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A Dimensional Splitting of ETD Schemes for Reaction-Diffusion Systems

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Abstract. Novel dimensional splitting techniques are developed for ETD Schemes which are second-order convergent and highly efficient. By using the ETD-Crank-Nicolson scheme we show that the proposed techniques can reduce the computational time for nonlinear reaction-diffusion systems by up to 70%. Numerical tests are performed to empirically validate the superior performance of the splitting methods.

AMS subject classifications: 65M12, 65M15, 65M20

Key words: Exponential time differencing, dimensional splitting, reaction diffusion equations.

1 Introduction

Reaction-diffusion systems are mathematical models with a wide variety of applications, *e.g.* [10, 11, 13, 15, 16]. Reaction-diffusion systems can be reduced to systems of stiff ordinary differential equations by employing a method-of-lines discretization. Numerous time-discretization methods for solving the resulting stiff ODE systems have been reported in the literature. Among these are linearly implicit methods [8, 19, 20], semi-implicit methods [17], and projection methods [18]. More recently Exponential Time Differencing (ETD) schemes have been developed [1, 12, 15], which make use of a single step representation of the evolutionary dynamics followed by an appropriate discretization of the exponentials that arise. Variants of the proposed ETD scheme have been developed over the years which adopt different approximations to the integral resulting from the nonlinear reaction term [2, 3, 5, 6, 12].

For multidimensional systems, the discretized diffusion matrices are often largely sparse with wide bands, which tend to slow down direct solvers during evolution. In [7] an ETD-LOD scheme was developed to reduce the storage requirements for solving

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higher dimensional problems and speed up the evolution through extrapolation of a first order ETD scheme and a simple type of locally one-dimensional (LOD) splitting to achieve second order accuracy. In the present article we are concerned with speed-up through different types of locally one-dimensional-splitting and without using extrapolation, instead, directly working with a second order ETD scheme. In the first approach we break up the PDE into sub-problems and apply a Strang composition of the sub-solution operators to recover the solution. The second method uses an integrating factor substitution to achieve a natural splitting of the PDE along its spatial dimensions. We apply the technique to split the second order ETD-CN [5] and examine its performance in discretizing several two dimensional problems.

2 ETD-schemes

Consider the following semilinear parabolic initial-boundary value problem:

$$\begin{aligned} &u_t+\mathcal{A}u=F(t,u) \quad \text{in } \Omega, \quad t\in(0,T], \\ &u(\cdot,0)=u_0 \qquad\quad \text{in } \Omega, \\ &\mathcal{L}u(\cdot,t)=0 \qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded domain in $\Re^d$ with Lipschitz continuous boundary, $\mathcal{A}$ represents a uniformly elliptic operator and F is a sufficiently smooth, nonlinear reaction term. The operator $\mathcal{A}$ is usually of the form,

$$\mathcal{A}:=-\sum_{j,k=1}^{d}\frac{\partial}{\partial x_j}\left(a_{j,k}(x)\frac{\partial}{\partial x_k}\right)+\sum_{j=1}^{d}b_j(x)\frac{\partial}{\partial x_j}+b_0(x),$$

where the coefficients $a_{j,k}$ and b_j are sufficiently smooth functions on $\overline{\Omega}$, $a_{j,k}=a_{k,j}$, $b_0\geq 0$, and for some $c_0>0$

$$\sum_{j,k=1}^{d}a_{j,k}(\cdot)\xi_j\xi_k\geq c_0|\xi|^2, \quad \text{on } \overline{\Omega}, \quad \text{for all } \xi\in\mathbb{R}^d.$$

The boundary operator is $\mathcal{L}u=u$ (Dirichlet) or $\frac{\partial u}{\partial n}$ (Neumann) boundary conditions. From the Duhamel principle, the solution for (2.1) can be expressed in terms of the recurrence relation

$$u(t_{n+1})=e^{-\mathcal{A}k}u(t_n)+\int_{t_n}^{t_{n+1}}e^{-\mathcal{A}(t_{n+1}-s)}F(s,u(s))ds,$$

which can be simplified further to

$$u(t_{n+1})=e^{-\mathcal{A}k}u(t_n)+\int_{0}^{k}e^{-\mathcal{A}(k-\tau)}F(t_n+\tau,u(t_n+\tau))d\tau, \tag{2.2}$$

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by making the substitution $t_n = nk$, and $s = t_n + \tau$ where $0 \le k \le k_0$ for some $k_0 \in \Re$ and $0 \le n \le N$. Eq. (2.2) is the basis for deriving many ETD schemes.

Methods of various order can be obtained by different discretizations of the integral. Here, we focus on second order ETD schemes, which employ a linear approximation of the nonlinear function

$$F(t_n+\tau,u(t_n+\tau)) \approx F(t_n,u(t_n)) + \tau\left(\frac{F(t_{n+1},u(t_{n+1})) - F(t_n,u(t_n))}{k}\right)$$

to obtain the semi-discrete scheme

$$u_{n+1} = e^{-\mathcal{A}k}u_n + \mathcal{A}^{-1}(I-e^{-\mathcal{A}k})F(u_n) + \frac{\mathcal{A}^{-2}}{k}(k\mathcal{A}-I+e^{-\mathcal{A}k})[F(u_{n+1})-F(u_n)].$$

To achieve a fully explicit scheme a first order approximation of u_{n+1},

$$u^* = e^{-\mathcal{A}k}u_n + \mathcal{A}^{-1}(I-e^{-\mathcal{A}k})F(u_n) \tag{2.3}$$

is used in the reaction term. The semi-discrete scheme then becomes

$$u_{n+1} = e^{-Ak}u_n + A^{-1}(I-e^{-Ak})F(u_n) + \frac{A^{-2}}{k}(kA-I+e^{-Ak})[F(u^*)-F(u_n)]. \tag{2.4}$$

At this point we assume the spatial operator $\mathcal{A}$ has been discretized to a matrix A, through any standard technique such as finite differences, finite elements, spectral methods, etc. There are many approaches to discretize the matrix exponentials that arise above. Here, we focus on using Padé schemes with particularly advantageous properties, *e.g.*, (1,1)-Padé or (0,1)-Padé [5, 6].

3 Dimensional splitting of ETD schemes

We propose two approaches for splitting the system (2.3)-(2.4) for the case where $\mathcal{A}$ is a diffusion operator. In 2D, let $\mathcal{A}u = -Du_{xx} - Du_{yy} =: \mathcal{A}_1u + \mathcal{A}_2u$, where $D \in \Re^m$ is the diffusion coefficient matrix for a system of m reacting species. Denote by A_1 and A_2 the finite difference approximations to $\mathcal{A}_1$ and $\mathcal{A}_2$, which can be expressed in Kronecker product notation as

$$A_1 = I_{m_2} \otimes B_{m_1} \otimes D,$$
$$A_2 = B_{m_2} \otimes I_{m_1} \otimes D$$

on an $m_1 \times m_2$ Cartesian grid. B_n is the usual one-dimensional difference matrix, which has the form

$$B_n = -\frac{1}{h^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix} \in \Re^{n\times n}, \quad h = \frac{1}{n+1}$$

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in the case of homogeneous Dirichlet boundary conditions, and the corresponding form

$$B_n = -\frac{1}{h^2}\begin{pmatrix} -2 & 2 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 2 & -2 \end{pmatrix} \in \Re^{n\times n}, \quad h = \frac{1}{n+1}$$

for homogeneous Neumann boundary conditions. I_{m_2} and I_{m_1} are identity matrices. The diffusion coefficient matrix can be expressed as

$$D = \begin{pmatrix} d_1 & 0 & & \\ 0 & d_2 & \ddots & \\ & \ddots & \ddots & 0 \\ & & 0 & d_m \end{pmatrix}.$$

3.1 Strang splitting

In the first approach we split (2.3)-(2.4) by distributing the nonlinear function evenly in each coordinate direction as

$$u_t = \left(-\mathcal{A}_1 u + \frac{1}{2}F(t,u)\right) + \left(-\mathcal{A}_2 u + \frac{1}{2}F(t,u)\right)$$

and reformulate the initial value problem into the subproblems

$$w_t + \mathcal{A}_1 w = \frac{1}{2}F(t,w), \qquad t_n < t \le t_{n+1}, \quad w(t_n) = w_n, \tag{3.1}$$

$$v_t + \mathcal{A}_2 v = \frac{1}{2}F(t,v), \qquad t_n < t \le t_{n+1}, \quad v(t_n) = w(t_{n+1}), \tag{3.2}$$

with $w_0 = u_0$. Applying Duhamel's principle to each of these subproblems results in the corresponding solution operators:

$$S^1_{\Delta t} w(t_n) = w(t_{n+1}) = e^{-k\mathcal{A}_1} w(t_n) + \frac{1}{2}\int_0^k e^{-(k-\tau)\mathcal{A}_1} F(t_n+\tau, w(t_n+\tau))d\tau,$$

$$S^2_{\Delta t} v(t_n) = v(t_{n+1}) = e^{-k\mathcal{A}_2} v(t_n) + \frac{1}{2}\int_0^k e^{-(k-\tau)\mathcal{A}_2} F(t_n+\tau, v(t_n+\tau)).$$

The one-step solution to (2.3)-(2.4) can then be estimated as

$$u(t_{n+1}) \approx [S^2_{\Delta t} \circ S^1_{\Delta t}] w(t_n),$$

which is known to be first order accurate, referred to here as Strang-simple splitting. The second order accurate estimate

$$u(t_{n+1}) \approx [S^1_{\frac{\Delta t}{2}} S^2_{\Delta t} S^1_{\frac{\Delta t}{2}}] w(t_n)$$

is referred to here as Strang-symmetric splitting. In each case the solution operators can be approximated by an appropriate ETD Scheme to obtain a fully discrete scheme.

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3.2 Integrating factor method

Here we make use of a change of variable to obtain a natural dimensional splitting of ETD schemes. Consider the change of variable

$$v = e^{\mathcal{B}t}u \qquad \text{and} \qquad v_t = e^{\mathcal{B}t}u_t + \mathcal{B}e^{\mathcal{B}t}u.$$

Substituting, $v_t = e^{\mathcal{B}t}F(u) - e^{\mathcal{B}t}\mathcal{A}u + \mathcal{B}e^{\mathcal{B}t}u$. Suppose we choose $\mathcal{B}$ such that $\mathcal{A}$ and $\mathcal{B}$ commute, then

$$v_t + (\mathcal{A}-\mathcal{B})e^{\mathcal{B}t}u = e^{\mathcal{B}t}F(u),$$

or,

$$v_t + (\mathcal{A}-\mathcal{B})v = e^{\mathcal{B}t}F(e^{-\mathcal{B}t}v).$$

In a two dimensional domain the splitting can be done as $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, and, with $\mathcal{B} = \mathcal{A}_1$,

$$v_t + \mathcal{A}_2 v = e^{\mathcal{A}_1 t}F(e^{-\mathcal{A}_1 t}v).$$

Set $e^{\mathcal{A}_1 t}F(e^{-\mathcal{A}_1 t}v) =: \tilde{F}(v)$, thus

$$\begin{aligned} &v_t + \mathcal{A}_2 v = \tilde{F}(v), \quad v(0) = u_0, \\ &u = e^{-\mathcal{A}_1 t}v. \end{aligned} \tag{3.3}$$

By Duhamel's principle on (3.3) and with change of variable $t = t_n + \tau$ it follows that

$$v(t_{n+1}) = e^{-k\mathcal{A}_2}v(t_n) + \int_0^k e^{-\mathcal{A}_2(k-\tau)}\tilde{F}(v(t_n+\tau))d\tau.$$

By a linear approximation of the nonlinear function $\tilde{F}(v(t_n+\tau))$ we generate the second order semi-discrete scheme

$$\begin{aligned} v_{n+1} = {} & e^{-kA_2}v_n + A_2^{-1}(I - e^{-kA_2})\tilde{F}(v_n) + \frac{A_2^{-2}}{k}(A_2 k - I + e^{-A_2 k}) \\ & \times [\tilde{F}(v_{n+1}) - \tilde{F}(v_n)], \end{aligned} \tag{3.4}$$

where $v_n \approx v(t_n)$. From (3.3) $u_n = e^{-A_1 nk}v_n$, we can rewrite the semidiscrete equation in terms of the original variable u_n:

$$\begin{aligned} u_{n+1} = {} & e^{-kA_1}e^{-kA_2}u_n + A_2^{-1}(I - e^{-kA_2})e^{-kA_1}F(u_n) \\ & + \frac{A_2^{-2}}{k}(A_2 k - I + e^{-A_2 k})[F(u^*) - e^{-kA_1}F(u_n)], \\ u^* = {} & e^{-kA_1}e^{-kA_2}u_n + A_2^{-1}(I - e^{-kA_2})e^{-kA_1}F(u_n). \end{aligned} \tag{3.5}$$

The matrix exponentials can be approximated using an appropriate rational approximation to obtain a fully discrete scheme.

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We show that $A_1A_2 = A_2A_1$ by using standard properties of the Kronecker product.

$$\begin{aligned} A_1A_2 &= (I_{m_2}\otimes B_{m_1}\otimes D)(B_{m_2}\otimes I_{m_1}\otimes D) \\ &= I_{m_2}B_{m_2}\otimes B_{m_1}I_{m_1}\otimes D^2 \\ &= B_{m_2}\otimes B_{m_1}\otimes D^2. \end{aligned}$$

Similarly,

$$\begin{aligned} A_2A_1 &= (B_{m_2}\otimes I_{m_1}\otimes D)(I_{m_2}\otimes B_{m_1}\otimes D) \\ &= B_{m_2}I_{m_2}\otimes I_{m_1}B_{m_1}\otimes D^2 \\ &= B_{m_2}\otimes B_{m_1}\otimes D^2. \end{aligned}$$

Application to Crank-Nicolson scheme

In this work we will explore the performance of our splitting techniques with the ETD-CN scheme. By applying the (1,1)-Padé approximation to the matrix exponential we obtain the following estimates

$$\begin{aligned} e^{-Ak} &\approx \left(I-\frac{1}{2}kA\right)\left(I+\frac{1}{2}kA\right)^{-1}, \\ A^{-1}(I-e^{-Ak}) &\approx A^{-1}\left(I-\left(I-\frac{1}{2}kA\right)\left(I+\frac{1}{2}kA\right)^{-1}\right) \\ &= A^{-1}\left(I+\frac{1}{2}kA-I+\frac{1}{2}kA\right)\left(I+\frac{1}{2}kA\right)^{-1} \\ &= k\left(I+\frac{1}{2}kA\right)^{-1}, \end{aligned}$$

and

$$\begin{aligned} \frac{A^{-2}}{k}(Ak-I+e^{-Ak}) &\approx \frac{A^{-2}}{k}\left(Ak-I+\left(I-\frac{1}{2}kA\right)\left(I+\frac{1}{2}kA\right)^{-1}\right) \\ &= \frac{A^{-2}}{k}(Ak+\frac{1}{2}k^2A^2-I-\frac{1}{2}kA+I-\frac{1}{2}kA)\left(I+\frac{1}{2}kA\right)^{-1} \\ &= \frac{1}{2}k\left(I+\frac{1}{2}kA\right)^{-1}. \end{aligned}$$

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Substituting these estimates into (3.5) yields the fully discrete split-ETD-CN scheme:

$$\begin{aligned}
w_{n+1} &= \left(I-\frac{1}{2}kA_1\right)\left(I+\frac{1}{2}kA_1\right)^{-1}\left(I-\frac{1}{2}kA_2\right)\left(I+\frac{1}{2}kA_2\right)^{-1}w_n \\
&\quad +k\left(I+\frac{1}{2}kA_2\right)^{-1}\left(I-\frac{1}{2}kA_1\right)\left(I+\frac{1}{2}kA_1\right)^{-1}F(w_n) \\
&\quad +\frac{k}{2}\left(I+\frac{1}{2}kA_2\right)^{-1}\left[F(w^*)-\left(I-\frac{1}{2}kA_1\right)\left(I+\frac{1}{2}kA_1\right)^{-1}F(w_n)\right], \qquad (3.6)\\
w^* &= \left(I-\frac{1}{2}kA_1\right)\left(I+\frac{1}{2}kA_1\right)^{-1}\left(I-\frac{1}{2}kA_2\right)\left(I+\frac{1}{2}kA_2\right)^{-1}w_n \\
&\quad +k\left(I+\frac{1}{2}kA_2\right)^{-1}\left(I-\frac{1}{2}kA_1\right)\left(I+\frac{1}{2}kA_1\right)^{-1}F(w_n),
\end{aligned}$$

with $w_n \approx u_n$. To reduce the number of matrix solves we reorganize the scheme, taking advantage of commutativity, to arrive at the final scheme:

$$\begin{aligned}
w_{n+1} &= \left(I+\frac{1}{2}kA_2\right)^{-1}\left[\left(I-\frac{1}{2}kA_2\right)\left(I-\frac{1}{2}kA_1\right)\left(I+\frac{1}{2}kA_1\right)^{-1}w_n\right. \\
&\quad \left.+k\left(I-\frac{1}{2}kA_1\right)\left(I+\frac{1}{2}kA_1\right)^{-1}F(w_n)\right] \\
&\quad +\frac{k}{2}\left(I+\frac{1}{2}kA_2\right)^{-1}\left[F(w^*)-\left(I-\frac{1}{2}kA_1\right)\left(I+\frac{1}{2}kA_1\right)^{-1}F(w_n)\right].
\end{aligned}$$

Implementation

We implement the above scheme as follows,

$$\begin{aligned}
&\left(I+\frac{1}{2}kA_1\right)a = \left(I-\frac{1}{2}kA_1\right)w_n, \\
&\left(I+\frac{1}{2}kA_1\right)b = \left(I-\frac{1}{2}kA_1\right)F(w_n), \\
&\left(I+\frac{1}{2}kA_2\right)w^* = \left[\left(I-\frac{1}{2}kA_2\right)a+kb\right], \\
&\left(I+\frac{1}{2}kA_2\right)c = (F(w^*)-b), \\
&w_{n+1} = w^*+\frac{k}{2}c.
\end{aligned}$$

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4 Numerical experiments

We evaluate the performance of the new dimensional splitting techniques for the ETD-CN scheme through empirical tests of convergence and comparing efficiency. We consider problems with both Neumann and Dirichlet boundary conditions in 2D. The accuracy of the scheme in all cases was measured using the L_∞ norm and the rates of convergence calculated by a standard comparison to the next solution in the table. The order is taken as

$$p=\frac{\log(\frac{\tilde{E}(k)}{\tilde{E}(\frac{k}{2})})}{\log(2)}.$$

4.1 Enzyme kinematics

We consider a two-dimensional enzyme kinetics of Michaelis-Menten type:

$$\frac{\partial u}{\partial t}=d\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{u}{(1+u)},\quad 0<x,y<1,\ \ t>0, \tag{4.1}$$

with homogeneous Dirichlet boundary conditions and initial condition

$$u(x,y,0)=\sin(\pi x)\sin(\pi y),\quad 0\leq x,y\leq 1.$$

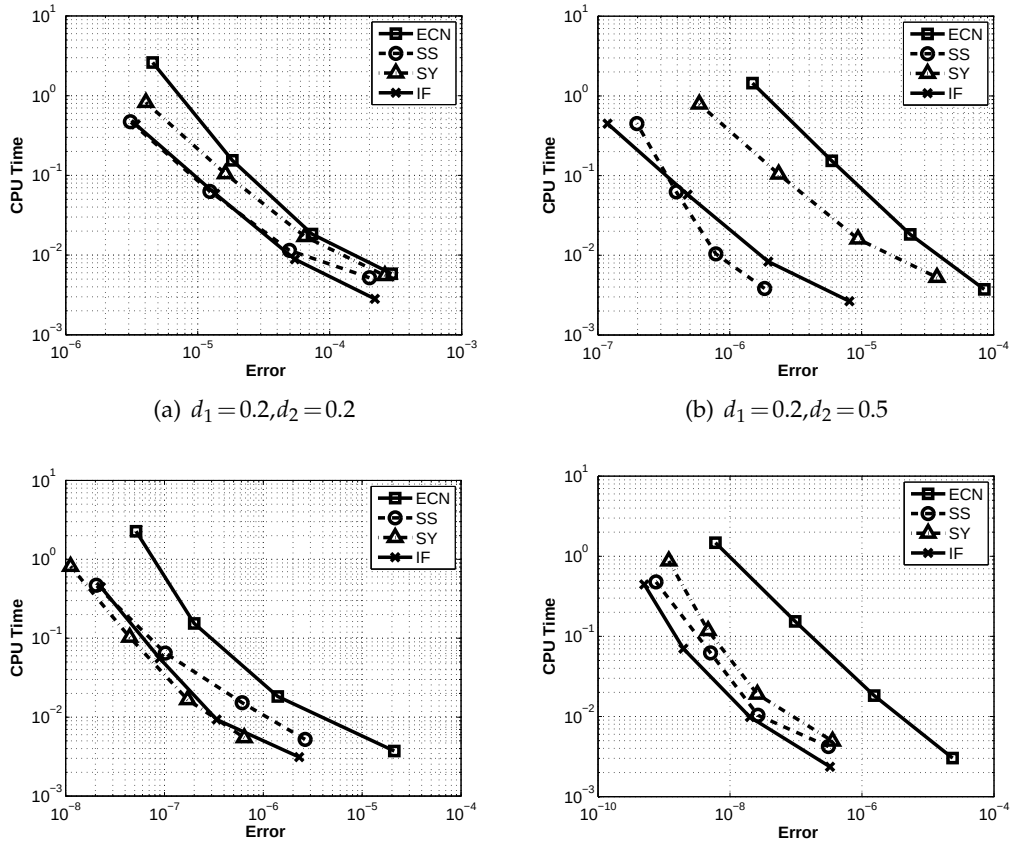
The splitting techniques show second order convergence and are significantly faster than the regular ETD-CN scheme.

As we vary the diffusion coefficient the Strang-simple splitting sometimes becomes first order accurate, as shown in Fig. 1(b). Table 1 shows convergence results for all the schemes (diffusion coefficient, $d=0.2$).

Table 1: Convergence of ETD splitting methods for enzyme kinetics. The subscript ECN refers to ETD-CN scheme, SS the Strang-simple splitting, SY the Strang-symmetric and IF the integrating factor method.

$k=h$	$Error_{ECN}$	Cpu Time	Rate	$Error_{SS}$	Cpu Time	Rate
0.1000	2.9447×10^{-4}	0.00383	-	2.001×10^{-4}	0.00280	-
0.0500	7.3406×10^{-5}	0.01829	2.00	4.9598×10^{-5}	0.00988	2.01
0.0250	1.8283×10^{-5}	0.15366	2.01	1.2372×10^{-5}	0.06054	2.00
0.0125	4.5594×10^{-6}	2.28658	2.00	3.0914×10^{-6}	0.44592	2.00
$k=h$	$Error_{SY}$	Cpu Time	Rate	$Error_{IF}$	Cpu Time	Rate
0.1000	2.6199×10^{-4}	0.00422	-	2.1989×10^{-4}	0.00204	-
0.0500	6.4963×10^{-5}	0.01641	2.01	5.4457×10^{-5}	0.00854	2.01
0.0250	1.6204×10^{-5}	0.09897	2.00	1.3581×10^{-5}	0.0552	2.00
0.0125	4.0483×10^{-6}	0.77917	2.00	3.3929×10^{-6}	0.43117	2.00

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(a) $d_1=0.2, d_2=0.2$ (b) $d_1=0.2, d_2=0.5$

(c) $d_1=0.2, d_2=1.0$ (d) $d_1=0.5, d_2=1.0$

Figure 1: Log-log efficiency plot for enzyme kinetics equation after 1 sec with varying diffusion coefficient. ECN refers to ETD-CN scheme, SS the Strang-simple splitting, SY the Strang-symmetric and IF the integrating factor method

4.2 The 2D Brusselator

We also investigate the computational efficiency of our splitting techniques for Brusselator equation, which is a reaction-diffusion system with Neumann boundary conditions:

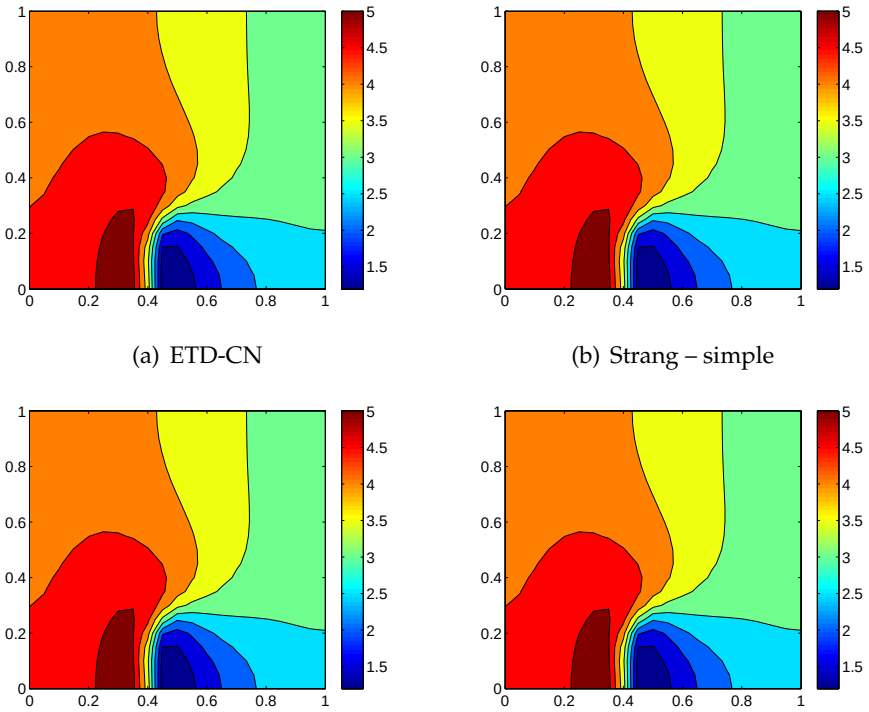
$$u_t = \epsilon_1 \Delta u + A + u^2 v - (B+1)u,$$
$$v_t = \epsilon_2 \Delta v + Bu - u^2 v,$$

where the diffusion coefficients are $\epsilon_1 = \epsilon_2 = 2.10^{-3}$, (chemical parameters) $A=1.0, B=3.4$, and boundary conditions

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = \frac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0, \qquad \text{with} \quad \Omega = [0,1]^2.$$

The initial conditions are $u(x,y,0) = \frac{1}{2} + y$, $v(x,y,0) = 1 + 5x$.

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(a) ETD-CN (b) Strang – simple

(c) Strang – symmetric (d) Integrating Factor

Figure 2: V solution profiles for Brusselator equation after 11 secs.

The splitting methods correctly capture the solution profile (Fig. 2) and attain the expected second order convergence (Table 2). On average we observe an 85% reduction in

Table 2: Convergence of ETD splitting methods for Brusselator after 2 secs. The subscript ECN refers to ETD-CN scheme, SS the Strang-simple splitting, SY the Strang-symmetric and IF the integrating factor method.

$k=h$	$Error_{ECN}$	Cpu Time	Rate	$Error_{SS}$	Cpu Time	Rate
0.1000	3.4490×10^{-1}	0.00373	-	2.7937×10^{-1}	0.01203	-
0.0500	2.3918×10^{-1}	0.01281	0.53	2.1463×10^{-1}	0.01203	0.38
0.0250	8.2716×10^{-2}	0.10076	1.53	7.1545×10^{-2}	0.05287	1.58
0.0125	2.1302×10^{-2}	1.46030	1.96	1.8012×10^{-2}	0.29253	1.99
$k=h$	$Error_{SY}$	Cpu Time	Rate	$Error_{IF}$	Cpu Time	Rate
0.1000	2.7352×10^{-1}	0.00670	-	3.4380×10^{-1}	0.00270	-
0.0500	2.1028×10^{-1}	0.01802	0.38	2.2550×10^{-1}	0.00806	0.61
0.0250	7.2922×10^{-2}	0.07947	1.53	7.6429×10^{-2}	0.03883	1.56
0.0125	1.8468×10^{-2}	0.43406	1.98	1.9321×10^{-2}	0.22363	1.98

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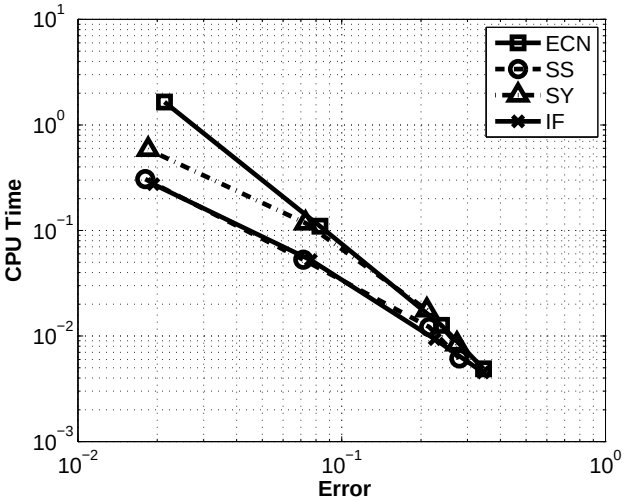


Figure 3: Log-log efficiency plot for the Brusselator equation after 2 secs. ECN refers to ETD-CN scheme, SS the Strang-simple splitting, SY the Strang-symmetric and IF the integrating factor method.

the computational time across all the methods due to splitting. Fig. 3 shows the improved efficiency in using the splitting methods.

4.3 Schnakenberg model

Finally, we examine performance of our scheme on the 2D Schnakenberg reaction-diffusion system. It is an auto catalytic chemical reaction,

$$\begin{aligned} u_t &= d_1\Delta u+\gamma(a-u+u^2v), \\ v_t &= d_2\Delta v+\gamma(b-u^2v), \end{aligned} \tag{4.2}$$

where the diffusion coefficients are $d_1=1.0$, $d_2=10$, and the chemical parameters are $a=0.126779$, $b=0.792366$, $\gamma=1000$. At the boundary of the spatial domain Ω Neumann conditions are imposed:

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega}=\frac{\partial v}{\partial n}\Big|_{\partial\Omega}=0, \qquad \text{with} \quad \Omega=[0,1]^2.$$

The initial conditions are

$$u(x,y,0)=0.919145+0.0016\cos(2\pi(x+y))+0.01\sum_{j=1}^{8}\cos(2\pi jx),$$

$$v(x,y,0)=0.937903+0.0016\cos(2\pi(x+y))+0.01\sum_{j=1}^{8}\cos(2\pi jx).$$

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Table 3: Convergence of ETD splitting methods for Schnakenberg after 0.2 secs. The subscript ECN refers to ETD-CN scheme, SS the Strang-simple splitting, SY the Strang-symmetric and IF the integrating factor method.

k	$Error_{ECN}$	Cpu Time	Rate	$Error_{SS}$	Cpu Time	Rate
$(0.025)^3$	7.1328×10^{-5}	15.4974	-	4.2561×10^{-4}	10.6411	-
$\frac{(0.025)^3}{2}$	1.8112×10^{-5}	30.3794	1.98	2.2344×10^{-4}	19.4923	0.93
$\frac{(0.025)^3}{4}$	4.5632×10^{-6}	60.8496	1.99	1.1432×10^{-4}	39.9136	0.97
$\frac{(0.025)^3}{8}$	1.1453×10^{-6}	119.7363	1.96	5.7801×10^{-5}	76.8786	0.98
k	$Error_{SY}$	Cpu Time	Rate	$Error_{IF}$	Cpu Time	Rate
$(0.025)^3$	3.2709×10^{-5}	18.0818	-	2.9885×10^{-3}	14.5705	-
$\frac{(0.025)^3}{2}$	8.8089×10^{-6}	33.1315	1.89	7.5931×10^{-4}	15.5848	1.98
$\frac{(0.025)^3}{4}$	2.2842×10^{-6}	61.6392	1.95	1.9165×10^{-4}	30.2865	1.99
$\frac{(0.025)^3}{8}$	5.8149×10^{-6}	120.9249	1.97	4.8162×10^{-5}	57.56133	1.99

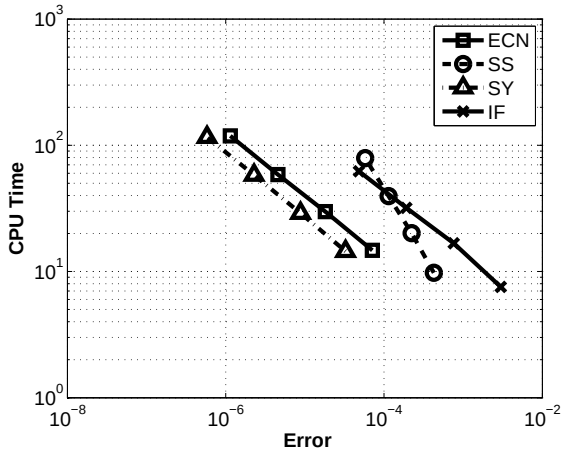


Figure 4: Log-log efficiency plot for the Schnakenberg equation after 0.2 secs. ECN refers to ETD-CN scheme, SS the Strang-simple splitting, SY the Strang-symmetric and IF the integrating factor method.

Again, the splitting methods all replicated the solution profile and yield the expected orders of convergence (Table 3). The Strang-simple splitting was only first order convergent. The integrating factor method remains second order convergent, but had larger errors compared to the Strang-symmetric splitting. We associate this to the affect of diffusion coefficients.

5 Conclusion

We have derived two different techniques for dimensional splitting of exponential time differencing schemes. By comparing to the standard ETD-CN scheme on several test

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problems, we demonstrated empirically that the methods converge at the expected rates of convergence. The proposed splitting methods significantly reduce the computational time required in evolving reaction-diffusion systems.

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