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CONCAVITY PROPERTIES OF EXTENSIONS OF THE PARALLEL VOLUME

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Abstract. In this paper we establish concavity properties of two extensions of the classical notion of the outer parallel volume. On the one hand, we replace the Lebesgue measure by more general measures. On the other hand, we consider a functional version of the outer parallel sets.

§1. *Introduction.* As an analogue of the famous *concavity of entropy power* in information theory (see e.g. [13, 31]), Costa and Cover [14] conjectured that the $1/n$-power of the *parallel volume* $|A + tB_2^n|$ is a concave function, where $|\cdot|$ denotes the Lebesgue measure and where B_2^n denotes the Euclidean closed unit ball.

CONJECTURE 1.1 (Costa–Cover [14]). *Let A be a bounded measurable set in $\mathbb{R}^n$. Then the function $t \mapsto |A + tB_2^n|^{1/n}$ is concave on $\mathbb{R}_+$.*

This conjecture has been studied by the author and Fradelizi in [19], where it was shown to be true for any sets in dimension one and for any connected sets in dimension two. However, they showed that the conjecture fails for arbitrary sets in dimension two and for arbitrary connected sets in dimension greater than or equal to three.

The notion of parallel volume can be extended by considering measures that are more general than the Lebesgue measure. An extension, provided by Borell in [7, 8], follows from the Brunn–Minkowski inequality, which states that for every $\lambda \in [0, 1]$ and for every compact subsets A, B of $\mathbb{R}^n$,

$$|(1-\lambda)A + \lambda B|^{1/n} \geqslant (1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}. \tag{1}$$

Borell defined for $s \in [-\infty, +\infty]$ the *s-concave measures*, satisfying by definition a similar inequality to (1):

$$\mu((1-\lambda)A + \lambda B) \geqslant M_s^\lambda(\mu(A), \mu(B))$$

for every $\lambda \in [0, 1]$ and for all compact subsets A, B of $\mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$, where $M_s^\lambda(a, b)$ denotes the s-mean of the non-negative real numbers a, b with weights $1 - \lambda$ and λ, defined as

$$M_s^\lambda(a, b) = ((1-\lambda)a^s + \lambda b^s)^{1/s} \quad \text{if } s \notin \{-\infty, 0, +\infty\},$$

$$M_{-\infty}^\lambda(a, b) = \min(a, b),\ M_0^\lambda(a, b) = a^{1-\lambda}b^\lambda,\ M_{+\infty}^\lambda(a, b) = \max(a, b).$$

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Basic properties of the s-concave measures are given in the next section. The case $s = 0$ corresponds to log-concave measures. The most famous example of a log-concave measure is the *standard multivariate Gaussian measure*

$$d\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}\, dx,$$

where $|\cdot|$ stands for the Euclidean norm. These measures are of interest. For example, isoperimetric inequalities have been established for the Gaussian measure γ_n by Borell [**9**] and independently by Sudakov and Cirel'son [**30**], which state that among sets of given Gauss measure, half-spaces minimize the Gauss surface area (see also [**24**, **3**, **11**, **15**]).

In this paper, we pursue the study of these measures by considering the following problem.

Problem 1. Let $s \in [-\infty, +\infty]$. Let μ be an s-concave measure on $\mathbb{R}^n$ and let A be a compact subset of $\mathbb{R}^n$. Is the function $t \mapsto \mu(A + tB_2^n)$ s-concave on $\mathbb{R}_+$?

Notice that from the Brunn–Minkowski inequality (1), the n-dimensional Lebesgue measure is $(1/n)$-concave. Thus Problem 1 generalizes the Costa–Cover conjecture.

Other extensions of geometric inequalities can be set up by considering functional versions. The most famous extension of this type in the Brunn–Minkowski theory is certainly the Prékopa–Leindler inequality (see [**27**, **29**, **8**]). Functional versions provide new proofs of geometric inequalities as well as new applications. Other examples of such extensions are a functional version of the Blaschke–Santaló inequality and a functional version of the Mahler conjecture (see e.g. [**2**, **1**, **20**, **21**, **26**]).

To establish a functional version of Conjecture 1.1, we consider a functional version of *parallel sets* $A + tB_2^n$. We set up the following problem (the notion of γ-concave functions is defined in the next section).

Problem 2. Let $\gamma \geqslant -1/n$. Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a measurable non-negative function and $g : \mathbb{R}^n \to \mathbb{R}_+$ be a γ-concave function. Let

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0;\ g(y)>0}} (f(x)^\gamma + t g(y)^\gamma)^{1/\gamma} \quad \text{and} \quad h_t^{(0)}(z) = \sup_{z=x+ty} f(x) g(y)^t.$$

Is the function $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z)\, dz$ $(\gamma/(1+\gamma n))$-concave on $\mathbb{R}_+$?

The Costa–Cover conjecture is a particular case of Problem 2 by taking $f = 1_A$, $g = 1_{B_2^n}$ and $\gamma = 0$.

For $\gamma < 0$, one can connect the function $h_t^{(\gamma)}$ with the Hopf–Lax solution of the Hamilton–Jacobi equation (see §4 for precise definitions). The Hopf–Lax solution is of interest. For example, it can be used to show that the *hypercontractivity* of this solution is equivalent to the *log-Sobolev inequalities*

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(see e.g. [4, 22]). In Problem 2, we pursue the study of this solution by investigating concavity properties with respect to the time variable of the Hopf–Lax solution of the Hamilton–Jacobi equation.

We shall prove that both Problems 1 and 2 have positive answers in dimension one. However, the Costa–Cover conjecture fails in dimension $n \geqslant 2$ in such a generality, so we won't expect positive answers of these more general problems in dimension $n \geqslant 2$. Using the geometric localization theorem of Kannan *et al* [24] in the form established by Fradelizi and Guédon [18] (see Theorem 3.2 below), we prove the following theorem.

THEOREM 1. *Let $s \in [-\infty, \frac{1}{2}] \cup [1, +\infty]$. Let μ be an s-concave measure on $\mathbb{R}$ and let A be a compact subset of $\mathbb{R}$. Then the function $t \mapsto \mu(A + t[-1, 1])$ is s-concave on $\mathbb{R}_+$.*

Moreover, for $s \in (\frac{1}{2}, 1)$ there exists a compact subset A of $\mathbb{R}$ such that $t \mapsto \mu(A + t[-1, 1])$ is not s-concave on $\mathbb{R}_+$ (see Proposition 3.3 below).

Using a precise analysis of the Hopf–Lax solution, we prove the following.

THEOREM 2. *Let $\gamma \in [-1, 0]$. Let $f : \mathbb{R} \to \mathbb{R}_+$ be such that f^γ (to be interpreted as $-\log(f)$ when $\gamma = 0$) is a bounded Lipschitz function. Define for every $y \in \mathbb{R}$, $V(y) = |y|^p/p$ with $p \geqslant 1$, and*

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0;\ V(y)>0}} (f(x)^\gamma + tV(y))^{1/\gamma}, \qquad h_t^{(0)}(z) = \sup_{z=x+ty} f(x)\mathrm{e}^{-tV(y)}.$$

If there exists $t_0 > 0$ such that for almost every $z \in \mathbb{R}$, $t \mapsto h_t^{(\gamma)}(z)$ is twice differentiable on $[0, t_0]$ with derivatives that are bounded by an integrable function, and if $\lim_{z\to\pm\infty} (\partial h_t^{(\gamma)}/\partial z)(z) = 0$, then the function $t \mapsto \int_{\mathbb{R}} h_t^{(\gamma)}(z)\, dz$ is concave on $[0, t_0]$.

Notice that the conclusion of Theorem 2 is stronger than the expected conclusion in Problem 2 (when restricted to log-concave functions of the form $|y|^p/p$, $p \geqslant 1$).

In the next section, we present some notation and explain basic properties of the s-concave measures. In §3, we give a complete answer to Problem 1. In §4, we give a partial answer to Problem 2. To conclude this paper, we derive a weighted Brascamp–Lieb-type inequality from our functional version.

§2. *Preliminaries.* We work in the Euclidean space $\mathbb{R}^n$, $n \geqslant 1$, equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the ℓ_2^n norm $|\cdot|$. The closed unit ball is denoted by B_2^n and the canonical basis by $(e_1, \ldots, e_n)$. We also denote by $|\cdot|$ the Lebesgue measure on $\mathbb{R}^n$. For non-empty sets $A, B \subset \mathbb{R}^n$ we define the *Minkowski sum*

$$A + B = \{a + b; a \in A, b \in B\}.$$

We denote by $\mathrm{int}(A)$ the *interior* of the set A, by $\overline{A}$ the *closure* of A and by $\partial A = \overline{A} \backslash \mathrm{int}(A)$ the *boundary* of A.

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For an arbitrary (non-negative) measure μ and a compact set A, we call the function defined on $\mathbb{R}_+$ by $t \mapsto V_A^\mu(t) = \mu(A + tB_2^n)$ the *outer parallel μ-volume* of A. We simply call this function the *outer parallel volume* when μ is the Lebesgue measure.

Let us recall basic properties of the s-concave measures. For $s \leqslant 1/n$, Borell showed in [8] that any s-concave measure which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$ admits a γ-concave density function, with

$$\gamma = \frac{s}{1 - sn} \in \left[-\frac{1}{n}, +\infty\right],$$

where a function f is said to be *γ-concave*, with $\gamma \in [-\infty, +\infty]$, if the inequality

$$f((1-\lambda)x + \lambda y) \geqslant M_\gamma^\lambda(f(x), f(y))$$

holds for every $\lambda \in [0, 1]$ and for every $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$. Notice that a 1-concave function is concave on its support, that the level sets of a $-\infty$-concave function are convex, and that a $+\infty$-concave function is constant on its support. For $s > 1$, Borell showed that every (non-null) s-concave measure is a Dirac measure. Notice that an s-concave measure is r-concave for all $r \leqslant s$.

§3. *The s-concavity of the parallel μ-volume.* In this section we solve Problem 1.

Let us note that since the function $t \mapsto \mu(A + tB_2^n)$ is non-decreasing, it follows that the answer to Problem 1 is positive for $s = -\infty$ (for every non-negative measure). Note also that Problem 1 is solved for convex sets. Indeed, if A is a compact convex subset of $\mathbb{R}^n$, then for every $\lambda \in [0, 1]$ and for every $t_1, t_2 \in \mathbb{R}_+$, one has

$$\begin{aligned}\mu(A + ((1-\lambda)t_1 + \lambda t_2)B_2^n) &= \mu((1-\lambda)(A + t_1B_2^n) + \lambda(A + t_2B_2^n)) \\ &\geqslant ((1-\lambda)\mu(A + t_1B_2^n)^s + \lambda\mu(A + t_2B_2^n)^s)^{1/s}.\end{aligned}$$

Before proving Theorem 1, we establish a preliminary lemma in dimension one.

LEMMA 3.1. *Let μ be an s-concave measure on $\mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure and let A be a compact subset of $\mathbb{R}$. Then the function $t \mapsto V_A^\mu(t) = \mu(A + t[-1, 1])$ admits left and right derivatives on $(0, +\infty)$, denoted by $(V_A^\mu)'_-$ and $(V_A^\mu)'_+$ respectively, which satisfy for every $t > 0$, $(V_A^\mu)'_-(t) \geqslant (V_A^\mu)'_+(t)$.*

Proof. Let us denote by ψ the density of the measure μ and let us denote by $[\alpha, \beta]$ the support of μ, with $-\infty \leqslant \alpha < \beta \leqslant +\infty$. Notice that for every $t > 0$, the set $A + t[-1, 1]$ is a finite disjoint union of intervals. Thus one can assume that $A = \bigcup_{i=1}^N [a_i, b_i]$, with $\alpha \leqslant a_1 < b_1 < \cdots < a_N < b_N \leqslant \beta$, and $a_1 = \alpha$ when $\alpha \in A$, $b_N = \beta$ when $\beta \in A$. Let $t_i = (a_{i+1} - b_i)/2$, $i \in \{0, \ldots, N\}$, with

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the convention that $b_0 = 2\alpha - a_1$ and that $a_{N+1} = 2\beta - b_N$. Notice that V_A^μ is differentiable on $(0, +\infty)\backslash\{t_0, \dots, t_N\}$ and that for every $t > 0$, one has

$$(V_A^\mu)'_+(t) = \sum_{a \in \partial(A+t[-1,1])} \psi(a) 1_{(\alpha,\beta)}(a),$$
$$(V_A^\mu)'_-(t) = \sum_{a \in \partial(A+t(-1,1))} \psi(a) 1_{[\alpha,\beta]}(a).$$

Notice that $\overline{A + t[-1, 1]} = \overline{A + t(-1, 1)}$ thus $\partial(A + t[-1, 1]) \subset \partial(A + t(-1, 1))$. We conclude that for every $t > 0$, $(V_A^\mu)'_-(t) \geqslant (V_A^\mu)'_+(t)$. □

The proof of Theorem 1 uses the following localization theorem.

THEOREM 3.2 (Fradelizi and Guédon [18]). *Let n be a positive integer, let K be a compact convex set in $\mathbb{R}^n$ and denote by $\mathcal{P}(K)$ the set of probability measures on $\mathbb{R}^n$, with support contained in K. Let $f : K \to \mathbb{R}$ be an upper semi-continuous function, let $s \in [-\infty, \frac{1}{2}]$ and denote by P_f the set of s-concave probability measures μ, with support contained in K, satisfying $\int f \, d\mu \geqslant 0$. Let $\Phi : \mathcal{P}(K) \to \mathbb{R}$ be a convex $w*$-upper semi-continuous function. Then* $\sup\{\Phi(\mu); \mu \in P_f\}$ *is achieved at a probability measure ν which is either a Dirac measure at a point x such that $f(x) \geqslant 0$ or a probability measure which is s-affine on a segment $[\alpha, \beta] \subset K$ such that $\int f \, d\nu = 0$ and $\int_\alpha^x f \, d\nu > 0$ on (α, β) or $\int_x^\beta f \, d\nu > 0$ on (α, β).*

Proof of Theorem 1. Let μ be an s-concave measure on $\mathbb{R}$ and let A be a compact subset of $\mathbb{R}$.

If $s > 1$ then μ is a Dirac measure δ_x, $x \in \mathbb{R}$, and one can see that the function $t \mapsto \delta_x(A + t[-1, 1])$ is constant on its support. Thus this function is $+\infty$-concave on $\mathbb{R}_+$, which proves Theorem 1 for $s > 1$. Notice that this argument is also valid in higher dimensions.

If $s = 1$ then for every $x \in \mathbb{R}$, $d\mu(x) = C 1_{[\alpha,\beta]}(x) \, dx$ with $C > 0$ is a constant and $[\alpha, \beta]$ is an interval of $\mathbb{R}$. One can see by direct computation that $t \mapsto \mu(A + t[-1, 1])$ is 1-concave on $\mathbb{R}_+$.

If $s \leqslant \frac{1}{2}$ then μ admits a density with respect to the Lebesgue measure on $\mathbb{R}$ which is $s/(1 - s)$-concave. We assume that $s \neq 0$; the case $s = 0$ follows by continuity. Let $t_0 > 0$ such that the set $A + t_0[-1, 1]$ is an interval. Notice that the function $t \mapsto V_A^\mu(t) = \mu(A + t[-1, 1])$ is s-concave on $[t_0, +\infty)$.

To prove that V_A^μ is s-concave on $(0, t_0)$, we shall reduce the problem to the extremal measures ν described in Theorem 3.2. For these specific measures, the outer parallel volume is easy to compute and is twice differentiable except for a finite number of points. Then we prove by differentiation that the function $t \mapsto V_A^\nu(t) = \nu(A+t[-1, 1])$ is s-concave outside points of non-differentiability. To conclude global s-concavity, we use Lemma 3.1.

Step 1: Reduction to extremal measures. Let $t_1, t_2 \in (0, t_0)$ such that $\mu(A + t_1[-1, 1])\mu(A + t_2[-1, 1]) > 0$. We assume that $t_1 < t_2$ without loss of

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generality. We define $K = A + t_0[-1, 1]$ and we consider the restriction of μ over K. Thus it is a finite measure that we can assume to be a probability measure without loss of generality. For convenience, we still denote this measure by μ. Let $0 < \varepsilon < t_2 - t_1$. We apply Theorem 3.2 to $f : K \to \mathbb{R}$ defined by

$$f = 1_{A+t_2[-1,1]} - \tau_\varepsilon 1_{A+(t_1+\varepsilon)(-1,1)}$$

and $\Phi_\varepsilon : \mathcal{P}(K) \to \mathbb{R}$ defined by

$$\Phi_\varepsilon = \rho_\varepsilon 1_{A+t_1[-1,1]} - 1_{A+(((t_1+\varepsilon)+t_2)/2)(-1,1)}$$

where

$$\rho_\varepsilon = \left(\frac{1}{2}\left(\frac{\mu(A + t_2[-1, 1])^s}{\mu(A + (t_1 + \varepsilon)[-1, 1])^s} + 1\right)\right)^{1/s},$$
$$\tau_\varepsilon = \frac{\mu(A + t_2[-1, 1])}{\mu(A + (t_1 + \varepsilon)[-1, 1])}.$$

The choice of introducing ε and the open interval $(-1, 1)$ is a technical trick to get upper semi-continuous functions regarding f and Φ_ε and to make the argument work for Dirac measures in Step 2 below. We shall prove that $\Phi_\varepsilon(\mu) \leqslant 0$; thus by letting ε go to 0 and by using that V_A^μ is continuous on $(0, t_0)$, this will lead to the conclusion that V_A^μ is s-concave on $[t_1, t_2]$, for arbitrary $t_1, t_2 \in (0, t_0)$. To prove that $\Phi_\varepsilon(\mu) \leqslant 0$, we shall prove that $\Phi_\varepsilon(\nu) \leqslant 0$ for the extremal probability measures ν described in Theorem 3.2. First, notice that $\tau_\varepsilon \geqslant 1$. If $\tau_\varepsilon = 1$, then V_A^μ is constant on $[t_1 + \varepsilon, t_2]$. Thus V_A^μ is s-concave on $[t_1 + \varepsilon, t_2]$. Thereafter, we assume that $\tau_\varepsilon > 1$.

Step 2: s-concavity for extremal measures. Let ν be a Dirac measure δ_x with x such that $f(x) \geqslant 0$. The condition $f(x) \geqslant 0$ says that $1_{A+t_2[-1,1]}(x) \geqslant \tau_\varepsilon 1_{A+(t_1+\varepsilon)(-1,1)}(x)$. Since $\tau_\varepsilon > 1$, it follows that $x \notin A + (t_1 + \varepsilon)(-1, 1)$. Thus $x \notin A + t_1[-1, 1]$. Hence $\Phi_\varepsilon(\delta_x) \leqslant 0$.

Let ν be an s-affine measure with support $[\alpha, \beta]$, i.e. the density of the measure ν, denoted by ψ, satisfies for every $x \in \mathbb{R}$, $\psi(x) = (mx + p)^{1/\gamma} 1_{[\alpha,\beta]}(x)$, where $\gamma = s/(1-s) \in [-1, 1]\backslash\{0\}$ and where $m \in \mathbb{R}\backslash\{0\}$, $p \in \mathbb{R}$ are such that for every $x \in [\alpha, \beta]$, $mx + p \geqslant 0$. We assume that $\int_{[x,\beta]} f \, d\nu < 0$ on (α, β), which means that for every $x \in (\alpha, \beta)$,

$$\nu((A + t_2[-1, 1]) \cap [x, \beta]) < \tau_\varepsilon \nu((A + (t_1 + \varepsilon)[-1, 1]) \cap [x, \beta]). \quad (2)$$

If $\beta \notin A + (t_1 + \varepsilon)[-1, 1]$, then there exists $x \in (\alpha, \beta)$ such that $(A + (t_1 + \varepsilon)[-1, 1]) \cap [x, \beta] = \emptyset$. This contradicts (2). It follows that $\beta \in A + (t_1 + \varepsilon)[-1, 1]$.

Notice that the function V_A^ν is twice differentiable on $[t_1 + \varepsilon, t_2]$ outside a finite number of points $s_0, \ldots, s_\ell$, with $s_0 := t_1 + \varepsilon < \cdots < s_\ell < s_{\ell+1} := t_2$.

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Case 1. $\alpha \in A+(t_1+\varepsilon)[-1,1]$. Let $j \in \{0,\dots,\ell\}$. Notice that $A+s_j[-1,1]$ is a finite disjoint union of intervals containing α and β, so we can assume that $A + s_j[-1,1] = \bigcup_{i=1}^{N}[a_i, b_i]$, with $a_1 \leqslant \alpha < b_1 < \cdots < a_N < \beta \leqslant b_N$. We assume that $N \geqslant 2$, otherwise the result clearly holds. Let $t \in [s_j, s_{j+1})$. We get

$$\begin{aligned}
V_A^{\nu}(t) &= \sum_{i=1}^{N} \int_{a_i - t}^{b_i + t} (mx+p)^{1/\gamma} 1_{[\alpha,\beta]}(x)\,dx,\\
(V_A^{\nu})'(t) &= \sum_{i=2}^{N} ((m(b_{i-1}+t)+p)^{1/\gamma} + (m(a_i - t)+p)^{1/\gamma}),\\
(V_A^{\nu})''(t) &= \frac{m}{\gamma}\left(\sum_{i=2}^{N} ((m(b_{i-1}+t)+p)^{(1-\gamma)/\gamma} - (m(a_i - t)+p)^{(1-\gamma)/\gamma})\right)\\
&\leqslant 0.
\end{aligned}$$

Hence the function V_A^{ν} is concave on $[s_j, s_{j+1})$. We deduce that V_A^{ν} is piecewise s-concave on $[t_1+\varepsilon, t_2]$. From Lemma 3.1, we get that for every $t \in [t_1+\varepsilon, t_2]$,

$$\begin{aligned}
s((V_A^{\nu})^s)'_-(t) &= s^2 (V_A^{\nu})'_-(t)(V_A^{\nu})^{s-1}(t) \geqslant s^2 (V_A^{\nu})'_+(t)(V_A^{\nu})^{s-1}(t)\\
&= s((V_A^{\nu})^s)'_+(t).
\end{aligned}$$

We conclude that the function V_A^{ν} is s-concave on $[t_1+\varepsilon, t_2]$.

Case 2. $\alpha \notin A+(t_1+\varepsilon)[-1,1]$. Let $j \in \{0,\dots,\ell\}$. If $\alpha \in A+s_j[-1,1]$ then from the previous case we can conclude that V_A^{ν} is s-concave on $[s_j, s_{j+1})$. Thus we can assume that $A + s_j[-1,1] = \bigcup_{i=1}^{N}[a_i, b_i]$, with $\alpha < a_1 < b_1 < \cdots < a_N < \beta \leqslant b_N$ and that $\alpha \notin A + s_{j+1}[-1,1]$. We assume that $N \geqslant 2$, otherwise the result clearly holds. In the following, we define $a_i(t) = m(a_i - t) + p$, $b_i(t) = m(b_i+t)+p$, $1 \leqslant i \leqslant N-1$ and $a_N(t) = m(a_N - t)+p$, $b_N(t) = m\beta+p$.

Let $t \in [s_j, s_{j+1})$. We get

$$\begin{aligned}
V_A^{\nu}(t) &= \frac{1}{m}\frac{\gamma}{1+\gamma}\sum_{i=1}^{N}(b_i(t)^{(1+\gamma)/\gamma} - a_i(t)^{(1+\gamma)/\gamma}),\\
(V_A^{\nu})'(t) &= \sum_{i=1}^{N-1}(b_i(t)^{1/\gamma} + a_i(t)^{1/\gamma}) + a_N(t)^{1/\gamma},\\
(V_A^{\nu})''(t) &= \frac{m}{\gamma}\left(-a_1(t)^{(1-\gamma)/\gamma} + \sum_{i=2}^{N}(b_{i-1}(t)^{(1-\gamma)/\gamma} - a_i(t)^{(1-\gamma)/\gamma})\right).
\end{aligned}$$

Then the function V_A^{ν} is s-concave on $[s_j, s_{j+1})$ if and only if for every $t \in [s_j, s_{j+1})$, $V_A^{\nu}(t)(V_A^{\nu})''(t) \leqslant (1-s)(V_A^{\nu})'(t)^2$ if and only if

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$$\left(\sum_{i=1}^{N}(a_i(t)^{(1+\gamma)/\gamma} - b_i(t)^{(1+\gamma)/\gamma})\right) \times \left(\sum_{i=1}^{N-1}(a_i(t)^{(1-\gamma)/\gamma} - b_i(t)^{(1-\gamma)/\gamma}) + a_N(t)^{(1-\gamma)/\gamma}\right) \leqslant \left(\sum_{i=1}^{N-1}(b_i(t)^{1/\gamma} + a_i(t)^{1/\gamma}) + a_N(t)^{1/\gamma}\right)^2.$$

If $m\gamma > 0$, then one has

$$0 \leqslant a_1(t)^{(1-\gamma)/\gamma} < b_1(t)^{(1-\gamma)/\gamma} < \cdots < a_N(t)^{(1-\gamma)/\gamma} < b_N(t)^{(1-\gamma)/\gamma}.$$

Thus $(V_A^\nu)''(t) \leqslant 0$. Hence the function V_A^ν is concave on $[s_j, s_{j+1})$. Using the same argument as in Case 1, we conclude that the function V_A^ν is s-concave on $[t_1 + \varepsilon, t_2]$.

If $m\gamma < 0$, then one has

$$a_1(t)^{(1-\gamma)/\gamma} > b_1(t)^{(1-\gamma)/\gamma} > \cdots > a_N(t)^{(1-\gamma)/\gamma} > b_N(t)^{(1-\gamma)/\gamma} \geqslant 0.$$

We deduce that

$$\left(\sum_{i=1}^{N}(a_i(t)^{(1+\gamma)/\gamma} - b_i(t)^{(1+\gamma)/\gamma})\right) \times \left(\sum_{i=1}^{N-1}(a_i(t)^{(1-\gamma)/\gamma} - b_i(t)^{(1-\gamma)/\gamma}) + a_N(t)^{(1-\gamma)/\gamma}\right) \leqslant a_1(t)^{(1+\gamma)/\gamma}a_1(t)^{(1-\gamma)/\gamma} \leqslant \left(\sum_{i=1}^{N-1}(b_i(t)^{1/\gamma} + a_i(t)^{1/\gamma}) + a_N(t)^{1/\gamma}\right)^2.$$

Hence the function V_A^ν is s-concave on $[s_j, s_{j+1})$. We conclude that V_A^ν is s-concave on $[t_1 + \varepsilon, t_2]$.

Hence we get that $\Phi_\varepsilon(\nu) \leqslant 0$ and it follows that V_A^μ is s-concave on $(0, t_0)$. We have seen that V_A^μ is s-concave on $[t_0, +\infty)$ and using Lemma 3.1 we conclude that V_A^μ is s-concave on $(0, +\infty)$. Finally, from the non-decreasing property of V_A^μ, it follows that V_A^μ is s-concave on $\mathbb{R}_+$. □

Remark. The result clearly holds if we replace the interval $[-1, 1]$ by any symmetric interval. However, it is not necessarily true for an arbitrary interval. For example, let $0 < s \leqslant \frac{1}{2}$ and take $B = [0, 1]$, $A = [0, 1] \cup [2, 3]$ and $d\mu(x) = x^{1/\gamma} 1_{[0,3]}(x)\, dx$, with $\gamma = s/(1 - s)$. Then μ is an s-concave measure. For $t \in [0, \frac{1}{2})$ we get

$$V_A^\mu(t) = \mu(A + tB) = \frac{\gamma}{\gamma + 1}((1 + t)^{(\gamma+1)/\gamma} + 3^{(\gamma+1)/\gamma} - 2^{(\gamma+1)/\gamma}).$$

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Thus,

$$V_A^\mu(0)(V_A^\mu)''(0) - (1-s)(V_A^\mu)'(0)^2 = \frac{1}{\gamma+1}(3^{(\gamma+1)/\gamma} - 2^{(\gamma+1)/\gamma}) > 0.$$

Hence V_A^μ is not s-concave on $\mathbb{R}_+$. For $s = 0$, the same example works. For $s < 0$, one can take $B = [-1, 0]$, $A = [0, 1] \cup [2, 3]$ and $d\mu(x) = x^{1/\gamma} 1_{[\alpha,3]}(x)\, dx$, with $\gamma = s/(1-s)$ and α sufficiently small.

The localization theorem (Theorem 3.2) holds only for $s \leqslant \frac{1}{2}$. Thus this theorem could not be used to examine the case of s-concave measures, with $s > \frac{1}{2}$. For $s \in (\frac{1}{2}, 1)$, the answer to Problem 1 is negative in general as shown in Proposition 3.3 below, but under specific conditions regarding the support of the measure, we can show that a stronger positive answer holds. First, let us show that for $s \in (\frac{1}{2}, 1)$ the answer to Problem 1 is negative in dimension one.

PROPOSITION 3.3. *Let $s \in (\frac{1}{2}, 1)$ and let $\gamma = s/(1-s)$. Let $\beta = 10(1 - 2^{(1-\gamma)/\gamma})^{-1}$ and let μ be a measure such that $d\mu(x) = x^{1/\gamma} 1_{[0,\beta]}(x)\, dx$. Let us set $A = [0, 1] \cup [2, \beta]$. Then the function $t \mapsto V_A^\mu(t) = \mu(A + t[-1, 1])$ is not s-concave on $\mathbb{R}_+$.*

Proof. For every $t \in [0, \frac{1}{2})$,

$$\begin{aligned} V_A^\mu(t) &= \frac{\gamma}{\gamma+1}((1+t)^{(1+\gamma)/\gamma} + \beta^{(1+\gamma)/\gamma} - (2-t)^{(1+\gamma)/\gamma}), \\ (V_A^\mu)'(t) &= (1+t)^{1/\gamma} + (2-t)^{1/\gamma}, \\ (V_A^\mu)''(t) &= \frac{1}{\gamma}((1+t)^{(1-\gamma)/\gamma} - (2-t)^{(1-\gamma)/\gamma}). \end{aligned}$$

Hence,

$$\begin{aligned} &V_A^\mu(0)(V_A^\mu)''(0) - (1-s)(V_A^\mu)'(0)^2 \\ &\quad = \frac{1}{\gamma+1}(\beta^{(1+\gamma)/\gamma}(1 - 2^{(1-\gamma)/\gamma}) - 2^{(1-\gamma)/\gamma} - 2^{(1+2\gamma)/\gamma}). \end{aligned}$$

Since $1 - 2^{(1-\gamma)/\gamma} > 0$ and $\beta > ((2^{(1+2\gamma)/\gamma} + 2^{(1-\gamma)/\gamma})/(1 - 2^{(1-\gamma)/\gamma}))^{\gamma/(\gamma+1)}$, it follows that $V_A^\mu(0)(V_A^\mu)''(0) - (1-s)(V_A^\mu)'(0)^2 > 0$. We conclude that V_A^μ is not s-concave on $\mathbb{R}_+$. □

We denote by $\mathrm{supp}(\mu)$ the support of μ and by $\mathrm{dist}(A, \mathrm{supp}(\mu)^c)$ the distance between A and the complement of the support of μ. When the support of μ is $\mathbb{R}$, the distance will be equal to $+\infty$.

PROPOSITION 3.4. *Let $s \geqslant \frac{1}{2}$. Let μ be an s-concave measure on $\mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure. Let A be a compact subset of $\mathbb{R}$ such that $\mathrm{dist}(A, \mathrm{supp}(\mu)^c)) > 0$. Then the function $t \mapsto V_A^\mu(t) = \mu(A + t[-1, 1])$ is concave on $[0, \mathrm{dist}(A, \mathrm{supp}(\mu)^c))$.*

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Proof. First, we assume that $s = \frac{1}{2}$. Hence μ admits a 1-concave density function denoted by ψ. Notice that V_A^μ is differentiable outside a finite number of points $t_0, \dots, t_N$ and that for every $t \in [0, \operatorname{dist}(A, \operatorname{supp}(\mu)^c))\backslash\{t_0, \dots, t_N\}$ one has

$$V_A^\mu(t) = \sum_{i=1}^{N} \int_{a_i - t}^{b_i + t} \psi(x)\, dx, \qquad (V_A^\mu)'(t) = \sum_{i=1}^{N} (\psi(b_i + t) + \psi(a_i - t)).$$

Since ψ is concave, it follows that for every $i \in \{1, \dots, N\}$, the function $t \mapsto \psi(b_i + t) + \psi(a_i - t)$ is non-increasing. Thus $(V_A^\mu)'$ is piecewise non-increasing. We conclude that V_A^μ is piecewise concave on $[0, \operatorname{dist}(A, \operatorname{supp}(\mu)^c))$. From Lemma 3.1, we deduce that V_A^μ is concave on $[0, \operatorname{dist}(A, \operatorname{supp}(\mu)^c))$.

Finally, if μ is s-concave with $s \geqslant \frac{1}{2}$, then μ is $\frac{1}{2}$-concave and we conclude from the first part of the proof that V_A^μ is concave on $[0, \operatorname{dist}(A, \operatorname{supp}(\mu)^c))$. □

Now we turn to the study of Problem 1 in dimension $n \geqslant 2$. It was shown in [**19**] that the Costa–Cover conjecture is false in dimension $n \geqslant 2$, and thus the answer to Problem 1 is negative in general. Let us recall the counterexample.

Let $n \geqslant 2$. Let us set $A = B_2^n \cup \{2e_1\}$ and let us define $V_A(t) = |A + tB_2^n|$. Then for every $t \in [0, \frac{1}{2})$, one has

$$V_A(t) = |B_2^n \cup \{2e_1\} + tB_2^n| = |B_2^n + tB_2^n| + |tB_2^n| = |B_2^n|((1+t)^n + t^n).$$

Since the $1/n$-power of this function is not concave (it is strictly convex), V_A is not $1/n$-concave on $\mathbb{R}_+$ for $n \geqslant 2$.

It could appear surprising that Problem 1 has a negative answer in dimension $n \geqslant 2$ since we proved that this problem has a positive answer in dimension one with the localization theorem. This localization technique is usually used to reduce inequalities for general convex measures in dimension n to inequalities for measures for which the support is a segment. Thus the problem becomes one-dimensional (see e.g. [**17**] and references therein). Let us explain why one cannot reduce Problem 1 to dimension one. The reduction done in dimension one with localization works the same way in dimension n and we get the following equivalence for every compact set A of $\mathbb{R}^n$:

(i) V_A^μ is s-concave for every measure μ s-concave; and
(ii) V_A^ν is s-concave for every measure ν s-affine on a segment $[\alpha, \beta]$.

However, (ii) is not true in dimension $n \geqslant 2$. Indeed, we can construct an explicit counterexample to show that the function $t \mapsto |(A + tB_2^n) \cap [\alpha, \beta]|_1$ is not continuous everywhere inside its support and hence this function is not s-concave. For example, consider $A = \{(0,0)\} \cup \{(3,0)\} \cup \{(x,1); x \in [1,2]\}$ and $[\alpha, \beta] = \{(x,0); x \in [0,3]\}$.

In dimension two, it was shown in [**19**] that if A is a connected subset of $\mathbb{R}^2$ then the function $t \mapsto |A + tB_2^2|$ is $\frac{1}{2}$-concave on $\mathbb{R}_+$. However, the following proposition shows that this property fails in general if we replace the Lebesgue measure by an arbitrary s-concave measure.

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PROPOSITION 3.5. *In dimension two, there exists a connected set A and a $\frac{1}{2}$-concave measure μ such that $t \mapsto \mu(A + tB_2^2)$ is not $\frac{1}{2}$-concave on $\mathbb{R}_+$.*

Proof. We set $d\mu(x) = 1_{B_1^2}(x)\,dx$, where B_1^2 denotes the unit ball for the ℓ_1^2 norm. Hence μ is $\frac{1}{2}$-concave. We construct the points $B = (-1, 0)$, $C = (-0.5, -0.5)$, $D = (0.5, 0.5)$, $E = (0, 1)$, $F = (-2, 0)$, $G = (0, -2)$, $H = (0, -1)$, $I = (2, 0)$, $J = (1, 0)$. We set

$$A = \mathrm{conv}(BCDE) \cup [FB] \cup [FG] \cup [GH] \cup [GI] \cup [IJ].$$

Then A is connected and for every $t \in [0, \frac{1}{8}]$ we get

$$V_A^\mu(t) = \mu(A + tB_2^2) = \frac{\sqrt{2}}{2} + \sqrt{2}t + \frac{\pi}{2}t^2.$$

It follows that $(\sqrt{V_A^\mu})''(0) > 0$. We conclude that $t \mapsto \mu(A + tB_2^2)$ is not $\frac{1}{2}$-concave on $\mathbb{R}_+$. □

Remark. Notice that we can adapt the counterexample of Proposition 3.5 to show that there exists an s-concave measure μ on $\mathbb{R}^n$, $n \geqslant 2$, such that for every $r \in (-\infty, s)$ there exists a compact connected set $A \subset \mathbb{R}^n$ such that $t \mapsto \mu(A + tB_2^n)$ is not r-concave on $\mathbb{R}_+$.

Question. Does there exist $s \leqslant \frac{1}{2}$ such that for every s-concave measure μ on $\mathbb{R}^2$ and for every compact set $A \subset \mathbb{R}^2$ the function $t \mapsto \mu(A + tB_2^2)$ is s-concave for t so that the set $\mathrm{supp}(\mu) \cap (A + tB_2^2)$ is connected?

In particular, for every convex subset K of $\mathbb{R}^2$ and for every compact set $A \subset \mathbb{R}^2$, is the function $t \mapsto |(A + tB_2^2) \cap K|$ $\frac{1}{2}$-concave for t so that the set $K \cap (A + tB_2^2)$ is connected?

§4. *Functional version.* In this section, we give a partial answer to Problem 2.

As in §3, where Problem 1 was solved for convex sets, the next proposition shows that Problem 2 is solved for γ-concave functions, as is expected.

PROPOSITION 4.1. *Let $\gamma \geqslant -1/n$. Let $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be two γ-concave functions. Then the function $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z)\,dz$ is $\gamma/(1 + \gamma n)$-concave on $\mathbb{R}_+$, where*

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0;\ g(y)>0}} (f(x)^\gamma + tg(y)^\gamma)^{1/\gamma} \quad \text{and} \quad h_t^{(0)}(z) = \sup_{z=x+ty} f(x)g(y)^t.$$

Proof. We examine the case $\gamma \neq 0$. The case $\gamma = 0$ can be proved with the same argument.

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For convenience, let us define $h_t = h_t^{(\gamma)}$. Let $\lambda \in [0,1]$ and let $t_1, t_2 \in \mathbb{R}_+$. We want to show that

$$\int_{\mathbb{R}^n} h_{(1-\lambda)t_1+\lambda t_2} \geqslant \left((1-\lambda)\left(\int_{\mathbb{R}^n} h_{t_1}\right)^{\gamma/(1+\gamma n)} + \lambda\left(\int_{\mathbb{R}^n} h_{t_2}\right)^{\gamma/(1+\gamma n)}\right)^{(1+\gamma n)/\gamma}.$$

From the Borell–Brascamp–Lieb inequality [8, 10] (dimensional Prékopa inequality), it is sufficient to show that for every $z_1, z_2 \in \mathbb{R}^n$ one has

$$h_{(1-\lambda)t_1+\lambda t_2}((1-\lambda)z_1 + \lambda z_2) \geqslant ((1-\lambda)h_{t_1}(z_1)^\gamma + \lambda h_{t_2}(z_2)^\gamma)^{1/\gamma}.$$

Let $z_1, z_2 \in \mathbb{R}^n$. Let $x_1, x_2 \in \mathbb{R}^n$ such that

$$\text{for all } i \in \{1,2\}, \quad h_{t_i}(z_i) = \left(f(x_i)^\gamma + t_i g\left(\frac{z_i - x_i}{t_i}\right)^\gamma\right)^{1/\gamma}.$$

Let us define $h = h_{(1-\lambda)t_1+\lambda t_2}((1-\lambda)z_1 + \lambda z_2)$ and $t = (1-\lambda)t_1 + \lambda t_2$. We get

$$\begin{aligned}
h &= \sup_{x\in\mathbb{R}^n}\left(f(x)^\gamma + t g\left(\frac{(1-\lambda)z_1 + \lambda z_2 - x}{t}\right)^\gamma\right)^{1/\gamma} \\
&\geqslant \Bigg(f((1-\lambda)x_1 + \lambda x_2)^\gamma \\
&\quad + t g\left(\frac{(1-\lambda)z_1 + \lambda z_2 - ((1-\lambda)x_1 + \lambda x_2)}{t}\right)^\gamma\Bigg)^{1/\gamma} \\
&\geqslant \Bigg((1-\lambda) f(x_1)^\gamma + \lambda f(x_2)^\gamma + (1-\lambda)t_1 g\left(\frac{z_1 - x_1}{t_1}\right)^\gamma \\
&\quad + \lambda t_2 g\left(\frac{z_2 - x_2}{t_2}\right)^\gamma\Bigg)^{1/\gamma} \\
&= ((1-\lambda)h_{t_1}(z_1)^\gamma + \lambda h_{t_2}(z_2)^\gamma)^{1/\gamma}. \qquad \square
\end{aligned}$$

As a consequence of the Hölder inequality, if $f : \mathbb{R}^n \to \mathbb{R}_+$ is β-concave and if $g : \mathbb{R}^n \to \mathbb{R}_+$ is γ-concave, then fg is α-concave for every $\alpha, \beta, \gamma \in \mathbb{R}\cup\{+\infty\}$ such that $\beta+\gamma \geqslant 0$ and $1/\beta+1/\gamma = 1/\alpha$ (see e.g. [23]). A generalized form of Proposition 4.1 follows.

PROPOSITION 4.2. *Let $\gamma \geqslant -1/n$. If a measure μ has a β-concave density, with $\beta \geqslant -\gamma$, and if $f, g : \mathbb{R}^n \to \mathbb{R}_+$ are two γ-concave functions, then $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z)\, d\mu(z)$ is $(\alpha/(1+\alpha n))$-concave on $\mathbb{R}_+$, where $1/\beta + 1/\gamma = 1/\alpha$.*

In Problem 2, for $\gamma < 0$ the function $V = g^\gamma$ is convex by assumption and one can naturally connect the function $h_t^{(\gamma)}$ with the Hopf–Lax solution of the Hamilton–Jacobi equation:

$$h_t^{(\gamma)}(z) = \sup_{x\in\mathbb{R}^n}\left(f(x)^\gamma + t V\left(\frac{z-x}{t}\right)\right)^{1/\gamma} = (Q_t^{(V)} f^\gamma(z))^{1/\gamma},$$

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where for arbitrary convex function V and for arbitrary function u,

$$Q_t^{(V)}u(z) = \inf_{x\in\mathbb{R}^n}\left(u(x) + tV\left(\frac{z-x}{t}\right)\right).$$

Let us recall the basic properties of the Hopf–Lax solution of the Hamilton–Jacobi equation. For a convex function V such that $\lim_{|z|\to+\infty} V(z)/|z| = +\infty$ and for a Lipschitz function u, it is known that $Q_t^{(V)}u$ is the solution, called the *Hopf–Lax solution*, of the following partial differential equation, called the *Hamilton–Jacobi equation* (see e.g. [16]).

$$\begin{cases} \dfrac{\partial}{\partial t}h(t,z) + V^*(\nabla_z h(t,z)) = 0 & \text{on } (0,+\infty)\times\mathbb{R}^n, \\ h(t,z) = u(z) & \text{on } \{t=0\}\times\mathbb{R}^n, \end{cases}$$

where V^* is the Legendre transform of V defined on $\mathbb{R}^n$ by

$$V^*(y) = \sup_{x\in\mathbb{R}^n}(\langle x,y\rangle - V(x)).$$

It is shown in [16] that if u is Lipschitz on $\mathbb{R}^n$ then $Q_t^{(V)}u$ is Lipschitz on $[0,+\infty)\times\mathbb{R}^n$. However, for an arbitrary convex function V, $t\mapsto Q_t^{(V)}u$ is not necessarily continuous at 0.

Proof of Theorem 2. We examine the case $\gamma\neq 0$. The case $\gamma = 0$ can be proved with the same argument. Since $\gamma\in[-1,0)$, one has

$$h_t^{(\gamma)}(z) = \sup_{x\in\mathbb{R}}\left(f(x)^\gamma + tV\left(\frac{z-x}{t}\right)\right)^{1/\gamma} = (Q_t^{(V)}f^\gamma(z))^{1/\gamma}.$$

We define for $t\in\mathbb{R}_+$,

$$F(t) = \int_{\mathbb{R}} h_t^{(\gamma)}(z)\,dz = \int_{\mathbb{R}}(Q_t^{(V)}f^\gamma(z))^{1/\gamma}\,dz.$$

For $p=1$, the function F is constant. We then consider $p>1$. For convenience, we set $\phi(t,z) = Q_t^{(V)}f^\gamma(z)$ and $\phi' = \partial\phi/\partial z$. By assumption, there exists $t_0>0$ such that for almost every $z\in\mathbb{R}$, $t\mapsto h_t^{(\gamma)}(z)$ is twice differentiable on $[0,t_0]$ with derivatives that are bounded by an integrable function. Hence, we get for every $t\in[0,t_0]$,

$$F'(t) = -\frac{1}{\gamma}\int_{\mathbb{R}} V^*(\phi')\phi^{(1-\gamma)/\gamma},$$
$$F''(t) = \frac{1}{\gamma}\int_{\mathbb{R}}\phi''((V^*)'(\phi'))^2\phi^{(1-\gamma)/\gamma} + \frac{1-\gamma}{\gamma^2}\int_{\mathbb{R}}(V^*(\phi'))^2\phi^{(1-2\gamma)/\gamma}.$$

We assumed that $V(u) = |u|^p/p$. Hence $V^*(u) = |u|^q/q$, with $1/p+1/q=1$. It follows that

$$F''(t) = \frac{1}{\gamma}\int_{\mathbb{R}}\phi''(\phi')^{2q-2}\phi^{(1-\gamma)/\gamma} + \frac{1-\gamma}{\gamma^2}\int_{\mathbb{R}}\frac{(\phi')^{2q}}{q^2}\phi^{(1-2\gamma)/\gamma}.$$

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By assumption, $\lim_{z\to\pm\infty}(\partial h_t^{(\gamma)}/\partial z)(z)=0$. Thus integration by parts gives

$$\frac{2q-1}{\gamma}\int_{\mathbb{R}}\phi''(\phi')^{2q-2}\phi^{(1-\gamma)/\gamma}=-\frac{1-\gamma}{\gamma^2}\int_{\mathbb{R}}(\phi')^{2q}\phi^{(1-2\gamma)/\gamma}.$$

Finally,

$$F''(t)=-\frac{1-\gamma}{\gamma^2}\frac{(q-1)^2}{q^2(2q-1)}\int_{\mathbb{R}}(\phi')^{2q}\phi^{(1-2\gamma)/\gamma}\leqslant 0.$$

We conclude that $t\mapsto\int_{\mathbb{R}}h_t^{(\gamma)}(z)\,dz$ is concave on $[0,t_0]$. □

Open problem. Problem 2 is open in dimension one for arbitrary γ-concave function g.

§5. *Links with weighted Brascamp–Lieb-type inequalities.* In this section, we express the s-concavity of the function $t\mapsto\int_{\mathbb{R}^n}h_t^{(\gamma)}(z)\,dz$ in term of a weighted Brascamp–Lieb-type inequality.

PROPOSITION 5.1. *Let $\gamma\in[-1/n,0)$ and let $s\in\mathbb{R}$. Let $f:\mathbb{R}^n\to\mathbb{R}_+$ be such that f^γ is a bounded Lipschitz function. Let $V:\mathbb{R}^n\to\mathbb{R}_+$ be a convex function such that $\lim_{|z|\to+\infty}V(z)/|z|=+\infty$. Let us define for every $z\in\mathbb{R}^n$,*

$$h_t^{(\gamma)}(z)=\sup_{\substack{z=x+ty\\ f(x)>0;\ V(y)>0}}(f(x)^\gamma+tV(y))^{1/\gamma}.$$

If there exists $t_0>0$ such that for almost every $z\in\mathbb{R}$, $t\mapsto h_t^{(\gamma)}(z)$ is twice differentiable on $[0,t_0]$ with derivatives that are bounded by an integrable function, and if $G\in L^2(\mu)$ where $d\mu(z)=((Q_t^{(V)}f^\gamma(z))^{1/\gamma}/\int(Q_t^{(V)}f^\gamma)^{1/\gamma})\,dz$ and $G=V^*(\nabla_z Q_t^{(V)}f^\gamma)/Q_t^{(V)}f^\gamma$, then the function $t\mapsto\int_{\mathbb{R}^n}h_t^{(\gamma)}(z)\,dz$ is s-concave on $[0,t_0]$ if and only if

$$\begin{aligned}\mathrm{Var}_\mu(G)\leqslant&-\frac{\gamma}{1-\gamma}\\&\times\int\frac{\langle(\mathrm{Hess}\,Q_t^{(V)}f^\gamma)(\nabla_z V^*)(\nabla_z Q_t^{(V)}f^\gamma),(\nabla_z V^*)(\nabla_z Q_t^{(V)}f^\gamma)\rangle}{Q_t^{(V)}f^\gamma}\,d\mu\\&+\frac{\gamma-s}{1-\gamma}\left(\int G\,d\mu\right)^2.\end{aligned}$$

Proof. Recall that $h_t^{(\gamma)}(z)=(Q_t^{(V)}f^\gamma(z))^{1/\gamma}$ where, for arbitrary convex function V and arbitrary function u,

$$Q_t^{(V)}u(z)=\inf_{x\in\mathbb{R}^n}\left(u(x)+tV\left(\frac{z-x}{t}\right)\right).$$

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For convenience, let us define $\phi = f^\gamma$ and $Q_t = Q_t^{(V)}$. We get

$$\frac{\partial h_t^{(\gamma)}}{\partial t} = -\frac{1}{\gamma} V^*(\nabla_z Q_t\phi)(Q_t\phi)^{(1-\gamma)/\gamma},$$
$$\frac{\partial^2 h_t^{(\gamma)}}{\partial t^2} = \frac{1}{\gamma}\langle(\mathrm{Hess}\, Q_t\phi)(\nabla_z V^*)(\nabla_z Q_t\phi), (\nabla_z V^*)(\nabla_z Q_t\phi)\rangle(Q_t\phi)^{(1-\gamma)/\gamma} + \frac{1-\gamma}{\gamma^2}(V^*(\nabla_z Q_t\phi))^2(Q_t\phi)^{(1-2\gamma)/\gamma}.$$

Thus the function F is s-concave if and only if $F(t)F''(t) \leqslant (1-s)F'(t)^2$ if and only if

$$\mathrm{Var}_\mu(G) \leqslant -\frac{\gamma}{1-\gamma}\int \frac{\langle(\mathrm{Hess}\, Q_t\phi)(\nabla_z V^*)(\nabla_z Q_t\phi), (\nabla_z V^*)(\nabla_z Q_t\phi)\rangle}{Q_t\phi}\, d\mu + \frac{\gamma - s}{1-\gamma}\left(\int G\, d\mu\right)^2,$$

where $d\mu(z) = ((Q_t\phi(z))^{1/\gamma}/\int (Q_t\phi)^{1/\gamma})\, dz$ and $G = (V^*(\nabla_z Q_t\phi)/Q_t\phi)$. □

Remark. For $\gamma = 0$ and $V(u) = |u|^2/2$, one may use the same argument to get that $t \mapsto \int_{\mathbb{R}^n} h_t^{(0)}(z)\, dz$ is log-concave if and only if

$$\mathrm{Var}_\mu(|\nabla_z Q_t\phi|^2) \leqslant 4\int \langle(\mathrm{Hess}\, Q_t\phi)\nabla_z Q_t\phi, \nabla_z Q_t\phi\rangle\, d\mu,$$

where $\phi = -\log f$ and $d\mu(z) = (\mathrm{e}^{-Q_t\phi(z)}/\int \mathrm{e}^{-Q_t\phi})\, dz$.

Question. For which function u does the following inequality hold?

$$\mathrm{Var}_\mu(|\nabla_z u|^2) \leqslant 4\int \langle(\mathrm{Hess}\, u)\nabla_z u, \nabla_z u\rangle\, d\mu,$$

where $d\mu(z) = (\mathrm{e}^{-u(z)}/\int \mathrm{e}^{-u})\, dz$.

From Proposition 4.1, if f is γ-concave with $\gamma \in [-1/n, 0)$, then one may apply Proposition 5.1 to get the following weighted Brascamp–Lieb-type inequality by letting t go to 0.

COROLLARY 5.2. *Let $\gamma \in [-1/n, 0)$ and let $s = \gamma/(1+\gamma n)$. For every $V, \phi : \mathbb{R}^n \to \mathbb{R}_+$ convex such that $\lim_{|z|\to+\infty} V(z)/z = +\infty$, one has*

$$\mathrm{Var}_\mu(G) \leqslant -\frac{\gamma}{1-\gamma}\int \frac{\langle(\mathrm{Hess}\phi)^{-1}\nabla_z G\phi, \nabla_z G\phi\rangle}{\phi}\, d\mu + \frac{\gamma - s}{1-\gamma}\left(\int G\, d\mu\right)^2, \tag{3}$$

where $d\mu(z) = (\phi^{1/\gamma}(z)/\int \phi^{1/\gamma})\, dz$ and $G = V(\nabla_z\phi)/\phi$.

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We have reproved (for a smaller class of functions G) the result of Bobkov and Ledoux [6], who used the same idea, since inequality (3) is derived from the Borell–Brascamp–Lieb inequality (dimensional Prékopa inequality). Bobkov and Ledoux have already seen in [5] that one can deduce the classical Brascamp–Lieb inequality from the classical Prékopa inequality (corresponding to the log-concave case). This idea has been explored by Cordero-Erausquin and Klartag in [12], where they showed that the converse is true, i.e. one can derive the Prékopa inequality from the Brascamp–Lieb inequality. Thereafter, Nguyen [28] generalized the work by Cordero-Erausquin and Klartag to the case of s-concave measures (even for $s \geqslant 0$) and improved Bobkov–Ledoux's Brascamp–Lieb-Type inequality (inequality (3)). Recently, Kolesnikov and Milman [25] generalized the weighted Brascamp–Lieb-type inequalities obtained by Bobkov, Ledoux and Nguyen to the setting of Riemannian manifolds.

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