lambda_{\mathbb{T}^2}(R)$, and as $a(\cdot, N) \in K$ on R, a contradiction to equation (3) occurs.

Thus there exists a non-zero finite $z \in E(a)$. We fix $v \in \mathbb{T}$. Then for every $B \in \mathcal{B}(\mathbb{T}^2)$ with $\lambda_{\mathbb{T}^2}(B) > 0$ and every neighbourhood $\mathcal{U}(z)$ of z there exists an integer n such that $\mu(R_v B \cap \tau_\theta^{-n} R_v B \cap \{a(\cdot, n) \in \mathcal{U}(z)\}) > 0$. As the translation R_v and τ_θ commute, equation (1) implies that $\mu(B \cap \tau_\theta^{-n} B \cap \{e^{2\pi i v} a(\cdot, n) \in \mathcal{U}(z)\}) > 0$. Thus $z e^{-2\pi i v}$ is also an essential value of a. As $v \in \mathbb{T}$ was arbitrary and the essential range of a is a closed subgroup of $\mathbb{C}$, it is the whole of $\mathbb{C}$, and hence a is an ergodic cocycle. □

We start our study of the quadratic Weyl sums as measurable dynamical systems with the statement that they can never be coboundaries for an irrational θ. This assertion together with Lemma 1 will also be necessary in the proof of Theorem 2.

THEOREM 1. *If θ is an irrational, then the cocycle $a((x, y), n)$ cannot be a coboundary.*

After having established that the cocycle $a((x, y), n)$ can never be a coboundary for irrational θ, we claim that there exists a topologically large set of irrationals θ such that $E(a) = \mathbb{C}$ and the transformation $\mathbf{T}_\theta$ is ergodic.

THEOREM 2. *There exists a dense G_δ set $V \subset \mathbb{T}$ such that for each $\theta \in V$ the transformation $\mathbf{T}_\theta$ is ergodic.*

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

THEOREM 3. *If the transformation* $\mathbf{T}_\theta$ *is ergodic, then there exists a dense* G_δ *set of* $x \in [0, 1)$ *of full measure, such that the sequence of partial sums*

$$\sum_{k=0}^{n-1} e^{2\pi i(k^2\theta+2kx)}, \quad n = 1, 2, \ldots,$$

is dense in $\mathbb{C}$. *There thus exists a dense* G_δ *set of* $\theta \in [0, 1)$ *such that the sequence of partial sums* $\sum_{k=0}^{n-1} e^{2\pi i(k^2\theta+2kx)}$, $n = 1, 2, \ldots$, *is dense in* $\mathbb{C}$ *for a dense* G_δ *set of* $x \in [0, 1)$ *of full measure.*

For $x \in \mathbb{R}$ we denote by $\|x\|$ the distance of x to the closest integer and by $\{x\}$ the fractional part of x. We denote by $[0; a_1, a_2, \ldots]$ the continued fraction expansion of $\theta \in [0, 1)$, and we recall that the denominators q_n, $n \geq 1$, are defined inductively by

$$q_0 = 1, \quad q_1 = a_1, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \tag{4}$$

and that

$$1/(2q_{n+1}) \leq \|q_n\theta\| \leq 1/q_{n+1}. \tag{5}$$

From $a_i \geq 1$ it follows by induction that $q_n \geq (3/2)^{n-1}$. If q_n is even, then the rotation by $q_n\theta$ on $\mathbb{T}$ is equivalent to the translation by $\|q_n\theta\|$, and if q_n is odd, then it is equivalent to the translation by $-\|q_n\theta\|$. We shall later use that the irrational rotation by θ on $\mathbb{T}$ splits into pairs of Rokhlin towers in a rather simple way. If q_n is odd, then $\mathbb{T}$ splits into the towers defined by the following intervals:

- $[\{j\theta\}, \{(q_{n-1} + j)\theta\})$, for $j = 0, \ldots, q_n - 1$, is the bigger tower; and
- $[\{q_n\theta\}, 1), [\{(q_n + j)\theta\}, \{j\theta\})$, for $j = 1, \ldots, q_{n-1} - 1$, is the smaller tower.

If q_n is even, then we get the following intervals:

- $[\{q_{n-1}\theta\}, 1), [\{(q_{n-1} + j)\theta\}, \{j\theta\})$, for $j = 0, \ldots, q_n - 1$, is the bigger tower; and
- $[\{j\theta\}, \{(q_n + j)\theta\})$, $j = 1, \ldots, q_{n-1} - 1$, is the smaller tower.

By choosing n large enough, these intervals can be made arbitrarily small.

The irrationals $\theta = [0; a_1, a_2, \ldots]$ found in [**Fo2**], for which the sequence of $a(x, n)$ is dense in $\mathbb{C}$ for all x in a set of full measure in $[0, 1)$, have to satisfy the conditions that $\sum_{n=1}^{\infty} 1/a_n < \infty$ and $\liminf_{q\to\infty} q^{3+\epsilon}\|q\theta\| = 0$. Independently of us, and using different arguments, the ergodicity of $\mathbf{T}_\theta$ has recently been proved for this class of irrationals in [**Fa**].

On the one hand, the irrationals $\theta = [0; a_1, a_2, \ldots]$ in our paper are not required to fulfil the condition that $\sum_{n=1}^{\infty} 1/a_n < \infty$, and hence the sequence (a_n) can contain arbitrarily long finite bounded segments. On the other hand, we do not know how rapidly the partial quotients of our irrationals tend to infinity. We prove the existence of a dense set of such irrationals and do not give a sufficient condition for the ergodicity.

4. *Proofs of Theorems 1, 2, and 3*

First we prove Theorem 1, which will be necessary in the proof of Theorem 2.

Proof of Theorem 1. Suppose that the cocycle a is a coboundary and thus the distributions of $a(\cdot, n)$, $n \in \mathbb{N}$, are uniformly tight. For an irrational α, let $S(\alpha) = \{1/\alpha\}$ denote the Gauss map, and define inductively $\theta_0 = \theta$ and $\theta_{i+1} = S(\theta_i)$ for all $i \geq 0$.

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

In [**HL**, Equation (2.1381)], it is shown that there exists a constant $C > 0$ such that for all integers $n \geq 1$ and all $\nu \geq 1$ with $n\theta\theta_1 \cdots \theta_{\nu-1}\theta_\nu < 1$ it holds that

$$|a((x, 0), n)| < \frac{C}{\sqrt{\theta\theta_1 \cdots \theta_{\nu-1}\theta_\nu}}$$

for all $x \in \mathbb{T}$. As $\theta_i\theta_{i+1} < 1/2$ for all $i \geq 0$ (see [**HL**, Equation (2.135)]), for a given $N \geq 1$ we can choose integers $n \geq N$ and $\nu \geq 1$ such that $n\theta\theta_1 \cdots \theta_{\nu-1}\theta_\nu \in (1/2, 1)$, and thus

$$|a((x, 0), n)| < \frac{C\sqrt{n}}{\sqrt{n\theta\theta_1 \cdots \theta_{\nu-1}\theta_\nu}} < C\sqrt{2n}$$

for all $x \in \mathbb{T}$. Thus equation (2) shows that there are arbitrarily large integers n such that $\|a(\cdot, n)\|_\infty < C\sqrt{2n}$. The uniform tightness of the distributions implies that there exists a constant $K > 0$ with

$$\lambda_{\mathbb{T}^2}(\{(x, y) \in \mathbb{T}^2 : |a((x, y), k)| \geq K\}) < \frac{1}{4C^2}$$

for all integers k. On the one hand, for each integer n with $\|a(\cdot, n)\|_\infty < C\sqrt{2n}$ it is valid that

$$\|a(\cdot, n)\|_2^2 < K^2 + \frac{1}{4C^2}2C^2n = K^2 + n/2.$$

On the other hand, from equation (1) it follows that $a((x, y), n)\overline{a((x, y), n)} = a((x, 0), n)\overline{a((x, 0), n)}$, and thus the L^2-norm of $a(\cdot, n)$ can be computed by integrating with respect to the x-coordinate only. The orthogonality of the functions $e^{2\pi i 2kx}$ on $\mathbb{T}$ implies that $\|a(\cdot, n)\|_2^2 = n$, and a contradiction occurs for all $n > 2K^2$. □

Proof of Theorem 2. We let $\theta = [0; a_1, a_2, \ldots]$ be an irrational and denote its sequence of denominators by (q_n). We assume that there exist two increasing sequences of integers $1 \leq n_k \nearrow \infty$ and $1 \leq L_k \nearrow \infty$, and a sequence of sets $D_k \in \mathcal{B}(\mathbb{T})$ such that $\liminf_{k\to\infty} \lambda_{\mathbb{T}^2}((D_k \times \mathbb{T}) \cap B) \geq \lambda_{\mathbb{T}^2}(B)/2$ for each $B \in \mathcal{B}(\mathbb{T}^2)$. We put $d_k = 4^k$, $\eta_k = 2^{-k}$, and

$$\delta_k = L_k^2 \|q_{n_k}\theta\|.$$

We suppose that, for every $k \geq 1$, with $n = n_k$, the following hold:

(i) we have

$$\delta_k < \min\left\{\frac{\eta_k}{4\pi k L_k q_n^2}, \frac{1}{8\pi k L_k^2 q_n}\right\},$$

and hence $L_k^2\|q_n^2\theta\| < \eta_k/k$;

(ii) for each x from a set $B_k \in \mathcal{B}(\mathbb{T})$ with $\lambda_{\mathbb{T}}(B_k) > 1 - 1/k^2$, the maximal distance between neighbouring numbers,

$$q_n \frac{e^{4\pi i A q_n x} - 1}{e^{4\pi i q_n x} - 1} \quad \text{for } 0 \leq A \leq L_k,$$

is smaller than $1/k$, and moreover $\max\{|e^{4\pi i A q_n x} - 1|,\ 0 \leq A \leq L_k\} > 1/d_k$;

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

(iii) if, for each integer $0 \le A \le L_k$, we define a function

$$f_{A,k}(x) = \begin{cases} A \sum_{j=0}^{q_n-1} e^{2\pi i(j^2\theta+2jx)} & \text{if } e^{4\pi i q_n x} = 1, \\ \dfrac{e^{4\pi i A q_n x} - 1}{e^{4\pi i q_n x} - 1} \sum_{j=0}^{q_n-1} e^{2\pi i(j^2\theta+2jx)} & \text{if } e^{4\pi i q_n x} \neq 1, \end{cases}$$

then for each $0 \le A \le L_k$ and each $x \in \mathbb{T}$ we have

$$|a((x, 0), Aq_n) - f_{A,k}(x)| = \left| \sum_{j=0}^{Aq_n-1} e^{2\pi i(j^2\theta+2jx)} - f_{A,k}(x) \right| < \frac{1}{k};$$

(iv) for each $0 \le A \le L_k$, and for any $x', x'' \in \mathbb{T}$ with $\|x' - x''\| \le 2\delta_k$, it is valid that $|f_{A,k}(x') - f_{A,k}(x'')| < 1/k$;

(v) for each $x \in D_k$ we have

$$\left| \frac{1}{e^{4\pi i q_n x} - 1} \sum_{j=0}^{q_n-1} e^{2\pi i(j^2\theta+2jx)} \right| \ge d_k.$$

Assuming the above requirements on θ to be satisfied, we will now verify all the conditions in Lemma 2. For every $k \ge 1$ we define a covering $\mathcal{P}_k$ of $\mathbb{T}^2$ by rectangles

$$R_k(x, y) = [x - \delta_k, x + \delta_k] \times [y - \eta_k, y + \eta_k].$$

Let $B \in \mathcal{B}(\mathbb{T}^2)$ be of positive measure and let $\epsilon \in (0, 1)$. Lebesgue's density theorem in dimension two shows that $\lambda_{\mathbb{T}^2}(B|R_k(x, y)) \to \chi_B(x, y)$ *in measure*, as $k \to \infty$, because $\delta_k, \eta_k \searrow 0$. For all k large enough it follows that

$$\lambda_{\mathbb{T}^2}(\{(x, y) \in \mathbb{T}^2 : |\lambda_{\mathbb{T}^2}(B|R_k(x, y)) - \chi_B(x, y)| \ge \epsilon\}) < \lambda_{\mathbb{T}^2}(B)/8,$$

from (ii) and (iv) it follows that $\lambda_{\mathbb{T}^2}(B_k \times \mathbb{T}) > 1 - \lambda_{\mathbb{T}^2}(B)/8$, and from (v) that $\lambda_{\mathbb{T}^2}((D_k \times \mathbb{T}) \cap B) > (3/8)\lambda_{\mathbb{T}^2}(B)$. Thus we can fix an integer $k > 3/\epsilon$ with $L_k > 1/\epsilon$ and a point $(x, y) \in B \cap (D_k \times \mathbb{T}) \cap (B_k \times \mathbb{T})$ such that $\lambda(B|R_k(x, y)) > 1 - \epsilon$. Then for $n = n_k$ the following are true:

- for every $0 \le A \le L_k$ the function $a((\cdot, 0), Aq_n)$ is approximated by the function $f_{A,k}$ with an error less than $\epsilon/3$ (cf. (iii));
- for every $0 \le A \le L_k$ the function $f_{A,k}(z)$ remains, for $z \in [x - \delta_k, x + \delta_k]$, within a fixed ball of diameter $\epsilon/3$ (cf. (iv));
- for every $z \in B_k \cap D_k$ the maximal distance between the values of $f_{A,k}(z)$ for $0 \le A \le L_k$ is smaller than $\epsilon/3$, because $\left|\sum_{j=0}^{q_n-1} e^{2\pi i(j^2\theta+2jz)}\right| \le q_n$ independently of z and θ (cf. (ii)), and for some $0 \le A \le L_k$ it is true that $|f_{A,k}(z)| \ge 1$ (cf. (ii) and (v)).

It follows that we can fix some $0 \le A \le L_k$, such that $||f_{A,k}(x)| - 1| < \epsilon/3$ and therefore $||a(\cdot, Aq_n)| - 1| < \epsilon$ on $R_k(x, y)$. Then it remains to show that $\lambda_{\mathbb{T}^2}(\tau_\theta^{-Aq_n} R|R) > 1 - \epsilon$. Recall that

$$\tau_\theta^{-Aq_n}(x, y) = (x - Aq_n\theta, y - 2Aq_n x + A^2 q_n^2 \theta).$$

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

By the definition of δ_k, it is true that $\|Aq_n\theta\| \le L_k\|q_n\theta\| \le \delta_k/L_k < \delta_k\epsilon$, and by (i) that $\|A^2q_n^2\theta\| \le L_k^2\|q_n^2\theta\| < \eta_k/k < \eta_k\epsilon/3$. Because $|f_{A,k}(x)| < 1+\epsilon/3$, we have from (v) that

$$|e^{4\pi iAq_nx} - 1| \le \frac{1}{d_k}(1+\epsilon/3) < 2\eta_k^2 \le \eta_k/k < \eta_k\epsilon/3.$$

From $|e^{4\pi iAq_nx} - e^{4\pi iAq_nz}| \le 4\pi Aq_n\|x-z\|$, $\delta_k < \eta_k/(4\pi kL_kq_n)$, and $A \le L_n$, it follows for all $z \in [x-\delta_k, x+\delta_k]$ that $|e^{4\pi iAq_nz} - 1| < \eta_k 2\epsilon/3$ and thus that $\|2Aq_nz\| < \eta_k 2\epsilon/3$. We can conclude that the transformation $\tau_\theta^{-Aq_n}$ translates each point in the rectangle $R_k(x, y) = [x-\delta_k, x+\delta_k] \times [y-\eta_k, y+\eta_k]$ less than $\delta_k\epsilon$ in the first coordinate and less than $\eta_k\epsilon$ in the second coordinate, and therefore $\lambda_{\mathbb{T}^2}(\tau_\theta^{-Aq_n}R|R) > 1-\epsilon$. As $B \in \mathcal{B}(\mathbb{T}^2)$ and $\epsilon \in (0,1)$ were arbitrary, Lemma 2 proves the ergodicity of $\mathbf{T}_\theta$.

It remains, given irrational $\theta_0 \in \mathbb{T}$ and $\rho > 0$, to find an irrational θ with $\|\theta - \theta_0\| < \rho$ and the sequences (n_k), (L_k), and (D_k) such that the conditions (i)–(v) are satisfied for every $k \ge 1$. The irrational θ will be defined by successively choosing the partial quotients (pq). For an irrational $\theta = [0; a_1, a_2, \ldots]$ and $k \ge 1$ we put

$$\mathrm{pq}_k(\theta) = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

and let $n_0 \ge 1$ be such that every θ with $\mathrm{pq}_{n_0}(\theta) = \mathrm{pq}_{n_0}(\theta_0)$ fulfils $\|\theta - \theta_0\| < \rho$.

We shall prove inductively that for every $k \ge 0$ there exist an irrational θ_k with $\mathrm{pq}_{n_0}(\theta_k) = \mathrm{pq}_{n_0}(\theta_0)$ and finite sequences (n_j) and (L_j), for $1 \le j \le k$, such that the conditions (i) and (ii) hold for all $1 \le j \le k$. After this induction process we shall prove that (i) implies (iii) and (iv), and the sequence (D_k) will be defined then by means of the sequence (l_k) obtained in the induction process.

For $k = 0$ there is nothing left to prove. For an integer $k \ge 1$ let θ_{k-1} and (n_j) and (L_j) for $1 \le j \le k-1$ be given. We conclude from Theorem 1 and Lemma 1 that there exists a positive integer l_k such that the cocycle $a_{k-1}((x, y), n)$, which is defined by the irrational θ_{k-1}, fulfils

$$\lambda_{\mathbb{T}}(x : |a_{k-1}((x, 0), l_k)| > 2d_k + 4) > 1 - 2^{-k}.$$

By increasing the lower bound $2d_k + 4$ with a positive integer N, such that the set $\{\{n\theta_{k-1}\} : |n| \le N\}$ is 2^{-k}-dense in $\mathbb{T}$, and using that $|a_{k-1}((x, y), N)| \le N$, we can also assume that $\|l_k\theta_{k-1}\| < 2^{-k}$. We choose an integer $n_k > n_{k-1}$ such that the partial denominator $q_{n_k}^{(k-1)}$ of θ_{k-1} is bigger than l_k, and such that for every irrational θ with $\mathrm{pq}_{n_k}(\theta) = \mathrm{pq}_{n_k}(\theta_{k-1})$ the condition (i) is valid for all $1 \le j \le k-1$, and the inequalities

$$\lambda_{\mathbb{T}}(x : |a((x, 0), l_j)| > 2d_j + 4) > 1 - 2^{-j} \tag{6}$$

and $\|l_j\theta\| < 2^{-j}$ are valid for all $1 \le j \le k$. Indeed, as L_j and q_{n_j} are given, the conditions (i) and $\|l_j\theta\| < 2^{-j}$ are true for all θ in a neighbourhood of θ_{k-1}. Furthermore, the function $\theta \mapsto a((x, 0), l_j)$ is Lipschitz continuous in θ uniformly in x with a Lipschitz constant of at most $\sum_{m=0}^{l_j-1} 2\pi m^2 = \pi(l_j-1)l_j(2l_j-1)/3$, and thus for θ in a small enough neighbourhood of θ_{k-1} equation (6) holds for all $1 \le j \le k$. The condition (ii) for $1 \le j \le k-1$ is not affected here, because $q_{n_0}, \ldots, q_{n_{k-1}}$ are constant. We fix n_k and

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

observe that for all θ with $\mathrm{pq}_{n_k}(\theta) = \mathrm{pq}_{n_k}(\theta_{k-1})$ the partial denominator q_{n_k} is equal to $q_{n_k}^{(k-1)}$. For a set of $x \in \mathbb{T}$ of full measure the sequence $e^{4\pi i A q_{n_k} x}$, $A \geq 0$, is uniformly distributed on the unit circle in $\mathbb{C}$. Thus for $L_k > L_{k-1}$ large enough the condition (ii) holds for $x \in B_k \in \mathcal{B}(\mathbb{T})$ with $\lambda_{\mathbb{T}}(B_k) > 1 - 1/k^2$. Then we choose a_{n_k+1} such that (i) is satisfied. Equations (4) and (5) imply that $\|q_n\theta\| \leq 1/(a_{n+1}q_n + q_{n-1})$. We put $n = n_k$ and conclude that for each θ with

$$a_{n+1} \geq \max\{4\pi k L_k^3 q_n/\eta_k,\, 8\pi k L_k^4\}$$

the condition (i) is valid. If we select θ_k with $\mathrm{pq}_{n_k+1}(\theta_k) = (\mathrm{pq}_{n_k}(\theta_{k-1}), a_{n_k+1})$, then conditions (i), (ii), $\lambda_{\mathbb{T}}(x : |a_k((x,0), l_j)| > 2d_j + 4) > 1 - 2^{-j}$, $q_{n_j}^{(k)} > l_j$, and $\|l_j\theta_k\| < 2^{-j}$ are valid for all integers $1 \leq j \leq k$.

This induction process defines an irrational θ and sequences (n_k), (L_k), and (l_k) such that (i), (ii), $\lambda_{\mathbb{T}}(x : |a((x,0), l_k)| > 2d_k + 4) > 1 - 2^{-k}$, and $\|l_k\theta\| < 2^{-k}$ are fulfilled for every $k \geq 1$. For condition (iii), put $n = n_k$ and notice that for all $0 \leq j \leq q_n - 1$ and $0 \leq m \leq A \leq L_k$ condition (i) implies that $\|(mq_n + 2j)mq_n\theta\| \leq L_k^2 q_n \|q_n\theta\| + 2L_k q_n \|q_n\theta\| < 1/(2\pi k L_k q_n)$. Thus we have

$$\begin{aligned}&\left|\sum_{j=0}^{Aq_n-1} e^{2\pi i(j^2\theta+2jx)} - \frac{e^{4\pi i A q_n x}-1}{e^{4\pi i q_n x}-1}\sum_{j=0}^{q_n-1} e^{2\pi i(j^2\theta+2jx)}\right| \\ &\quad = \left|\sum_{m=0}^{A-1}\sum_{j=0}^{q_n-1} e^{2\pi i((mq_n+j)^2\theta+2(mq_n+j)x)} - \sum_{m=0}^{A-1}\sum_{j=0}^{q_n-1} e^{2\pi i(j^2\theta+2jx+2mq_nx)}\right| < 1/k,\end{aligned}$$

and (iii) is verified. For every $k \geq 1$ the function

$$f_{A,k}(x) = \sum_{l=0}^{A-1} e^{4\pi i q_n l x} \sum_{j=0}^{q_n-1} e^{2\pi i(j^2\theta+2jx)} = \sum_{l=0}^{A-1}\sum_{j=0}^{q_n-1} e^{2\pi i(j^2\theta+2(q_nl+j)x)}$$

is Lipschitz continuous in x uniformly in θ with a Lipschitz constant less than $4\pi \sum_{l=0}^{A-1} q_n(l+1) = 4\pi q_n A(A+1)/2$, and thus condition (i) implies condition (iv).

Now we want to define (D_k) for an arbitrary integer $k \geq 1$. Given a positive integer M, as in [**Fo2**, Lemma A4], we calculate that

$$\begin{aligned}a((x + l\theta, 0),\, M) &= \sum_{j=0}^{M-1} e^{2\pi i(j^2\theta+2jl\theta+2jx)} \\ &= \sum_{j=0}^{M-1} e^{2\pi i((j+l)^2\theta-l^2\theta+2(j+l)x-2lx)} \\ &= e^{2\pi i(-l^2\theta-2lx)} \sum_{j=l}^{M+l-1} e^{2\pi i(j^2\theta+2jx)}\end{aligned}$$

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

and

$$\begin{aligned}\sum_{j=l}^{M+l-1} e^{2\pi i(j^2\theta+2jx)} &= \sum_{j=0}^{M-1} e^{2\pi i(j^2\theta+2jx)} - \sum_{j=0}^{l-1} e^{2\pi i(j^2\theta+2jx)} + \sum_{j=0}^{l-1} e^{2\pi i((j+M)^2\theta+2(j+M)x)}\\ &= \sum_{j=0}^{M-1} e^{2\pi i(j^2\theta+2jx)} + \sum_{j=0}^{l-1} e^{2\pi i(j^2\theta+2jx)}(e^{2\pi i(M^2\theta+2jM\theta+2Mx)} - 1)\\ &= a((x,0),M) + a((x,0),l)(e^{4\pi iMx} - 1)\\ &\quad + e^{4\pi iMx} \sum_{j=0}^{l-1} e^{2\pi i(j^2\theta+2jx)}(e^{2\pi i(M^2\theta+2jM\theta)} - 1).\end{aligned}$$

Now it follows from $|e^{2\pi i(M^2\theta+2jM\theta)} - 1| \le 2\pi(2l+M)\|M\theta\|$ that

$$|a((x,0),l)| \cdot |e^{4\pi iMx} - 1| \le |a((x+l\theta,0),M)| + |a((x,0),M)| + 2\pi l(2l+M)\|M\theta\|.$$

We obtain for $M = q_{n_k}$ from (i) that $M^2\|M\theta\| < \eta_k/(4\pi k L_k^3)$, and with $l_k < M$ and $\lambda_{\mathbb{T}}(|e^{4\pi iMx} - 1| > 2\eta_k) > 1 - 2^{-k}$ this implies that

$$\lambda_{\mathbb{T}}(\{x : 2\pi l_k(2l_k+M)\|M\theta\|/|e^{4\pi iMx} - 1| < 1\}) > 1 - 2^{-k}.$$

Thus for x in a set $F_k \in \mathcal{B}(\mathbb{T})$ of measure $1 - 2^{-k+1}$ we have that

$$|a(x+l\theta,M)|/|e^{4\pi iMx} - 1| > d_k + 1 \quad \text{or} \quad |a(x,M)|/|e^{4\pi iMx} - 1| > d_k + 1.$$

The irrational rotation T by θ can be split into pairs of Rokhlin towers consisting of arbitrarily small intervals and the function $|a(\cdot,M)|/|e^{4\pi iMx} - 1|$ is piecewise continuous with $2M$ discontinuities. Hence we can fix a pair of towers, each of them higher than $2^k l_k$, such that the union G_k of all intervals on which the function $|a(\cdot,M)|/|e^{4\pi iMx} - 1|$ remains in a ball of diameter one has $\lambda_{\mathbb{T}}(G_k) > 1 - 2^{-k}$. Furthermore, the union H_k of all intervals which do not leave their tower under T^{l_k} has also $\lambda_{\mathbb{T}}(H_k) > 1 - 2^{-k}$, due to the height of both towers being larger than $2^k l_k$. Let $I, TI, \ldots, T^{K-1}I$ be one of the towers, i.e. $K = q_p$ or $K = q_{p-1} - 2$ for some positive integer p. For any $0 \le j < l_k + j \le K - 1$ with $T^j I \cap F_k \cap G_k \cap H_k \ne \emptyset$ we have that

$$|a(\cdot,M)|/|e^{4\pi iMx} - 1| > d_k \quad \text{on } T^j I \text{ or on } T^{j+l_k} I.$$

Thus we can find a set $D_k = F_k \cap G_k \cap H_k \in \mathcal{B}(\mathbb{T})$ with $\lambda_{\mathbb{T}}(D_k) > 1/2 - 2^{-k+3}$ such that $|a(x,M)|/|e^{4\pi iMx} - 1| > d_k$ on D_k. The σ-algebras generated by the partial partitions of $\mathbb{T}$ into couples $T^j I \cup T^{j+l_k} I$ of floors of the Rokhlin towers converge to $\mathcal{B}(\mathbb{T})$ as $k \to \infty$, because $\|l_k\theta\| < 2^{-k}$. Hence for any interval $J \subset \mathbb{T}$, $\limsup_{k\to\infty} \lambda_{\mathbb{T}}(D_k \cap J) \ge \lambda_{\mathbb{T}}(J)/2$, and this guarantees (v).

We therefore get a dense (but countable) set Θ of rotations θ for which the skew product $\mathbf{T}_\theta$ is ergodic. For each $\theta \in \Theta$ and $k \ge 1$ we have constructed a family $\mathcal{P}'_k = (D_k \times \mathbb{T}) \cap \mathcal{P}_k$ of rectangles $R_k(x,y) = [x - \delta_k, x + \delta_k] \times [y - \eta_k, y + \eta_k]$ such that $\lambda_{\mathbb{T}^2}(B \cap \bigcup \mathcal{P}'_k) \to \lambda_{\mathbb{T}^2}(B)/2$ for each $B \in \mathcal{B}(\mathbb{T}^2)$, and for each $R \in \mathcal{P}'_k$ there exists an integer $N \ge 1$ such that:

- $\lambda_{\mathbb{T}^2}(\tau_\theta^{-N} R | R)$ is close to 1; and
- $|a(\cdot,N)|$ is close to 1 on R.

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

Notice that for $j \gg k$ the sets $\bigcup \mathcal{P}'_j$ and $\bigcup \mathcal{P}'_k$ are practically independent. Taking a union of several $\mathcal{P}'_j$, $j \geq k$, we thus get a family $\mathcal{R}_k$ of rectangles such that:

- $\lambda_{\mathbb{T}^2}(\bigcup \mathcal{R}_k) > 1 - 1/k$;
- the sides of each $R \in \mathcal{R}_k$ are smaller than $1/k$;
- for each $R \in \mathcal{R}_k$ there exists an integer $N \geq 1$ such that $\lambda_{\mathbb{T}^2}({\tau_\theta}^{-N} R|R) \geq 1 - 1/(2k)$ and $||a(\cdot, N)| - 1| < 1/(2k)$ on R.

Therefore we have an open neighbourhood $U_k(\theta)$ of θ such that for each $\theta' \in U_k(\theta)$ and for each $R \in \mathcal{R}_k$ there exists an integer $N \geq 1$ such that $\lambda_{\mathbb{T}^2}({\tau_{\theta'}}^{-N} R|R) \geq 1 - 1/k$ and $||a(\cdot, N)| - 1| < 1/k$ on R. The sets $V_k = \bigcup_{\theta \in \Theta} U_k(\theta)$ for $k \geq 1$ are dense and open, and by the Baire theorem the intersection $V = \bigcap_{k \geq 1} V_k$ is a dense G_δ subset of $\mathbb{T}$.

Let $\theta \in V$ and $B \in \mathcal{B}(\mathbb{T}^2)$ with $\lambda_{\mathbb{T}^2}(B) > 0$. For an integer $k \geq 1$ sufficiently large and a partial partition $\mathcal{R}_k$ of $\mathbb{T}^2$ by rectangles with sides smaller than $1/k$ we get for most $R \in \mathcal{R}_k$ that $\lambda_{\mathbb{T}^2}(B|R) > 1 - 1/k$ or $\lambda_{\mathbb{T}^2}(B|R) < 1/k$. We have $\theta \in V_k$ and hence $\theta \in U_k(\theta_0)$ for some $\theta_0 \in \Theta$. Taking the partial partition $\mathcal{R}_k$ constructed above we can find a rectangle $R \in \mathcal{R}_k$ and an integer $N \geq 1$ such that $\lambda_{\mathbb{T}^2}(B|R) > 1 - 1/k$, $\lambda_{\mathbb{T}^2}({\tau_\theta}^{-N} R|R) \geq 1 - 1/k$, and $||a(\cdot, N)| - 1| < 1/k$ on R. As the set $B \in \mathcal{B}(\mathbb{T}^2)$ with $\lambda_{\mathbb{T}^2}(B) > 0$ was arbitrary and the integer $k \geq 1$ can be chosen arbitrarily large, it follows by Lemma 2 that the transformation $\mathbf{T}_\theta$ is ergodic.

This completes the proof of Theorem 2. □

Proof of Theorem 3. For each $z \in \mathbb{C}$ and $\epsilon > 0$, the set of $x \in \mathbb{T}$ for which $a((x, 0), n) \notin \mathcal{U}_\epsilon(z)$ for all $n \geq 0$ has measure zero ($\mathcal{U}_\epsilon(z)$ denotes the ϵ-neighbourhood of z). For any countable set C of complex numbers there thus exists a set of $B \subset \mathbb{T}$ of full measure such that for all $x \in B$, $z \in C$ and all $\epsilon > 0$ there exists an $n \geq 0$ with $a((x, 0), n) \in \mathcal{U}_\epsilon(z)$.

From Forrest's proofs in [**Fo1**] it follows that if for each $z \in \mathbb{C}$, for each set $B \in \mathcal{B}(\mathbb{T}^2)$ of positive measure and $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $\lambda_{\mathbb{T}^2}(B \cap {\tau_\theta}^{-n} B \cap \{a(x, n) \in \mathcal{U}_\epsilon(z)\}) > 0$, then the set of $a(x, n)$ is dense in $\mathbb{C}$ for a dense G_δ set of $x \in \mathbb{T}$. □

Acknowledgements. The authors would like to thank Professor K. Schmidt for advice and encouragement. They would also like to thank an unknown referee for helpful comments.

References

[Aa] J. Aaronson. *An Introduction to Infinite Ergodic Theory (Mathematical Surveys and Monographs, 50)*. American Mathematical Society, Providence, RI, 1997.

[AaLV] J. Aaronson, M. Lemańczyk and D. Volný. A cut salad of cocycles. *Fund. Math.* **157**(2–3) (1998), 99–119.

[Fa] B. Fayad. On the ergodicity of the Weyl sums cocycle. *Acta Arith.* **125**(4) (2006), 305–316.

[Fo1] A. H. Forrest. The limit points of Weyl sums and other continuous cocycles. *J. London Math. Soc. (2)* **54**(3) (1996), 440–452.

[Fo2] A. H. Forrest. Symmetric cocycles and classical exponential sums. *Colloq. Math.* **84/85**(1) (2000), 125–145.

[Fu] H. Furstenberg. Strict ergodicity and transformation of the torus. *Amer. J. Math.* **83** (1961), 573–601.

[HL] G. H. Hardy and J. E. Littlewood. The trigonometric series associated with the elliptic θ-functions. *Acta Math.* **37** (1914), 193–239.

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press

[S] K. Schmidt. *Cocycles on Ergodic Transformation Groups (Macmillan Lectures in Mathematics, 1)*. Macmillan Company of India, Ltd., Delhi, 1977.

[V] D. Volný. Completely squashable smooth ergodic cocycles over irrational rotations. *Topol. Methods Nonlinear Anal.* **22**(2) (2003), 331–344.

[VN] D. Volný and M. Nerurkar. On the ergodicity of Weyl sum cocycles. *Preprint*.

https://doi.org/10.1017/S0143385707000181 Published online by Cambridge University Press