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Affine distance-transitive graphs with quadratic forms

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1. *Introduction*

The classification of all finite primitive distance-transitive graphs is basically divided into two cases. In the one case, known as the almost simple case, we have an almost simple group acting primitively as a group of automorphisms on the graph. In the other case, known as the affine case, the vertices of the graph can be identified with the vectors of a finite-dimensional vector space over some finite field. In this case the automorphism group G of the graph Γ contains a normal p-subgroup N which is elementary Abelian and acts regularly on the set of vertices of Γ. Let G_0 be the subgroup of G that stabilizes a vertex. Identifying the vertices of Γ with G_0-cosets in G, one obtains a vector space V on which N acts as a group of translations, G_0 stabilizes 0 and, as Γ is primitive, G_0 acts irreducibly on V.

Recently it was shown that this group G_0 is essentially an almost simple group. Combining the classification of finite simple groups with a diameter bound as in [1], a list of groups and modules can be obtained. Thus the problem is reduced to a question about the orbit structure of finite non-Abelian simple groups acting on small-dimensional modules. It turned out that in many cases the action of G_0 on V leaves a unitary or orthogonal form invariant. Hence one can partition the G_0-orbits into two classes, the singular vectors and the non-singular vectors, those vectors on which the form vanishes and those on which it does not.

The aim of this paper is to exploit this geometric structure in order to classify all primitive affine distance-transitive graphs which leave some non-degenerate unitary or orthogonal form invariant. If the diameter of Γ is equal to 2, then the group G_0 has only 3 orbits on V; hence G is a so-called affine rank 3 group and is known (cf. Liebeck [7]). In fact there are two graphs associated with G, one being the complement of the other, constructed by calling two vectors adjacent whenever their difference is a non-singular or singular vector, respectively.

Theorem 1·1. *Suppose that* Γ *is a primitive affine distance-transitive graph with point stabilizer* G_0 *such that* G_0 *leaves an orthogonal form invariant over* $\mathrm{GF}(q)$, *up to scalar multiplication and field automorphisms, and* $\mathrm{GF}(q)^* \leqslant G_0$. *Then the diameter of* Γ *is* 2 *or* Γ *is one of the following: a Hamming graph, a folded cube, a halved cube or a halved folded cube.*

The proof is spread out over the next four sections, which study the various cases where $\Gamma_1(0)$ and $\Gamma_2(0)$ contain singular or non-singular vectors. The assumption that $\mathrm{GF}(q)^*$ is a subgroup of G_0 enables us to study orbits in the projective space corresponding to V. Although at this moment there is no general proof available for the fact that $\mathrm{GF}(q)^*$ is a subgroup of G_0, it does not give a severe restriction on the

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usage of Theorem 1·1. (For most choices for $G_0\,\mathrm{GF}(q)^*/\mathrm{GF}(q)^*$, it follows almost immediately that this must hold.)

As a non-degenerate unitary form on an n-dimensional $\mathrm{GF}(q^2)$-vector space leads to a non-degenerate orthogonal form on a $2n$-dimensional $\mathrm{GF}(q)$-vector space, we have the following corollary to the theorem.

COROLLARY 1·2. *Suppose that Γ is a primitive affine distance-transitive graph with point stabilizer G_0 such that G_0 leaves an orthogonal or unitary form invariant, up to scalar multiplication and field automorphisms, and $\mathrm{GF}(q)^* \leqslant G_0$. Then the diameter of Γ is 2 or Γ is one of the following: a Hamming graph, a folded cube, a halved cube or halved folded cube.*

Remark. In the case when $\mathrm{GF}(p)^*$ is the only subgroup of $\mathrm{GF}(q)^*$ contained in G_0, we can restrict the form to $\mathrm{GF}(p)$, where the theorem remains valid.

The notation is taken from the book by Brouwer, Cohen and Neumaier [**2**], which is also used for general reference. Throughout this paper let V be a vector space over $\mathrm{GF}(q)$, Q a form left invariant and f the corresponding bilinear form. An element x of V will be called a vector and we will refer to $\langle x\rangle$ as a point. Similarly we shall abuse language by calling 2-spaces of V lines. We shall switch freely from V to the corresponding projective space PV and vice versa without further reference. The reader should bear in mind that the primitivity assumption on Γ implies that each G_0-orbit on V contains a basis for it. Consequently we may assume throughout the paper that the dimension of V is at least 3.

The organization of this paper is as follows. Section 2 deals with the case where $\Gamma_1(0)$ contains singular vectors. In this section we rely on results involving locally polar spaces with affine planes. In Section 3 we handle the case where $\Gamma_1(0)$ is non-singular, $\Gamma_2(0)$ singular and $q = 2$. This comes down to the study of a special class of locally copolar spaces. In Section 4 we deal with the same situation as the previous section but now with $q \neq 2$. Finally in Section 5 we consider the situation where both $\Gamma_1(0)$ and $\Gamma_2(0)$ are non-singular.

2. *The set $\Gamma_1(0)$ is singular*

Throughout this section we shall assume that the vectors adjacent to 0 are singular. The case $q = 2$ will be dealt with separately. Let $\mathscr{P}$ denote the polar space embedded in PV with point set $P = \{\langle x\rangle \mid Q(x) = 0\}$ and line set $L = \{\langle x, y\rangle \mid y \notin \langle x\rangle$ and $Q(z) = 0$ for all $z \in \langle x, y\rangle\}$.

LEMMA 2·1. *The vectors at distance 2 from 0 are all non-singular. Moreover, the collection of 1-spaces in $\Gamma_1(0)$ forms a non-degenerate subspace $\mathscr{S}$ of $\mathscr{P}$.*

Proof. As G_0 acts transitively on $\Gamma_1(0)$, all vectors in $\Gamma_1(0)$ are singular. The subspace spanned by any two vectors of $\Gamma_1(0)$ is entirely contained in $\Gamma_{\leqslant 2}(0)$. Thus if $\Gamma_2(0)$ contains a singular vector, then $\Gamma_1(0)$ is a collection of singular vectors such that the 2-space spanned by any two of them is totally singular. As the form is orthogonal, any two vectors of $\Gamma_1(0)$ are orthogonal to each other. Hence the subspace spanned by $\Gamma_1(0)$ is totally singular, which contradicts primitivity. Thus $\Gamma_2(0)$ contains only non-singular vectors. Clearly if $v, w \in \Gamma_1(0)$ then all the singular vectors

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of $\langle v, w\rangle$ are in $\Gamma_{\leqslant 2}(0)$ and hence in $\Gamma_1(0)$. It follows that the 1-spaces in $\Gamma_1(0)$ form a subspace of the polar space $\mathscr{P}$, which must be non-degenerate by irreducibility of G_0. ∎

COROLLARY 2·2. *If $\mathscr{S}$ is of rank at least 2, then Γ is of diameter 2.*

Proof. Consider the geometry Ψ constructed as follows. The points of Ψ are the vectors of V, the translates of the 1-spaces contained in $\Gamma_1(0)$ are the lines, the translates of the 2-spaces contained in $\Gamma_1(0)$ are the planes, and so on. Then Ψ is a geometry with affine plane whose point-residues are isomorphic to $\mathscr{S}$. It follows now from Cohen and Schult [3], Cuypers and Pasini [5] and Cuypers [4] that Γ is as stated. ∎

In view of the corollary we may assume that $\Gamma_1(0)$, viewed as a subspace of $\mathscr{P}$, does not contain any lines; hence the 2-spaces spanned by any two vectors of $\Gamma_1(0)$ are hyperbolic. If the dimension of V is odd, then q must be odd too. But in that case, the non-singular points of PV fall into two different classes and a hyperbolic line intersects both classes non-trivially. This contradicts the transitive action of G_0 on $\Gamma_2(0)$. Thus for the remainder of this section we may assume that the dimension of V is even and is at least 4; moreover, we shall assume that the diameter of Γ is at least 3.

LEMMA 2·3. *If $q \neq 2$, then the vectors in $\Gamma_3(0)$ are singular.*

Proof. Suppose by way of contradiction that $\Gamma_3(0)$ contains non-singular vectors. Then for all $x, y, z \in \Gamma_1(0)$, all singular vectors of $\langle x, y, z\rangle$ are in $\Gamma_1(0)$. Let $u \in \Gamma_1(0)$, with $u \notin \langle x, y, z\rangle$. If l is a singular line in $\langle u, x, y, z\rangle$, then it will intersect each of the planes $\langle x, y, z\rangle$, $\langle x, y, u\rangle$, $\langle x, u, z\rangle$, $\langle u, y, z\rangle$ in a singular point. Among these intersections there are at least two different points. Hence there are two vectors in $\Gamma_1(0)$ which span this line, which is impossible. Thus $\langle u, x, y, z\rangle$ does not contain lines, that is, it is isomorphic to an O_4^--geometry. Now $\langle x, y, z\rangle$ is isomorphic to a O_3-geometry, so that all singular vectors of $\langle x, y, z\rangle$ are in $\Gamma_1(0)$. If q is odd, all non-singular vectors are in $\Gamma_2(0)$. Consequently, all vectors of $\langle u, x, y, z\rangle$ are at distance at most 3 from 0. Thus all singular vectors in a 4-space generated by four members of $\Gamma_1(0)$ are in $\Gamma_1(0)$. If the dimension of V is equal to 4, then this readily implies that the diameter of Γ is 2. Any 5-dimensional space contains a singular line. As this line will intersect the five 4-dimensional hyperplanes obtained by removing one of the vectors in at least two different points, this line is in fact contained in $\Gamma_1(0)$, a contradiction. Thus we may assume that q is even. Again all singular vectors of $\langle x, y, z\rangle$ are in $\Gamma_1(0)$, but now only the radical of $\langle x, y, z\rangle$ can be in $\Gamma_3(0)$. Let v be a singular vector in $\langle u, x, y, z\rangle$ contained neither in $\langle x, y, z\rangle$ nor in $\langle u\rangle$. As $q \neq 2$, there is a $w \in \Gamma_1(0) \cap \langle u, x, y\rangle$ with $w \notin \{u, x, y\}$. If $v \in \langle u, w\rangle$ then $v \in \Gamma_1(0)$ by the above. Otherwise at least one of $\langle u, v\rangle \cap \langle x, y, z\rangle$ or $\langle w, v\rangle \cap \langle x, y, z\rangle$ is not the radical of $\langle x, y, z\rangle$. Therefore $v \in \Gamma_{\leqslant 3}(0)$, whence $v \in \Gamma_1(0)$. It follows that, as $q \neq 2$, all the singular vectors in a space generated by four members of $\Gamma_1(0)$ are in $\Gamma_1(0)$, and we can finish as before. ∎

LEMMA 2·4. *If $q \neq 2$, then all non-singular vectors are in $\Gamma_2(0)$.*

Proof. Let H be a subspace generated by $x_1, x_2, \ldots, x_n \in \Gamma_1(0)$ such that all non-singular vectors are in $\Gamma_2(0)$. Let $v \in \Gamma_1(0) \backslash H$ and let $x \in \langle v, H\rangle \backslash (H \cup \{v\})$ be a non-

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singular vector. Moreover, let $y \in H \cap \langle v, x\rangle$. Then $x \in \langle y, v\rangle$. We want to show that $x \in \Gamma_2(0)$. If y is a singular vector in $\Gamma_1(0)$ or a non-singular vector, then $x \in \Gamma_{\leqslant 3}(0)$, so that $x \in \Gamma_2(0)$. Assume that y is singular. If there exist a x_i with $\langle x_i, y\rangle$ hyperbolic, then all the non-singular vectors of $\langle y, x_i\rangle$ and $\langle v, x_i\rangle$ are in $\Gamma_2(0)$. Let $z \in \langle x, x_i\rangle$ with z non-singular and contained neither in $\langle x\rangle$ nor in $\langle x_i\rangle$. As $\langle x_i\rangle$ and $\langle y\rangle$ are the only two singular points on $\langle y, x_i\rangle$, we have that $\langle v, z\rangle \cap \langle y, x_i\rangle$ is non-singular. Hence z is in $\Gamma_2(0)$, and as $x \in \langle z, x_i\rangle$, we must also have $x \in \Gamma_2(0)$. As there are at least $q-2 \geqslant 1$ choices for $\langle z\rangle$, we are done.

Assume from now on that y is orthogonal to all x_i. Then y is in the radical of H. Let z be a non-singular vector in $\langle x, x_1\rangle \backslash (\langle x\rangle \cup \langle x_1\rangle)$. Now $\langle v, y, z\rangle = \langle v, y, x_1\rangle$ and $u = \langle v, z\rangle \cap \langle y, x_1\rangle$ is a singular vector which is not in the radical of H. Hence there exists an x_i such that $\langle u, x_i\rangle$ is hyperbolic, which forces z to be in $\Gamma_2(0)$ by the previous paragraph. Finally, it follows that x is also in $\Gamma_2(0)$.

As all non-singular vectors of each subset of size three of $\Gamma_1(0)$ are in $\Gamma_2(0)$, it follows by induction on the dimension that all non-singular vectors are, in fact, in $\Gamma_2(0)$. ∎

From the above it follows that the diameter of Γ can be at most 4. The next lemma will show us that Γ must be strongly regular.

LEMMA 2·5. *If all non-singular vectors of O^{ϵ}_{2n} are in $\Gamma_2(0)$ then* $\operatorname{diam}(\Gamma) = 2$.

Proof. In this situation, we have $k_2 = q^{n-1}(q^n - \epsilon)(q-1)$. Moreover, if we fix a singular vector, there are $q^{n-1}(q^n - q^{n-1} - \epsilon)(q-1)$ non-singular vectors that differ by a non-singular vector from our fixed one. Hence this number is equal to $p^1_{2,2}$ and $p^3_{2,2}$. As $kp^1_{2,2} = k_2 p^2_{2,1}$ we have

$$(q^n - q^{n-1} - \epsilon)\, k = (q^n - \epsilon)\, a_2.$$

From the fact that $(q^n - q^{n-1} - \epsilon)$ and $(q^n - \epsilon)$ are relatively prime it follows that we can write

$$k = \alpha(q^n - \epsilon), \quad a_2 = \alpha(q^n - q^{n-1} - \epsilon) \quad \text{and} \quad b_2 + a_2 = \alpha q^{n-1}$$

for some natural number α. Recall that we may assume that $\Gamma_1(0)$, as a subspace of $\mathscr{P}$, contains no lines. This implies that $b_1 = k - (q-1)$. From $kb_1 = k_2 c_2$ we obtain

$$q^{n-1}(q-1)\, c_2 = \alpha(\alpha(q^n - \epsilon) - (q-1))$$

and, as the greatest common divisor of α and $(\alpha(q^n - \epsilon) - (q-1))$ divides $(q-1)$, we must have that q^{n-1} divides either α or $(\alpha(q^n - \epsilon) - (q-1))$. Thus there exists a natural number β with either $\alpha = \beta q^{n-1}$ or $\alpha = \beta q^{n-1} - \epsilon(q-1)$. Consequently $(q-1)\, c_2$ is equal to either $\beta(\alpha(q^n - \epsilon) - (q-1))$ or $\alpha(\beta(q^n - \epsilon) - q(q-1)\,\epsilon)$; both conclusions lead to a contradiction with $c_2 < \alpha q^{n-1}$. ∎

There remains the case of $q = 2$. Let n be the dimension of V. Clearly n must be even.

LEMMA 2·6. *If $q = 2$ then Γ is a Hamming graph or a folded $(n+1)$-cube, with $n/2$ even.*

Proof. Here we have the situation that $\Gamma_1(0)$ is a collection of singular vectors, no two of which are orthogonal. Thus if $x, y \in \Gamma_1(0)$, then $Q(x) = Q(y) = 0, f(x, x) = 0$ and

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$f(x,y) = 1$ if $x \neq y$. Fix a basis $x_1, x_2, \ldots, x_n$ of V in $\Gamma_1(0)$. As $Q(u+v) = Q(x)+Q(v)+f(u,v)$, it is straightforward to prove that if z is a vector which is the sum of r basis vectors, then $Q(z) = r(r-1)/2 \bmod 2$. If $\Gamma_1(0)$ is a basis for V, then Γ is a Hamming graph. So let $y \in \Gamma_1(0)$ with y not in the basis. Without loss of generality we may write $y = x_1 + \ldots + x_r$ for some $r \leqslant n$. If $r \neq n$ then $f(y, x_1) = r-1 \bmod 2$ and $f(y, x_n) = r \bmod 2$, a contradiction. Thus $r = n$ and, as y is singular, we have that $n/2$ is even. The graph thus obtained is the folded $(n+1)$-cube. ∎

Remark. The folded $(n+1)$-cubes with $n/2$ odd will appear at the end of Section 5.

3. *The set* $\Gamma_1(0)$ *is non-singular,* $\Gamma_2(0)$ *is singular and* $q = 2$

In this section we consider the case $q = 2$ with non-singular vectors at distance 1 and singular vectors at distance 2 from 0. The graph induced on $\Gamma_1(0)$ is isomorphic to the collinearity graph of a copolar space Δ, for two vectors $x, y \in \Gamma_1(0)$ are adjacent if and only if $x+y \in \Gamma_1(0)$. With the sets $\{x, y, x+y\}$ as lines, each vector of $\Gamma_1(0)$ is collinear with either 2 or 0 points of a line. In Δ two vectors are adjacent if and only if they are not orthogonal. As $q = 2$, we cannot have that Δ consists only of mutually orthogonal vectors. Hence the space Δ contains lines. Let $\Delta(x)$ denote the points of Δ collinear with x. Let $x, y \in \Delta$ with x not collinear with y. Then the copolar space Δ is called reduced if $\Delta(x) \neq \Delta(y)$. Here we find besides the diameter 2 examples also the halved and halved folded cubes. Throughout this section we assume that Γ is not of diameter 2.

LEMMA 3·1. *The copolar space* Δ *is reduced.*

Proof. Suppose that $x, y \in \Delta$ with $\Delta(x) = \Delta(y)$ and with x not adjacent to y. Then $Q(x) = Q(y) = 1$ and for all v in $\Delta \backslash \{x, y\}$ we have that $Q(x+v) = Q(y+v)$; in particular $f(x,v) = f(y,v)$ and $f(x+y, v) = 0$. As x is not adjacent to y, we have $Q(x+y) = 0$. Hence $Q(x+y+v) = 1$ for all v in $\Delta \backslash \{x+y\}$ and thus $x+y$ is in the radical of Δ. This contradicts the irreducible action of G_0 of Δ. ∎

LEMMA 3·2. *The space* Δ *is either connected or a disjoint union of lines.*

Proof. Suppose that $\Delta = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_n$ with $n \geqslant 2$ and Δ_i the connected components. Clearly each component is again a copolar space and each vector in Δ_i is orthogonal to all vectors in Δ_j for $i \neq j$.

Let $u, v \in \Delta_1$ with u is adjacent to w. If $\Delta_1 = \{u, w, u+w\}$ then the space Δ is the disjoint union of lines. If $\Delta_1 \neq \{u, w, u+w\}$ then there is vector $v \in \Delta_1 \backslash \{u, w, u+w\}$ such that v is collinear with exactly 2 vectors of $\{u, w, u+w\}$. Hence Δ_1 contains two intersecting lines. Consequently, each connected component contains at least 2 intersecting lines, and the diameter of a connected component is 2. Let $x \in \Delta_1$, $y \in \Delta_2$ and $z \in \Delta_1$ with z not adjacent to x, so that $x+z$ and $x+y$ are in $\Gamma_2(0)$. For each $v \in \Delta$ and $w \in \{x+y, x+z\}$ we have $Q(v+w) = 1+f(v,w)$. Notice that the distances between v and $x+y$ and between v and $x+z$ are both at most 3. As before, let $\Delta_i(u)$ be the set of vectors in Δ_i adjacent to u; it is either empty or $\Delta(u)$. We shall consider two cases.

Suppose first that $\Gamma_3(0)$ contains singular vectors. In this case $Q(v+w) = 1$ if and only if $v \in \Gamma_1(0) \cap \Gamma_1(w)$. As $Q(v+w) = 1$ if and only if $f(v,w) = 0$, it readily follows that $\Gamma_1(0) \cap \Gamma_1(x+y)$ is isomorphic to $\Delta \backslash (\Delta_1(x) \cup \Delta_2(y))$ and $\Gamma_1(0) \cap \Gamma_1(x+z)$ is

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isomorphic to $(\Delta_1(x) \cap \Delta_1(z)) \cup (\Delta \backslash (\Delta_1(x) \cup \Delta_1(z)))$; this contradicts distance-transitivity.

Finally suppose that $\Gamma_3(0)$ contains non-singular vectors. In this case we have $Q(v+w) = 0$ if and only if $v \in \Gamma_1(0) \cap \Gamma_2(w)$. As $Q(v+w) = 0$ if and only if $f(v, w) \neq 0$, it readily follows that $\Gamma_1(0) \cap \Gamma_2(x+y)$ is isomorphic to $\Delta_1(x) \cup \Delta_2(y)$ and $\Gamma_1(0) \cap \Gamma_2(x+z)$ is isomorphic to $(\Delta_1(x) \cup \Delta_1(z)) \backslash (\Delta_1(x) \cap \Delta_1(z))$; this contradicts distance-transitivity. ∎

LEMMA 3·3. *If Δ is the disjoint union of lines, then Γ is isomorphic to a Hamming graph.*

Proof. Let $\Delta = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_n$ with $n \geqslant 2$ and Δ_i the connected components. Write $\Delta_i = \{x_i, y_i, x_i + y_i\}$ and $V_i = \langle x_i, y_i \rangle$. Then as no two lines intersect, all vectors of V_i are orthogonal to all vectors of V_j if $i \neq j$. Hence G_0 preserves an orthogonal decomposition of V into 2-dimensional subspaces, so Γ is a Hamming graph (which can be viewed as a coming from a basis over GF(4)). ∎

LEMMA 3·4. *If Δ is reduced and connected, then Γ is isomorphic to a halved or folded halved cube.*

Proof. We are now in the situation that $\Gamma_1(0)$ is a reduced and connected subspace of the copolar space belonging to $\mathrm{PSp}(2n, 2)$. Moreover the vectors in this subspace span the whole vector space V. It follows now from a theorem of Hall [**6**] that Γ is as stated. The halved and folded halved cubes arise from the embedding of the symmetric groups in the orthogonal groups. ∎

4. *The set $\Gamma_1(0)$ is non-singular, $\Gamma_2(0)$ is singular and $q \neq 2$*

In this case, for any two vectors $u, v \in \Gamma_1(0)$, all non-singular vectors of $\langle u, v \rangle$ are also in $\Gamma_1(0)$. Throughout this section we will assume that $q \neq 2$. We shall use an induction argument to show that all non-singular vectors are in $\Gamma_1(0)$.

LEMMA 4·1. *Suppose that for each collection $\{x_1, x_2, \ldots, x_{n-1}\}$ of $n-1$ vectors in $\Gamma_1(0)$ with $n \geqslant 4$, all the non-singular vectors of $\langle x_1, x_2, \ldots, x_{n-1} \rangle$ are also in $\Gamma_1(0)$. Then this is also true for any set of size n.*

Proof. Let $x_1, x_2, \ldots, x_n \in \Gamma_1(0)$. If this set is not linearly independent, then the statement is obviously true. Let $W_i = \langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle$ with $1 \leqslant i \leqslant n$. Then all the non-singular vectors of W_i are in $\Gamma_1(0)$. Let $W = \langle x_1, \ldots, x_n \rangle$ and $x = \sum a_i x_i$ with $a_i \neq 0$ for all $1 \leqslant i \leqslant n$. Now $\langle x, x_3, \ldots, x_n \rangle$ is a hyperplane of W which intersects W_n in the subhyperplane $H = \langle a_1 x_1 + a_2 x_2, x_3, \ldots, x_{n-1} \rangle$. By assumption, all non-singular vectors of $H \leqslant W_n$ and $\langle x_3, \ldots, x_n \rangle$ are in $\Gamma_1(0)$. As $q \neq 2$, either $a_1 x_1 + a_2 x_2$ is non-singular or there must be a $\lambda \in \mathrm{GF}(q)^*$ such that $a_1 x_1 + a_2 x_2 + \lambda x_3$ is non-singular. Replacing $a_1 x_1 + a_2 x_2$ by $a_1 x_1 + a_2 x_2 + \lambda x_3$, if necessary, we see that H is spanned by $n-2$ non-singular vectors of $\Gamma_1(0)$. From the induction hypothesis it now follows that all non-singular vectors of $\langle H, x_n \rangle$ are also in $\Gamma_1(0)$. In particular, x is in $\Gamma_1(0)$. ∎

We shall use this induction argument in the following lemma.

LEMMA 4·2. *If q is odd, then all non-singular vectors are in $\Gamma_1(0)$.*

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Proof. Let $u, v, w \in \Gamma_1(0)$ be such that $\langle u, v, w\rangle$ is a 3-dimensional space. Let $x \in \langle u, v, w\rangle$ be non-singular with x not in $\langle u, v\rangle$, $\langle u, w\rangle$ or $\langle v, w\rangle$. We will show that $x \in \Gamma_1(0)$. Any elliptic line on $\langle x\rangle$ will intersect $\langle u, v\rangle$, $\langle u, w\rangle$ and $\langle v, w\rangle$ in a non-singular point; thus there are at least two vectors $\alpha, \beta \in \Gamma_1(0)$ with $x \in \langle \alpha, \beta\rangle$, that is, $x \in \Gamma_1(0)$. So we may assume that x is not on any elliptic line in $\langle u, v, w\rangle$; thus this space is degenerate. Let rad be the radical of this space.

If the dimension of rad is two, then each non-singular point is on $q+1$ tangents. Let c be in the radical but c not on $\langle u, v\rangle$, $\langle v, w\rangle$ or $\langle v, x\rangle$. Then $\langle c, x\rangle$ must intersect both $\langle u, v\rangle$ and $\langle v, w\rangle$ in a non-singular point different from v. Hence there exist two vectors in $\Gamma_1(0)$ such that x is on the tangent they generate; and so $x \in \Gamma_1(0)$.

If the dimension of rad is one, say $\mathrm{rad} = \langle \alpha\rangle$, then $\langle u, v, w\rangle / \langle \alpha\rangle$ must be hyperbolic. Let $y \in \langle v, u\rangle \backslash \langle v\rangle$ be a non-singular vector. The line $\langle y, x\rangle$ will intersect $\langle v, w\rangle \backslash \langle v\rangle$; this gives a bijection from $\langle v, u\rangle \backslash \langle v\rangle$ to $\langle v, w\rangle \backslash \langle v\rangle$. If $\langle u, v\rangle$ is a tangent, then there are $q-1$ choices for $\langle y\rangle$. If $\langle u, v\rangle$ is hyperbolic, then there are $q-2$ choices for $\langle y\rangle$. Hence if $q \geqslant 5$ or both lines are tangents, then there exist two vectors in $\Gamma_1(0)$ such that x is on the line they generate. Thus $q = 3$.

Let Q be the quadratic form and let $\mathscr{A} = \{x \in \Gamma_1(0) \mid Q(x) = 1\}$ and $\mathscr{B} = \{x \in \Gamma_1(0) \mid Q(x) = -1\}$. For any two vectors in $\mathscr{A}$, we have that the line they generate is elliptic or a tangent. By the previous paragraphs, all non-singular vectors of $\langle \mathscr{A}\rangle$ are in $\Gamma_1(0)$. The same holds for $\langle \mathscr{B}\rangle$. If $\langle \mathscr{A}\rangle = \langle \mathscr{B}\rangle$, then we are done, and so assume that these two subspaces are different. As the commutator subgroup of G_0 stabilizes these two subspaces, they must intersect trivially. Hence the lines generated by an element of $\mathscr{A}$ and one of $\mathscr{B}$ are hyperbolic, that is to say, the two spaces are orthogonal to each other and G_0 respects this decomposition. This leads to a contradiction. ∎

LEMMA 4·3. *If q is even, then all non-singular vectors are in $\Gamma_1(0)$.*

Proof. Choose $u, v, w \in \Gamma_1(0)$ such that $\langle u, v, w\rangle$ is a 3-dimensional space. Let $x \in \langle u, v, w\rangle$ be non-singular with x not in $\langle u, v\rangle$, $\langle u, w\rangle$ or $\langle v, w\rangle$. We will show that $x \in \Gamma_1(0)$. Any elliptic line on $\langle x\rangle$ will intersect $\langle u, v\rangle$, $\langle u, w\rangle$ and $\langle v, w\rangle$ in a non-singular point; hence there are at least two vectors $\alpha, \beta \in \Gamma_1(0)$ with $x \in \langle \alpha, \beta\rangle$, that is to say, $x \in \Gamma_1(0)$. Thus we may assume that x is not any elliptic line in $\langle u, v, w\rangle$. If this space is non-degenerate then x must have been the radical. But as all non-singular points different from x are in $\Gamma_1(0)$ and tangents on x contain at least 3 other non-singular points, we see that x must also be in $\Gamma_1(0)$. Let r be a singular vector in the radical of $W = \langle u, v, w\rangle$. There are three possible situations. If $W/\langle r\rangle$ is elliptic, then we have only tangents through $\langle r\rangle$ and the other lines are elliptic, which contradicts our assumption. If $W/\langle r\rangle$ is a tangent, then all lines containing non-singular vectors are tangents. As the singular points on the lines $\langle u, v\rangle$, $\langle u, w\rangle$ and $\langle v, w\rangle$ are different, we see that any tangent on x but not on $\{u, v, w\}$ must intersect at least two of the above lines in distinct non-singular points, whence x is in $\Gamma_1(0)$. Finally, if $W/\langle r\rangle$ is hyperbolic then all lines not on $\langle r\rangle$ are hyperbolic and contain at least 3 non-singular points. Without loss of generality we may assume that $\langle u, v\rangle$ and $\langle u, w\rangle$ are hyperbolic. As $q \geqslant 4$, we can replace w, if necessary, so that $\langle v, w\rangle$ is also hyperbolic. Now the tangent on x will intersect these 3 hyperbolic lines in at least 2 distinct non-singular points. Thus $x \in \Gamma_1(0)$. ∎

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COROLLARY 4·4. *The diameter of* Γ *is* 2.

Proof. This is obvious from the preceding lemmas. ∎

5. *Both sets* $\Gamma_1(0)$ *and* $\Gamma_2(0)$ *are non-singular*

In this case, for each pair of vectors $x, y \in \Gamma_1(0)$ the line $\langle x, y\rangle$ is elliptic. If $z \in \Gamma_1(0) \backslash \langle x, y\rangle$ then $\langle x, y, z\rangle$ will contain a singular vector, whence $\Gamma_3(0)$ consists of singular vectors.

First we shall deal with the case when the dimension of V is odd.

LEMMA 5·1. *If the dimension of* V *is odd, then* $q = 3$ *and* Γ *is a Hamming graph.*

Proof. By irreducibility, q is odd. Now there are two types of non-singular vectors. As an elliptic line contains both types, we must have one type at distance one and the other type at distance two from 0. Let $x, y, z \in \Gamma_1(0)$. Then all non-singular vectors of $\langle x, y, z\rangle$ are in $\Gamma_{\leqslant 2}(0)$. If there exists a tangent in $\langle x, y, z\rangle$ on x, then all non-singular vectors on this tangent are of the same type and hence must be in $\Gamma_1(0)$; thus we find a singular vector at distance two from 0, a contradiction. Consequently, $\langle x, y, z\rangle$ is not degenerate and x, y, z are not on a tangent in this O_3-subspace. We will call the type of a non-singular point in this O_3-subspace ϵ if the set of perpendicular points is isomorphic to O_2^ϵ. Now all vectors of type -1 of this space are in $\Gamma_1(0)$. For the same reason as before, no two such elements can be on a hyperbolic line. Hence $q = 3$ and $\Gamma_1(0)$ must be a basis for V. This proves the lemma. ∎

From now on we shall assume that the dimension of V is even. We shall deal with the case $q \leqslant 3$ separately.

LEMMA 5·2. *If* $q \geqslant 4$, *then all non-singular vectors are in* $\Gamma_{\leqslant 2}(0)$ *and* diam $(\Gamma) \leqslant 3$.

Proof. We wish to prove by induction that, for all $x_1, \ldots, x_m$ in $\Gamma_1(0)$, we have $\langle x_1, \ldots, x_m\rangle \subseteq \Gamma_{\leqslant 3}(0)$. This is clear for $m \leqslant 3$. Assume that it is true for $m = n-1$. Let $x_1, x_2, \ldots, x_{n-1}, x_n \in \Gamma_1(0)$ and let $H = \langle x_1, x_2, \ldots, x_{n-1}\rangle$ be such that all non-singular vectors of H are in $\Gamma_{\leqslant 2}(0)$. Let $z \in \langle H, x_n\rangle$ and $\langle y\rangle = H \cap \langle x, z\rangle$. We want to show that $z \in \Gamma_{\leqslant 3}(0)$. Without loss of generality we may assume that $x_n \notin H$ and $z \notin H \cup \{x_n\}$.

If y is non-singular then $y \in \Gamma_{\leqslant 2}(0)$ and thus $\langle x_n, y\rangle \subseteq \Gamma_{\leqslant 3}(0)$; hence $z \in \Gamma_{\leqslant 3}(0)$.

If y is singular, then take a non-singular vector $v \in H \cap \Gamma_1(0)$ and consider the space $\langle v, y, x_n\rangle$. As $\langle v, y\rangle$ is in H, all non-singular vectors of this 2-dimensional subspace are in $\Gamma_{\leqslant 2}(0)$, so all non-singular vectors in $\langle x, x_n\rangle$ are in $\Gamma_{\leqslant 2}(0)$, where x is a non-singular vector of $\langle v, y\rangle$. Hence, if $\langle v, z\rangle \cap \langle x, x_n\rangle$ is non-singular, then this vector is in $\Gamma_{\leqslant 2}(0)$. Consequently z is in $\Gamma_{\leqslant 3}(0)$. It is straightforward to see that x can be chosen in such a way that the intersection of the two lines is non-singular, except when $q = 4$, z is non-singular and $\langle v, y\rangle$ is hyperbolic. If $\langle x_n, y\rangle$ is a tangent, then we can replace z by z' so that the intersection will be a non-singular vector. Thus z' must be in $\Gamma_{\leqslant 2}(0)$, which forces z in $\Gamma_{\leqslant 3}(0)$. The only configuration left is where $\langle x_n, y\rangle$ and $\langle v, y\rangle$ are hyperbolic and, for each non-singular vector $a \in \langle x_n, y\rangle \backslash \langle x_n\rangle$ and non-singular vector in $b \in \langle v, y\rangle \backslash \langle v\rangle$, the intersection $\langle a, v\rangle \cap \langle x_n, b\rangle$ is singular. As the line $\langle v, x_n\rangle$ is elliptic we see that the corresponding projective plane contains 7 or 8 singular points, a contradiction. ∎

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LEMMA 5·3. *If* diam $(\Gamma) = 3$, *then* $q < 4$.

Proof. Suppose by way of contradiction that $q \geqslant 4$. Now we have all singular vectors at distance 3 from 0. If we fix a non-singular vector, then there are $q^{n-1}(q^{n-1}+\epsilon)$ singular vectors differing by a singular vector from the fixed one, and $(q^{n-1}+\epsilon)(q^n-q^{n-1}-\epsilon)$ differing by a non-singular one. Moreover, if we fix a singular vector, then there are $q^{2n-2}(q-1)$ singular vectors differing by a non-singular one. Thus we have

$$p^1_{3,3} = p^2_{3,3} = q^{n-1}(q^{n-1}+\epsilon), \quad k_3 = (q^n-\epsilon)(q^{n-1}+\epsilon),$$
$$k_1+k_2 = q^{n-1}(q^n-\epsilon)(q-1), \quad p^1_{3,2} = (q^{n-1}+\epsilon)(q^n-q^{n-1}-\epsilon),$$

and

$$p^3_{3,1}+p^3_{3,2} = q^{2n-2}(q-1).$$

As $kp^1_{3,3} = k_3 p^3_{3,1}$ we have $kq^{n-1} = (q^n-\epsilon)a_3$. Hence there exists a natural number α with

$$k = \alpha(q^n-\epsilon), \quad a_3 = \alpha q^{n-1} \quad \text{and} \quad c_3 = \alpha(q^n-q^{n-1}-\epsilon).$$

Similarly it follows from $k_2 p^2_{3,3} = k_3 p^3_{3,2}$ that there exists a natural number β with

$$k_2 = \beta(q^n-\epsilon) \quad \text{and} \quad p^3_{3,2} = \beta q^{n-1}.$$

Hence $\alpha+\beta = q^{n-1}(q-1)$. By proposition 4·2·18 of [2], we have $c_3+b_2 = k+1$ so that $b_2 = \alpha q^{n-1}+1$. From the relations $p^1_{2,3} = b_1 b_2/c_2$ and $b_1/c_2 = \beta/\alpha$, we obtain that

$$\alpha^2+\epsilon(q-2)\alpha+1-q = 0.$$

Hence if $\epsilon = 1$ then $\alpha = 1$, which leads to a contradiction with $c_2 \geqslant 2$. If $\epsilon = -1$ then $\alpha = q-1$ and $\beta = (q^{n-1}-1)(q-1)$. Hence

$$a_3 = q^{n-1}(q-1), \quad c_3 = (q^n-q^{n-1}+1)(q-1), \quad b_2 = (q^n-q^{n-1}+1)$$

and

$$b_1 = (q^{n-1}-1)c_2.$$

Now the vector $(1, -1/k, -1/k, c_3/(kb_2))$ is an eigenvector with multiplicity $q^{n-1}k$ belonging to the eigenvalue -1. Let f, g be the two remaining multiplicities unequal to 1. Then $f+g = (q^n+1)(q^{n-1}-1)$. The vector

$$u^s = \left(1, \frac{\theta_s}{k}, \frac{c_2}{kb_1}(a_3-\theta_s), \frac{-c_2 c_3}{kb_1}\right),$$

where $s \in \{f, g\}$, is an eigenvector belonging to the eigenvalues θ_s, with multiplicity s, where $\theta_s^2+(c_2-\lambda)\theta_s-k = c_2 a_3$. (Note that a_3 is an eigenvalue of the affine graph obtained by calling two vectors adjacent if and only if they differ by a non-singular vector.) As

$$s = \frac{q^{2n}}{\sum_{i=0}^4 k_i (u_i^s)^2}$$

it follows that $\sum_{i=0}^4 k_i(u_i^s)^2$ is rational. As

$$\sum_{i=0}^4 k_i(u_i^s)^2 = \frac{q^n}{(q^n+1)(q^{n-1}-1)(q-1)}$$
$$\times((q-1)(q^n+q^{n-1}+1+q^{n-2}c_2)+(q^n-q^{n-1}-1-q^{n-2}c_2)\theta_s),$$

θ_s must also be a rational number if $n \geqslant 3$. Straightforward computation yields that

$$\theta_s = \tfrac{1}{2}(q^{n+1}-q^n+q-2-q^{n-1}c_2 \pm q\sqrt{((q^n-q^{n-1}+1-q^{n-2}c_2)^2+4q^{n-2}c_2)}).$$

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As GF(q)* acts on Γ, we have $\lambda \geqslant q-2$; hence if $n \geqslant 3$, then $c_2 \leqslant q(q-1)$. If $n \geqslant 3$, write

$$s^2 = (q^n - q^{n-1} + 1 - q^{n-2}c_2)^2 + 4q^{n-2}c_2, \quad a = q^n - q^{n-1} + 1 \quad \text{and} \quad b = q^{n-2}c_2.$$

Then

$$s^2 = (a-b)^2 + 4b, \quad a > b \quad \text{and} \quad s = a - b + r$$

for some $r \geqslant 1$. Hence $4b = 2(a-b)r + r^2$, so that $r = 2\delta$ for some $\delta \geqslant 1$ and $b = (a-b)\delta + \delta^2$. Reducing the last equality modulo q^{n-2} yields $\delta \equiv 0, -1 \bmod q^{n-2}$. This leads to a contradiction to the fact that c_2 is an integer and $c_2 \leqslant q(q-1)$. It follows from the subgroup structure of $O_4^-(q) \cong L_2(q^2)$ that $n = 2$ cannot occur. ∎

In order to finish this section and the proof of the main theorem, we only need to handle the cases when $q = 2$ and $q = 3$. This will be done in the following two lemmas.

LEMMA 5·4. *If $q = 3$, then Γ is a Hamming graph.*

Proof. Let $x_1, x_2 \in \Gamma_1(0)$ be linearly independent with $Q(x_1) = Q(x_2)$. Then $\langle x_1, x_2 \rangle$ must be elliptic. Hence x_1, x_2 are orthogonal. Let

$$\mathscr{A} = \{x \in \Gamma_1(0) \mid Q(x) = 1\} \quad \text{and} \quad \mathscr{B} = \{x \in \Gamma_1(0) \mid Q(x) = -1\}.$$

Then two vectors $u, v \in \mathscr{A}$, with $u \neq \pm v$ are mutually orthogonal, and the same holds for $\mathscr{B}$. However, an element of $\mathscr{A}$ and an element of $\mathscr{B}$ are never orthogonal. By distance-transitivity, either the two sets have the same size or one of them is empty. If one of these sets is empty, then the other is projectively a basis for V and Γ would be a Hamming graph. Suppose that the dimension of V is not 4. Then both sets contain three linear independent vectors. Let x_1, x_2, x_3 be three linear independent vectors of $\mathscr{A}$ and y_1, y_2, y_3 be three linear independent vectors of $\mathscr{B}$. As three vectors of $\mathscr{A}$ generate a non-degenerate 3-space, we cannot have $z_1 = x_1 + x_2$ in $\mathscr{B}$. As both $\mathscr{A}$ and $\mathscr{B}$ are projectively cocliques, we must have $z_2 = x_1 + y_1$ also at distance 2 from 0. Let $z \in \{z_1, z_2\}$. The distance between z and an element $v \in \Gamma_1(0)$ is 3 if and only if $Q(z-v) = 0$. Consider the set $\mathscr{P} = \Gamma_1(0) \cap \Gamma_3(z) \cap \Gamma_3(-z)$. Then $v \in \Gamma_1(0)$ is in $\mathscr{P}$ if and only if $\langle v, z \rangle$ is hyperbolic, that is, if and only if $Q(v) = -Q(z)$ and $f(v, z) = 0$. So for z_1, we have $\mathscr{P} = \mathscr{A} - \{\pm x_1, \pm x_2\}$ while for z_2, $\mathscr{P}$ is equal to either $\{\pm x_1\}$ or $\{\pm y_1\}$. Distance-transitivity forces $|\mathscr{A}| = 6$, so $\dim(V) \leqslant 6$. If $\dim(V) = 6$ then Γ is again a Hamming graph. The case that the dimension of V is 4 remains. Now the set $\mathscr{A}$ can only contain 4, 6 or 8 vectors. If the sets $\mathscr{A}$ and $\mathscr{B}$ contain only four vectors, then again $\Gamma_1(0)$ must be a basis and one can finish as before. If $\mathscr{A}$ is a basis for V, then the induced geometry is isomorphic to O_4^+, so any element of $\mathscr{B}$ can be written as the sum of two elements of $\mathscr{A}$; hence there will be a singular vector at distance 2 from 0, a contradiction. Thus $|\mathscr{A}| = 6$. Write $\mathscr{A} = \{\pm x_1, \pm x_2, \pm x_3\}$ and let $\mathscr{B} = \{\pm y_1, \pm y_2, \pm y_3\}$. Moreover, let x_4 be orthogonal to all elements of $\mathscr{A}$. Then $\{x_1, x_2, x_3, x_4\}$ is a basis for V and $\langle x_1, x_2, x_3 \rangle$ is isomorphic to an O_3-geometry. If $Q(x_4) = 1$, then each element $\mathscr{B}$ can be written as the sum of $\pm x_4$ and one element of $\mathscr{A}$. Hence at a distance 2, one can find a vector which is the sum of three linearly independent mutually orthogonal vectors on which Q takes the same value. Clearly this vector must be singular, a contradiction. If $Q(x_4) = -1$ and $y \in \mathscr{B}$, then we must have $Q(y) = -1$ and $f(y, x_i) \neq 0$ for $i \in \{1, 2, 3\}$. Without loss of generality, we may assume that

$$y_1 = x_1 + x_2 + x_3 + x_4, \quad y_2 = a_1 x_1 + a_2 x_2 + a_3 x_3 + x_4 \quad \text{and} \quad y_3 = b_1 x_1 + b_2 x_2 + b_3 x_3 + x_4$$

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with $a_i, b_j \in \{\pm 1\}$ for all i,j. As $f(y_1, y_2) = f(y_1, y_2) = f(y_2, y_3) = 0$, we have

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 1 \quad \text{and} \quad a_1 b_1 + a_2 b_2 + a_3 b_3 = 1,$$

which leads to a contradiction. ∎

LEMMA 5·5. *If $q = 2$, then Γ is a Hamming graph or a folded $(n+1)$-cube with $n/2$ odd.*

Proof. Let $x_1, x_2, x_3 \in \Gamma_1(0)$. These three vectors span a Fano plane and it readily follows that $x_1 + x_2 + x_3$ is the only singular vector in this plane and that the sum of any two elements of $\Gamma_1(0)$ is in $\Gamma_2(0)$. Hence $\lambda = 0$ and $c_2 = 2$. Moreover, if f is a bilinear form belonging to Q, then for all $x, y \in \Gamma_1(0)$ we have $f(x, y) = 0$ if and only if $x = y$. Let $x_1, \ldots, x_n$ be a basis in $\Gamma_1(0)$. If Γ is not a Hamming graph, there exists an element $y \in \Gamma_1(0)$ not in this basis. Write $y = x_1 + x_2 + \ldots + x_r$. If $r \neq n$ then $f(x_1, y) = r - 1$ and $f(x_n, y) = r$, a contradiction. Hence $r = n$. So $\Gamma_1(0)$ contains $n+1$ vectors and it is straightforward to see that this graph is the folded $(n+1)$-cube, and as $x_1 + \ldots + x_{n-1}$ must be non-singular, $n/2$ must be odd. ∎

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