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# Finiteness of compact maximal flats of bounded volume

HEE OH†
*Mathematics Department, Princeton University, Princeton, NJ 08544, USA*
*and*
*School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA*
*(e-mail: heeoh@its.caltech.edu)*

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*Abstract*. Let $M$ be a complete Riemannian locally symmetric space of non-positive curvature and of finite volume. We show that there are only finitely many compact maximal flats in $M$ of volume bounded by a given number. As a corollary in the case $M = \mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$, we give a different proof of a theorem of Remak that for any $n \in \mathbb{N}$, there are only finitely many totally real number fields of degree $n$ whose regulator is less than a given number.

## 1. *Introduction*

Let $M$ be a complete Riemannian locally symmetric space of non-positive curvature and of finite volume. A flat in $M$ is a complete totally geodesic submanifold of sectional curvature zero. A maximal flat means a flat of maximal dimension, i.e. of dimension equal to the rank of $M$. On each flat of $M$, we have an induced volume form.

The main aim of this paper is to prove the following.

THEOREM 1.1. *For any given $c > 0$, the number of compact maximal flats in $M$ of volume less than $c$ is finite.*

In the case when $M$ is compact, Theorem 1.1 was proved earlier by Spatzier [**Sp**].

In proving the above theorem, by applying the theorem of Eberlein [**Eb**, Theorem 7.3.3], we may assume that $M$ has no local Euclidean de Rham factor. We may further assume, without loss of generality, that $M$ is irreducible, in the sense that there exists no finite covering of $M$ which is a direct product (as Riemannian manifolds) of locally symmetric spaces of positive dimension [**He**].

Then $M$ is of the form $\Gamma\backslash G/K$, where $G$ is a connected center free semisimple real algebraic group with no compact factors, $K$ is a maximal compact subgroup of $G$ and $\Gamma$ is

† Current address: 253-37 Mathematics, Caltech, Pasadena, CA 91125, USA.

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an irreducible torsion free lattice in $G$, and the metric on $M$ is induced from a left-invariant Riemannian metric on $G/K$ [**He**].

In the case when the real rank of $G$ is one, that is when $M$ has negative curvature, a compact maximal flat in $M$ is just a primitive closed geodesic in $M$. In this case, even the precise asymptotic for the number of (primitive) closed geodesics of length less than $c$, as $c \to \infty$, is known by Margulis [**Ma1**] for $M$ compact and by Gangolli and Warner [**GW**] otherwise. The latter is deduced from Selberg trace formula.

Hence we may assume that the real rank of $G$ is at least two. By Margulis's arithmeticity theorem [**Ma2**], $\Gamma$ is then an arithmetic subgroup of $G$, that is there exists a semisimple algebraic group $H$ defined over $\mathbb{Q}$ and an epimorphism $f : H(\mathbb{R})^0 \to G$ with compact kernel such that $\Gamma$ is commensurable with $f(H(\mathbb{Z}) \cap H(\mathbb{R})^0)$. Since any such $\Gamma$ has a torsion free subgroup of finite index by Selberg's lemma [**Se**], there exist a maximal compact subgroup $K_0$ of $H(\mathbb{R})^0$ and a torsion free arithmetic subgroup $\Delta$ such that $\Delta\backslash H(\mathbb{R})^0/K_0$ is a finite covering space of $M$.

In what follows, a real (respectively connected) algebraic group defined over $\mathbb{Q}$ means the group of real points of (respectively the identity component of) a connected algebraic subgroup defined over $\mathbb{Q}$.

Summarizing the above, to show Theorem 1.1 it suffices to show the following.

THEOREM 1.2. *Let $G$ be a connected semisimple real algebraic group defined over $\mathbb{Q}$ and $K$ be a maximal compact subgroup of $G$. Let $\Gamma$ be a torsion free arithmetic subgroup of $G$ with respect to the given $\mathbb{Q}$-structure of $G$. For any given $c > 0$, the number of compact maximal flats in $\Gamma\backslash G/K$ of volume less than $c$ is finite.*

In Section 4, we obtain a number theoretic application of our theorem.

COROLLARY 1.3. *Denote by $\Omega_n$ the set of all orders in totally real number fields of degree $n$. For any $c > 0$,*

$$\#\{\mathcal{D} \in \Omega_n : \mathrm{Reg}(\mathcal{D}) < c\} < \infty,$$

*where* $\mathrm{Reg}(\mathcal{D})$ *denotes the regulator of the order $\mathcal{D}$.*

In particular, this implies that there are only finitely many totally real number fields of degree $n$ with regulator less than a given number. This is a special case of a theorem of Remak who proved the same statement for any number fields which are not CM. A number field is called CM if it is a totally imaginary quadratic extension of a totally real number field. He obtained this as a consequence of a lower bound for the regulator in terms of the discriminant of the field [**Re**] (see also [**Si**]).

## 2. *Compact maximal flats in* $\Gamma\backslash G/K$

Let $G$ be a connected center free semisimple linear real algebraic group. Let $K$ be a maximal compact subgroup in $G$ and $\Gamma$ a torsion free lattice in $G$. Let

$$G = (\exp \mathfrak{p})K$$

be the Cartan decomposition and consider the Riemannian symmetric space $G/K$ with a fixed left invariant Riemannian metric.

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LEMMA 2.1. *There exists a maximal real split torus of $G$, whose identity component will be denoted by $S$, such that*

$$C_G(S)^0 = S(C_G(S)^0 \cap K) \quad \textit{(as a direct product)}$$

*where $C_G(S)$ denotes the centralizer of $S$ in $G$.*

*Proof.* Let $S$ be the identity component of any maximal real split torus of $G$. Then $C_G(S)^0$ is a reductive algebraic subgroup and hence for some $g \in G$, $gC_G(S)^0g^{-1}$ has a Cartan decomposition compatible with the one fixed for $G$ [**Mo**]. On the other hand, since $S$ is a maximal real split torus, $gC_G(S)^0g^{-1}$ is a direct product of $gSg^{-1}$ with $gC_G(S)^0g^{-1} \cap K$. Hence $gSg^{-1}$ satisfies the desired properties. □

Fix $S$ as in the above lemma and set

$$A = C_G(S)^0 \quad \text{and} \quad M = A \cap K.$$

We refer to [**Mo**] for some known facts about maximal flats in the discussion below. Any maximal flat in the Riemannian symmetric space $G/K$ is of the form $gSK/K$ for $g \in G$. We say that a maximal flat $\mathcal{F}$ in $G/K$ is $\Gamma$-compact if its image $\Gamma\backslash\Gamma\mathcal{F}$ under the projection map $G/K \to \Gamma\backslash G/K$ is compact. A compact maximal flat in $\Gamma\backslash G/K$ is then of the form $\Gamma\backslash\Gamma\mathcal{F}$, where $\mathcal{F}$ is a $\Gamma$-compact maximal flat in $G/K$.

Since $SK = AK$, we have $gSK/K = gAK/K$. Moreover by [**Mo**, Lemma 5.1],

$$gSK/K = hSK/K \quad \text{if and only if } h^{-1}g \in N_G(S), \tag{2.2}$$

where $N_G(S)$ denotes the normalizer of $S$ in $G$.

For a subgroup $H$ of $G$, the notation $\bar{g}H$ denotes the orbit of $\bar{g} \in \Gamma\backslash G$ under the right translation action of $H$ and the notation $[H]$ denotes the $\Gamma$-conjugacy class of subgroups of $G$ containing $H$.

Now set

$$X := \{[gSg^{-1}] : gSK/K \text{ is } \Gamma\text{-compact}\}$$

and

$$Y := \{[gAg^{-1} \cap \Gamma] : \bar{g}A \text{ is compact}\}.$$

By (2.2), the set $X$ is in bijection with the space of compact maximal flats in $\Gamma\backslash G/K$.

Moreover, we have the following.

THEOREM 2.3. *The map*

$$[gSg^{-1}] \mapsto [gAg^{-1} \cap \Gamma]$$

*defines a bijection from $X$ to $Y$.*

*Proof.* It can be easily checked that $N_G(S) \subset N_G(A)$. Hence $[gSg^{-1}] = [hSh^{-1}]$ implies $[gAg^{-1} \cap \Gamma] = [hAh^{-1} \cap \Gamma]$. Now supposing $gSK/K$ is $\Gamma$-compact, we show that $A \cap g^{-1}\Gamma g$ is co-compact in $A$, or equivalently $\bar{g}A$ is compact in $\Gamma\backslash G$. Set $\mathcal{F} = gSK/K$. By (2.2), the stabilizer of the flat $\mathcal{F}$ in $\Gamma$ is equal to $gN_G(S)g^{-1} \cap \Gamma$ and hence

$$\Gamma\backslash\Gamma\mathcal{F} = (gN_G(S)g^{-1} \cap \Gamma)\backslash\mathcal{F}.$$

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Since $gAg^{-1} \cap \Gamma$ has finite index in $gN_G(S)g^{-1} \cap \Gamma$ and $\mathcal{F}$ is $\Gamma$-compact, we have that $(gAg^{-1} \cap \Gamma)\backslash\mathcal{F}$ is compact as well. Since $gSg^{-1} = gAg^{-1}/gMg^{-1}$ and it acts simply transitively on $\mathcal{F}$,

$$(gAg^{-1} \cap \Gamma)\backslash\mathcal{F} = (gAg^{-1} \cap \Gamma)\backslash gAg^{-1}/gMg^{-1}.$$

Since $M$ is compact, the canonical projection map

$$(gAg^{-1} \cap \Gamma)\backslash G \to (gAg^{-1} \cap \Gamma)\backslash G/gMg^{-1}$$

is proper and hence it follows that $(gAg^{-1} \cap \Gamma)\backslash gAg^{-1}$ is compact, being the preimage of $(gAg^{-1} \cap \Gamma)\backslash gAg^{-1}/gMg^{-1}$. This proves that the map is well defined.

To show that the map is injective, first note that if $gAg^{-1} \cap \Gamma$ is co-compact in $gAg^{-1}$, then the image of $gAg^{-1} \cap \Gamma$, under the natural projection of $gAg^{-1}$ onto $gSg^{-1}$, is co-compact in $gSg^{-1}$ as well, since $A = S/M$ with $M$ compact. Since $S$ is $\mathbb{R}$-split, it follows that the Zariski closure of $gAg^{-1} \cap \Gamma$ contains $gSg^{-1}$ for any $gSg^{-1} \in X$. Since $S$ is the unique maximal real split torus of $A$, it follows that $gAg^{-1} \cap \Gamma = hAh^{-1} \cap \Gamma$ implies $gSg^{-1} = hSh^{-1}$, proving the claim. Since $gSK/K = gAK/K$, the surjectivity is clear. □

Note that for $G = \mathrm{PSL}_2(\mathbb{R})$, $K = \mathrm{PSO}_2$ and $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, the above theorem is precisely the well known one-to-one correspondence between the set of primitive closed geodesics and the conjugacy classes of primitive hyperbolic elements in $\mathrm{PSL}_2(\mathbb{Z})$.

On the group $G$, we fix a left $G$-invariant and right $K$-invariant Riemannian metric which is compatible with the left invariant metric on $G/K$. For each closed subgroup $H$ of $G$, this induces volume forms on $H$ as well as on closed orbits $\bar{g}H$ in $\Gamma\backslash G$. If $\bar{g}H$ is compact, the volume of $\bar{g}H$ is then equal to $\mathrm{vol}((H \cap g^{-1}\Gamma g)\backslash H)$.

LEMMA 2.4. *If $g \in G$ is such that $\mathcal{F} = gSK/K$ is $\Gamma$-compact, then*

$$\mathrm{vol}(\Gamma\backslash\Gamma\mathcal{F}) \geq r \cdot \mathrm{vol}(\bar{g}A),$$

*where $r > 0$ is a constant independent of $\mathcal{F}$.*

*Proof.* If $\Omega$ is a fundamental domain for $\Gamma$ in $\mathcal{F}$, then $g^{-1}\Omega$ is a fundamental domain for $g^{-1}\Gamma g$ in $g^{-1}\mathcal{F} = SK/K$. Since the metric on $G/K$ is left invariant,

$$\mathrm{vol}(\Gamma\backslash\Gamma\mathcal{F}) = \mathrm{vol}((N_G(S) \cap g^{-1}\Gamma g)\backslash g^{-1}\mathcal{F}).$$

Since $[N_G(S) \cap \Gamma : A \cap \Gamma] \leq [N_G(S) : A]$ and

$$(A \cap g^{-1}\Gamma g)\backslash g^{-1}\mathcal{F} = (A \cap g^{-1}\Gamma g)M\backslash A,$$

we have

$$\mathrm{vol}(\Gamma\backslash\Gamma\mathcal{F}) \geq \frac{1}{[N_G(S) : A]} \mathrm{vol}((A \cap g^{-1}\Gamma g)M\backslash A).$$

If $\Omega$ is a subset of $A$ which bijectively maps to a fundamental domain in $A/M$ for the action of $A \cap g^{-1}\Gamma g$, then the product $\Omega \times M$ is a fundamental domain in $A$ for the action of $A \cap g^{-1}\Gamma g$, which can be shown using the torsion free property of $\Gamma$. Therefore,

$$\mathrm{vol}(\bar{g}A) = \mathrm{vol}((A \cap g^{-1}\Gamma g)\backslash A) = \mathrm{vol}((A \cap g^{-1}\Gamma g)M\backslash A) \cdot \mathrm{vol}(M)$$

Hence we deduce

$$\mathrm{vol}(\Gamma\backslash\Gamma\mathcal{F}) \geq \frac{\mathrm{vol}(\bar{g}A)}{\mathrm{vol}(M) \cdot [N_G(S) : A]}.$$ □

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3. *Proof of Theorem 1.2*

Let $G$ be a connected linear semisimple real algebraic group defined over $\mathbb{Q}$ and $\Gamma$ be an arithmetic subgroup with respect to this $\mathbb{Q}$-structure. Fix a left invariant Riemannian metric on $G$ and hence induced volume forms on closed subgroups of $G$ as well as on closed orbits of the form $\Gamma\backslash\Gamma H$ for $H$ closed subgroups.

By Theorem 2.3 and Lemma 2.4, Theorem 1.2 follows from the following.

THEOREM 3.1. *Let $S$ be the identity component of a maximal $\mathbb{R}$-split torus of $G$. Denote by $A$ the identity component of the centralizer of $S$ in $G$. For any $c > 0$,*

$$\#\{[gAg^{-1} \cap \Gamma] : \bar{g}A \text{ is compact, } \mathrm{vol}(\bar{g}A) < c\} < \infty. \tag{3.2}$$

*Moreover, if $G$ is $\mathbb{R}$-split, then*

$$\#\{\bar{g}S : \bar{g}S \text{ is compact, } \mathrm{vol}(\bar{g}S) < c\} < \infty.$$

Our main tools in proving the above theorem are the following.

THEOREM 3.3. [**DM**, Theorem 5.1] *Let $G$ be a connected linear Lie group and $\Gamma$ a discrete subgroup of $G$. For any $c > 0$, there are only finitely many subgroups of the form $H \cap \Gamma$ such that $H$ is a closed subgroup of $G$, $\Gamma\backslash\Gamma H$ is closed and the co-volume of $H \cap \Gamma$ in $H$ is less than $c$.*

THEOREM 3.4. [**TW**, Theorem 1.3] *Let $G$ be a real algebraic group defined over $\mathbb{Q}$ and $S$ be a maximal $\mathbb{R}$-split torus of $G$ containing a maximal $\mathbb{Q}$-split torus of $G$. Let $\Gamma$ be an arithmetic subgroup of $G$. Then there exists a compact subset $K \subset \Gamma\backslash G$ such that for any $g \in G$,*

$$\bar{g}S \cap K \neq \emptyset.$$

For $G = SL_n(\mathbb{R})$ and $\Gamma = SL_n(\mathbb{Z})$, this theorem was proved earlier by Margulis.

*Proof of Theorem 3.1.* The subgroup $A$ is a direct product of $S$ and its unique maximal compact subgroup. First note that in $G$ there exists a maximal real split torus containing a maximal $\mathbb{Q}$-split torus, e.g., take a maximal real split torus in the centralizer of a maximal $\mathbb{Q}$-split torus. Since all maximal real split tori of $G$ are conjugate with each other, we may apply Theorem 3.4 to $S$ to conclude that there is a compact subset $K_0 \subset G$ such that any $A$-orbit is of the form $\bar{g}A$ where $g \in K_0$.

Clearly the left-hand side in (3.2) is bounded above by

$$\#\{gAg^{-1} \cap \Gamma : g \in K_0, \bar{g}A \text{ is compact, } \mathrm{vol}(\bar{g}A) < c\}. \tag{3.5}$$

Note that for any compact $A$-orbit $\bar{g}A$, the volume of $\bar{g}A$ is given by $\mathrm{vol}((g^{-1}\Gamma g \cap A)\backslash A)$ and

$$\mathrm{vol}((\Gamma \cap gAg^{-1})\backslash gAg^{-1}) = \mathrm{vol}((g^{-1}\Gamma g \cap A)\backslash A) \cdot \delta_g,$$

where $\delta_g$ denotes the factor by which volumes of subsets get multiplied under the transformation $a \to gag^{-1}$ for all $a \in A$ (here volumes are computed with respect to the induced metric on the submanifolds $A$ and $gAg^{-1}$ of $G$).

Since $\delta_g$ is a continuous function on $G$, we have

$$d := \max_{g \in K_0} \delta_g < \infty.$$

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We now have, for any $g \in K_0$ such that $\bar{g}A$ is compact,

$$\mathrm{vol}((\Gamma \cap gAg^{-1})\backslash gAg^{-1}) \leq d \cdot \mathrm{vol}(\bar{g}A). \tag{3.6}$$

On the other hand, by Theorem 3.3, there are only finitely many subgroups of the form $gAg^{-1} \cap \Gamma$ such that $\bar{g}A$ is compact and

$$\mathrm{vol}((\Gamma \cap gAg^{-1})\backslash gAg^{-1}) < c \cdot d.$$

Hence (3.5) is finite, proving the first claim. To see the second claim, if $G$ is $\mathbb{R}$-split, we have $A = S$ and $gSg^{-1} \cap \Gamma$ is Zariski dense in $gSg^{-1}$ for any compact $\bar{g}S$. Hence the map from $\{[gSg^{-1}] : \bar{g}S \text{ is compact}\}$ to $\{[gSg^{-1} \cap \Gamma] : \bar{g}S \text{ is compact}\}$ induced by $gSg^{-1} \to gSg^{-1} \cap \Gamma$ is a bijection. Therefore, the first part of the theorem implies that

$$\#\{[gSg^{-1}] : \bar{g}S \text{ is compact, } \mathrm{vol}(\bar{g}S) < c\} < \infty.$$

On the other hand, the cardinality of the fiber of the map $\bar{g}S \mapsto gSg^{-1}$ is $[N_G(S) : S]$, which is finite, since $S = A$. Therefore the number of compact $\bar{g}S$ with volume less than $c$ is bounded above by

$$[N_G(S) : S] \cdot \#\{[gSg^{-1}] : \bar{g}S \text{ is compact, } \mathrm{vol}(\bar{g}S) < c\} < \infty.$$

This proves the claim.

## 4. *Finiteness of orders of bounded regulators*

We set up some notation and recall some basic definitions in number theory (cf. [**BS**]).

*Notation.* Let $k$ be a number field of degree $n$ which is totally real, that is any field embedding of $k$ into $\mathbb{C}$ takes values in $\mathbb{R}$.

(1) We set $k^*$ to be the set of totally positive elements in $k$, i.e.

$$k^* = \{x \in k : \sigma(x) > 0 \text{ for any embedding } \sigma : k \to \mathbb{R}\}.$$

(2) Denote by $\mathcal{O}_k$ the ring of algebraic integers in $k$. An order of $k$ is a subring of $\mathcal{O}_k$ containing 1 which has rank $n$ as a $\mathbb{Z}$-submodule. Note that $\mathcal{O}_k$ is the unique maximal order in $k$.

(3) For an order $\mathcal{D}$ of $k$, the notation $\mathcal{D}^*$ denotes the group of units in $\mathcal{D}$. Then

$$\mathcal{D}^* = \{u \in k : u\mathcal{D} = \mathcal{D}\} = \{u \in \mathcal{D} : \mathrm{N}_{k/\mathbb{Q}}(u) = \pm 1\},$$

where $\mathrm{N}_{k/\mathbb{Q}}(u)$ denotes the norm of $u \in k$.

(4) The regulator $\mathrm{Reg}(\mathcal{D})$ of an order $\mathcal{D}$ of $k$ is defined as the covolume in $\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\right\}$ of the discrete subgroup

$$\{(\log|\sigma_1(u)|, \ldots, \log|\sigma_n(u)|) : u \in \mathcal{D}^*\}$$

where $\sigma_1, \ldots, \sigma_n$ are $n$ different embeddings of $k$ into $\mathbb{R}$.
By the Dirichlet unit theorem, $\mathrm{Reg}(\mathcal{D}) < \infty$ for any order $\mathcal{D}$ of $k$.

(5) The regulator of a field $k$ means the regulator of its maximal order $\mathcal{O}_k$.

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We now begin the proof of Theorem 1.3. Let $S$ denote the identity component of the diagonal subgroup of $\mathrm{SL}_n(\mathbb{R})$, i.e.

$$S = \{\mathrm{diag}(a_1, \ldots, a_n) \in \mathrm{SL}_n(\mathbb{R}) : a_i > 0\}.$$

Consider the space $Z\,\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$, where $Z$ denotes the center of $\mathrm{GL}_n(\mathbb{R})$. Note that an $S$-orbit $\bar{g}S$, $g \in \mathrm{GL}_n(\mathbb{R})$, is compact in $Z\,\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$ if and only if the discrete subgroup $S \cap g^{-1}\,\mathrm{SL}_n(\mathbb{Z})g$ is co-compact in $S$.

We may identify $Z\,\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$ with $\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})$ by the canonical isomorphism. Consider a left invariant Riemannian metric on $\mathrm{SL}_n(\mathbb{R})$. Since each compact $S$-orbit $\bar{g}S$ is a submanifold in $\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})$, we have an induced volume form such that $\mathrm{vol}(\bar{g}S)$ is given by $\mathrm{vol}(S \cap g^{-1}\,\mathrm{SL}_n(\mathbb{Z})g\backslash S)$.

Setting $\mathcal{C}$ to be the set of all compact $S$-orbits in $Z\,\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$, we first establish an injective map from $\Omega_n$ to $\mathcal{C}$ (recall that $\Omega_n$ denotes the set of all orders in totally real number fields of degree $n$). To do so, for each totally real number field $k$ of degree $n$, fix an $n$-tuple $\sigma_k := (\sigma_1, \ldots, \sigma_n)$ of different embeddings of $k$ into $\mathbb{R}$ with $\sigma_1$ being the identity. For an order $\mathcal{D}$ of $k$, we set

$$\phi(\mathcal{D}) = \bar{g}_\xi S,$$

where $g_\xi := (\sigma_j(\xi_i))_{ij}$ for a $\mathbb{Z}$-basis $\xi = (\xi_1, \ldots, \xi_n)$ of $\mathcal{D}$. Clearly $\phi(\mathcal{D})$ does not depend on the choice of $\mathbb{Z}$-basis of $\mathcal{D}$ and hence $\phi$ is a well defined map from $\Omega_n$ to the set of $S$-orbits in $Z\,\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$.

To show that $\phi$ is injective, suppose that $\phi(\mathcal{D}) = \phi(\mathcal{D}')$ for some $\mathcal{D}, \mathcal{D}' \in \Omega_n$. Then

$$g_{\xi'} \in Z\,\mathrm{GL}_n(\mathbb{Z})g_\xi S$$

where $\xi = (\xi_1, \ldots, \xi_n)$ and $\xi' = (\xi_1, \ldots, \xi_n')$ are $\mathbb{Z}$-bases for $\mathcal{D}$ and $\mathcal{D}'$, respectively.

By comparing the first columns, we obtain that

$$\mathcal{D}' = a\mathcal{D} \quad \text{for some non-zero } a \in \mathbb{R}.$$

Since $1 \in \mathcal{D}$, we have $a \in \mathcal{D}'$. Therefore,

$$a\mathcal{D}' \subset \mathcal{D}' = a\mathcal{D}; \quad \text{hence } \mathcal{D}' \subset \mathcal{D}.$$

On the other hand, since $1 \in \mathcal{D}'$, $a^{-1} \in \mathcal{D}$. Hence,

$$\mathcal{D} = aa^{-1}\mathcal{D} \subset a\mathcal{D} = \mathcal{D}'.$$

Therefore, $\mathcal{D} = \mathcal{D}'$ showing that $\phi$ is injective.

LEMMA 4.1. *We have*

$$S \cap g_\xi^{-1}\,\mathrm{SL}_n(\mathbb{Z})g_\xi = \{\mathrm{diag}(\sigma_1(u), \ldots, \sigma_n(u)) : u \in \mathcal{D}^* \cap k^*\},$$

*where $\xi$ is a $\mathbb{Z}$-basis of $\mathcal{D}$.*

*Proof.* Note that $\mathrm{diag}(a_1, \ldots, a_n) \in S \cap g_\xi^{-1}\,\mathrm{SL}_n(\mathbb{Z})g_\xi$ if and only if

$$\mathbb{Z}^n\, g_\xi\, \mathrm{diag}(a_1, \ldots, a_n) = \mathbb{Z}^n\, g_\xi, \tag{4.2}$$

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where $\mathbb{Z}^n$ are integral row vectors. This is again the same as saying that for any non-zero $x \in \mathcal{D}$, there exists $y \in \mathcal{D}$ such that

$$a_j \sigma_j(x) = \sigma_j(y) \quad \text{for all } 1 \le j \le n.$$

Setting $u = yx^{-1}$, we have $u \in k$ and $a_j = \sigma_j(u)$. Since $a_j \sigma_j(\mathcal{D}) = \sigma_j(\mathcal{D})$ from (4.2), $u\mathcal{D} = \mathcal{D}$ and hence $u \in \mathcal{D}^*$. Moreover, $\sigma_j(u) = a_j > 0$ for each $1 \le j \le n$ and hence $u \in k^*$. This proves $\subset$. The other inclusion is clear as well. □

Since $\{\mathrm{diag}(\sigma_1(u), \dots, \sigma_n(u)) : u \in \mathcal{D}^* \cap k^*\}$ has index at most $2^n$ in the subgroup

$$\{\mathrm{diag}(\sigma_1(u), \dots, \sigma_n(u)) : u \in \mathcal{D}^*\},$$

by the Dirichlet unit theorem and the above lemma, we have $\phi(\Omega_n) \subset \mathcal{C}$.

Summarizing the above, we have the following.

LEMMA 4.3. *The map $\phi : \Omega_n \to \mathcal{C}$ is a well-defined injective map and for any $\mathcal{D} \in \Omega_n$,*

$$c \cdot \mathrm{Reg}(\mathcal{D}) \le \mathrm{vol}(\phi(\mathcal{D})) \le c \cdot 2^n \cdot \mathrm{Reg}(\mathcal{D}),$$

*where $c > 0$ is a constant depending only on the volume form on $S$.*

Now Theorem 1.3 follows from Theorem 3.1.

*Remarks*

- In fact, any compact $S$-orbit in $\mathcal{C}$ can be constructed from a rank $n$ free $\mathbb{Z}$-submodule (not necessarily order) in a totally real number field of degree $n$. We refer to [**Oh**] for a precise description of the bijection between $\mathcal{C}$ and *module classes*.
- To get an injective map into the maximal flats in $\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$ rather than into $\mathcal{C}$, we need to put an equivalence relation on the set $\Omega_n$ so that $\mathcal{D} \sim \mathcal{D}'$ if and only if there exists an isomorphism between the fields containing $\mathcal{D}$ and $\mathcal{D}'$ as orders which maps $\mathcal{D}$ to $\mathcal{D}'$.

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