Macroeconomic Dynamics, **16**, 2012, 625–656. Printed in the United States of America.
doi:10.1017/S1365100510000775

FACTOR SUBSTITUTION AND ECONOMIC GROWTH: A UNIFIED APPROACH

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This paper provides a unified approach to characterizing the relation between factor substitution and economic growth in different one-sector growth models (namely, the Solow, Ramsey, and Diamond models). Our main finding is that if better factor substitution raises savings in the steady state, then a higher per capita income results. There are two channels by which factor substitution affects savings: the positive efficiency effect via income and the ambiguous distribution effect via factor income shares. If the efficiency effect dominates, then a higher elasticity of substitution leads to a higher level of per capita steady-state income. In transition, factor substitution affects the rate of convergence both directly and through the equilibrium profit share. The former arises from diminishing marginal productivity of capital whereas the latter reflects its relative scarcity. Depending on the interaction of these effects, the net outcomes are characterized.

Keywords: Elasticity of Substitution, Normalized CES Production Function, Rate of Convergence

1. INTRODUCTION

From a standard neoclassical aggregate production function where output Y_t is produced by capital K_t and labor L_t,

$$Y_t = A_t F(K_t, L_t),$$

we can conclude that there are three sources of economic growth:

(1) accumulation of factor inputs (K_t and L_t),
(2) total factor productivity (A_t),
(3) substitution between factor inputs (K_t and L_t).

We would like to thank an anonymous referee, Costas Azariadis, Kazuo Mino, Ted Palivos, Chris Papageorgiou, John Seater, Steve Turnovsky, Ping Wang, and participants from the 2010 Taipei International Conference on Growth, Trade and Dynamics (GTD2010), and the 10th SAET Conference on Current Trends in Economics in Singapore for comments and suggestions. Of course, errors remaining are our sole responsibility. Address correspondence to: Chong K. Yip, Department of Economics, The Chinese University of Hong Kong, Shatin, Hong Kong; e-mail: chongkeeyip@cuhk.edu.hk.

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https://doi.org/10.1017/S1365100510000775 Published online by Cambridge University Press

Although the existing growth literature focuses mainly on the study of the first two sources of growth, there has been rising interest on the third source recently. For instance, in one-sector convex growth models, Palivos and Karagiannis (2010) have shown that an elasticity of substitution that is asymptotically larger than unity can lead to perpetual growth. However, to study the relation between factor substitution and economic growth, we need a systematic comparison between technologies with different elasticities of substitution. To this end, there is a literature studying the normalized specification of the constant elasticity of substitution (CES) production function [see, for example, de La Grandville (1989), Klump and de La Grandville (2000), Miyagiwa and Papageorgiou (2003)]. Specifically, Klump and de La Grandville (2000) proposed two theorems on growth and factor substitution in the Solow (1956) growth model using the normalized CES production function. Other things equal, we have that

(1) an economy with a higher elasticity of substitution will have a higher level of per capita income (Theorem 1),
(2) an economy with a higher elasticity of substitution will have a higher capital intensity and a higher level of per capita income in the steady state (Theorem 2).

The first theorem is a property of the normalized CES production function that is always valid, whereas the second theorem is specific to the Solow growth model and may not be robust. For instance, Klump (2001) finds that when the initial steady-state level of per capita capital is below the normalized level, the second theorem above may not hold. Specifically, Klump and Saam (2008) find "counterintuitive results" that the steady-state level of capital declines with higher elasticity of substitution in this case. In an overlapping-generations setting of Diamond (1965), Miyagiwa and Papageorgiou (2003) show that, if the initial steady-state level of per capita capital is above the normalized level, then an economy with a higher elasticity of substitution can have a lower level of capital and output per worker in the steady state.[1]

In this paper, we provide a unified approach to characterizing the relation between factor substitution and economic growth both in the steady state and in transition. Our analysis focuses on second-order derivatives of the normalized CES function and applies the resulting properties to study the effects of the elasticity of substitution on the steady-state capital and the speed of convergence in different one-sector models of economic growth (namely, the Solow, Ramsey, and Diamond models). Our main finding is that, among these popular one-sector growth models, the effects of factor substitution on the steady-state levels of capital (hence output) depend on the amount of savings being channeled into investment in the steady state. If an improvement in factor substitution raises the amount of savings in the steady state, then it leads to a higher equilibrium level of per capita income. There are two channels by which the factor-substitution mechanism affects savings. We denote them as the efficiency and distribution effects of savings. The former captures the direct effect of factor substitution on savings via income, which

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is always positive, whereas the latter captures the effects on savings via factor income shares, which can go in either direction. It is shown that if the efficiency effect dominates, then a higher elasticity of substitution leads to a higher level of per capita steady-state income. In the Solow model, all incomes (both labor and capital incomes) are saved, so only the efficiency effect is at work at equilibrium. In the other growth models, only one type of income is saved, so that there is a distribution effect at work when factor substitution adjusts. Then the choice of the normalized level of per capita capital is no longer innocuous, because it determines the direction of the distribution effect. This then explains why we obtain contrary conclusions in the literature when different growth models are adopted in the analysis.

In transition, among all three models of economic growth, factor substitution affects the rate of convergence through the equilibrium capital income share. As identified by Klump and Preissler (2000) in the analysis of the Solow model, this profit-share channel of factor substitution reflects the relative scarcity of capital and depends on the relative magnitude between the equilibrium and normalized levels of capital. But there is an additional direct effect of factor substitution on the rate of convergence in the optimizing growth models. When the steady-state level of capital depends on its marginal product, the direct effect captures the positive impact of the elasticity of substitution on the diminishing marginal product of capital. The presence of the direct effect is due to the fact that only one type of factor income is saved in the optimizing growth models. Depending on the interaction of the direct effect and the relative scarcity effect, the net outcomes are characterized according to the choice of the normalized level of per capita capital.

The organization of the paper is as follows. In the next section, we describe the basic model and derive the properties of the normalized CES production function. Section 3 investigates the relation between the steady-state level of capital and the elasticity of substitution in different growth models. We then study the effects of factor substitution on the rates of convergence in Section 4. Section 5 summarizes our findings with a parameterized example, and there is a sixth concluding section.

2. THE BASICS

Before we consider the details of the growth models, we first highlight the common feature that lies across them—the production technology. As we focus on the relation between factor substitution and economic growth, the constant–elasticity of substitution (CES) production function is naturally adopted for the technology specification. However, it is well known that there are variants of the CES production functions used in the economic literature. As pointed out by Klump and Preissler (2000), different variants of the CES functions all share a common structure that can be characterized by the different distribution and efficiency parameters. The specification of these parameters depends on the elasticity of substitution, which relies on three arbitrarily given baseline values:

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(1) a given capital intensity or capital–labor ratio,
(2) a given level of per capita income, and
(3) a given marginal rate of substitution.

For a given set of these baseline values, different members of a family of CES production functions can then be identified by the different elasticity of substitution. This is the idea of normalizing the CES production function introduced by de La Grandville (1989). This procedure is crucial in studying the effects of factor substitution.

2.1. The Klump–de La Grandville Normalized Constant–Elasticity of Substitution Production Function

Consider the well-known CES production function proposed by Arrow et al. (1961),

$$y_t = A\left[\alpha k_t^{\psi} + (1-\alpha)\right]^{\frac{1}{\psi}}, \tag{1}$$

where y_t is per capita output and k_t is per capita capital; ψ, A, and α are the substitution, the efficiency, and the distribution "parameter," respectively. The elasticity of substitution is given by $\sigma = (1-\psi)^{-1}$. To study the effects of a change in σ, we follow the normalization procedure proposed by Klump and de La Grandville (2000) and choose particular baseline values for the following three variables: the capital–labor ratio (or capital intensity) $\overline{k}$, the per capita income $\overline{y} = f(\overline{k})$, and the marginal rate of substitution $\overline{m} = [f(\overline{k}) - \overline{k} f_1(\overline{k})]/f_1(\overline{k})$. We then have[2]

$$y_t = A(\sigma)\left\{\alpha(\sigma) k_t^{\psi} + [1-\alpha(\sigma)]\right\}^{\frac{1}{\psi}}, \tag{2}$$

$$A(\sigma) = \overline{y}[\overline{\pi}\overline{k}^{-\psi} + (1-\overline{\pi})]^{\frac{1}{\psi}}, \tag{3}$$

$$\alpha(\sigma) = \frac{\overline{\pi}\overline{k}^{-\psi}}{\overline{\pi}\overline{k}^{-\psi} + (1-\overline{\pi})}, \tag{4}$$

where $\overline{\pi}$ is the baseline capital income share at $\overline{k}$, which equals

$$\overline{\pi} = \frac{\overline{k} f_1(\overline{k})}{f(\overline{k})} = \frac{\overline{k}}{\overline{m} + \overline{k}}. \tag{5}$$

After substituting A and α into (1), we obtain the normalized CES function

$$f(k_t, \sigma) = \overline{y}\left[\overline{\pi}\left(\frac{k_t}{\overline{k}}\right)^{\psi} + (1-\overline{\pi})\right]^{\frac{1}{\psi}}. \tag{6}$$

The marginal product of capital is positive because

$$f_1(k_t, \sigma) = \frac{\partial f(k_t, \sigma)}{\partial k_t} = \pi(k_t, \sigma)\frac{f(k_t, \sigma)}{k_t} > 0. \tag{7}$$

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In addition, according to Klump and de La Grandville (2000), the normalized CES production function (6) is an increasing function of σ:

$$\begin{aligned} f_2(k_t, \sigma) &= \frac{\partial f(k_t, \sigma)}{\partial \sigma} \\ &= -\frac{1}{\sigma^2}\frac{1}{\psi^2} y_t \left[\pi_t \ln\left(\frac{\overline{\pi}}{\pi_t}\right) + (1-\pi_t)\ln\left(\frac{1-\overline{\pi}}{1-\pi_t}\right)\right] \\ &= \frac{1}{\sigma^2}\frac{1}{\psi^2} y_t \Phi_\sigma > 0, \end{aligned} \tag{8}$$

where

$$\Phi_\sigma = -\left[\pi_t \ln\left(\frac{\overline{\pi}}{\pi_t}\right) + (1-\pi_t)\ln\left(\frac{1-\overline{\pi}}{1-\pi_t}\right)\right] > 0.$$

Following Irmen and Klump (2009), we denote the term Φ_σ that captures the direct effect of σ on per capita output as the efficiency effect, which is always positive.

Next, we define the capital income share π as

$$\pi(k_t, \sigma) = \frac{k_t f_1(k_t, \sigma)}{f(k_t, \sigma)} = \frac{\alpha k_t^\psi}{\alpha k_t^\psi + (1-\alpha)} = \frac{\overline{k}^{1-\psi} k_t^\psi}{\overline{k}^{1-\psi} k_t^\psi + \overline{m}}. \tag{9}$$

Then Klump and de La Grandville (2000) show that π_t has the following properties:

$$\pi_1(k_t, \sigma) = \frac{\partial \pi_t}{\partial k_t} = \frac{\psi}{k_t}\pi_t(1-\pi_t), \tag{10}$$

$$\pi_2(k_t, \sigma) = \frac{\partial \pi_t}{\partial \sigma} = \frac{1}{\sigma^2}\pi_t(1-\pi_t)\ln\frac{k_t}{\overline{k}} = \frac{\pi_t(1-\pi_t)}{(\sigma-1)^2}\Phi_k, \tag{11}$$

where $\Phi_k = \psi^2 \ln(k_t/\overline{k}) \gtrless 0$ if and only if $k_t \gtrless \overline{k}$. Similarly to the treatment of Φ_σ, we denote the term Φ_k that captures the direct effect of σ on π as the distribution effect, which can be of either sign.

Letting w_t and r_t denote the wage rate and the rental rate for capital, a competitive factor market yields

$$w_t = f(k_t, \sigma) - k_t f_1(k_t, \sigma) \text{ and } r_t = f_1(k_t, \sigma). \tag{12}$$

We are ready to describe the structure of different models of economic growth.

2.2. Growth Models with Constant Saving Rates

We first study the class of growth models where per capita savings are a fixed proportion of per capita incomes in the steady state.[3] This class of growth models includes the Solow model, the Ramsey–Cass–Koopman model, and the Diamond overlapping-generations model with logarithmic preferences.[4] It is shown that as long as the efficiency effect dominates, a higher elasticity of substitution leads to a higher level of per capita income in the steady state.

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The Solow Model. We consider the discrete variant of the one-sector model à la Solow (1956), in which a homogeneous good, Y_t, is produced using capital, K_t, and labor, L_t.[5] The homogeneous good can be either consumed or saved with a fixed fraction denoted by $s \in (0, 1)$, and therefore the aggregate saving at time t is sY_t. The gross investment at t is the sum of new and replacement investment; that is, $I_t = K_{t+1} - K_t + \delta K_t$, where δ is the depreciation rate. The equality of investment and saving yields the fundamental accumulation equation for the motion of per capita capital,

$$k_{t+1} = sy_t + (1 - \delta)\, k_t,$$

where $y_t = f(k_t, \sigma)$ is the technology described by a normalized CES production function. In the steady state, where $k_{t+1} = k_t = k^*$, we have

$$\frac{sf(k^*, \sigma)}{k^*} = \delta. \tag{13}$$

Direct differentiation of (13) then yields

$$\frac{dk^*}{d\sigma} = \frac{k^* f_2(k^*, \sigma)}{w(k^*, \sigma)} = \frac{k^* \Phi_\sigma}{\sigma^2 \psi^2 [1 - \pi(k^*, \sigma)]} > 0.$$

This is the finding of Klump and de la Grandville (2000).

The Ramsey–Cass–Koopman Model. Consider the Ramsey–Cass–Koopman model where the representative agent maximizes a lifetime utility function of the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where the utility function takes the constant–intertemporal elasticity of substitution form so that $u(c) = (c^{1-\theta} - 1)/(1 - \theta)$ with $\theta > 0$ and the discount factor $\beta \in (0, 1)$. The agent faces the budget constraint

$$k_{t+1} = r_t k_t + (1 - \delta)\, k_t + w_t - c_t$$

and an initial condition for capital k_0 and the usual no–Ponzi scheme constraint. From the first-order conditions of the utility maximization problem and equation (12), we get the two-equation system

$$\begin{aligned} k_{t+1} &= f(k_t, \sigma) + (1 - \delta)\, k_t - c_t, \\ \frac{c_t^{-\theta}}{\beta c_{t+1}^{-\theta}} &= 1 - \delta + f_1(k_{t+1}, \sigma), \end{aligned}$$

which characterizes the equilibrium situation. If a steady state exists, we can solve out (k^*, c^*) from the above equation system,

$$f_1(k^*, \sigma) = \rho + \delta, \tag{14}$$

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$$f(k^*, \sigma) = \delta k^* + c^*, \tag{15}$$

where $\rho = \beta^{-1} - 1$. By substituting the expression of (7) into (14), we have

$$\frac{f(k^*, \sigma)}{k^*}\pi(k^*, \sigma) = \rho + \delta. \tag{16}$$

Direct differentiation of (16) then yields

$$\frac{dk^*}{d\sigma} = \frac{k^*}{\sigma\psi^2}\left[\frac{\Phi_\sigma}{1 - \pi\ (k^*, \sigma)} + \Phi_k\right] > 0 \quad \text{if } k^* > \bar{k}.$$

This is the finding of Klump (2001).

The Diamond Model with Logarithmic Preferences. The third constant–saving rate growth model that we consider here is the Diamond (1965) overlapping-generations model of logarithmic preferences. Suppose that L_t individuals are born in period t and $L_t = L_{t-1}$, as population growth rate is zero. Individuals are identical and live for just two periods. In the first period, each individual supplies one unit of labor and divides the wage income into consumption in the first period and saving; in the second period, the individual retires and consumes his/her saving in the first period plus the interest earned. Let C_{1t} and C_{2t+1} denote the consumption in the first and second period, respectively. The budget constraint for the individual is given by

$$C_{1t} + \frac{1}{1 + r_{t+1}}C_{2t+1} = w_t.$$

The lifetime utility function for the individual is a CRRA utility function

$$U_t = \ln C_{1t} + \beta \ln C_{2t+1},$$

where $\beta > 0$. Solving the optimization problem of the individual yields the well-known Euler equation. Together with the budget constraint, we can express C_{1t} in terms of the labor income and the real interest rate as

$$C_{1t} = \frac{1}{1 + \beta}w_t,$$

so that the constant saving rate is given by

$$s = \frac{\beta}{1 + \beta}. \tag{17}$$

Because the per capita capital stock in period $t + 1$, k_{t+1}, is the amount saved by young individuals in period t, we obtain

$$k_{t+1} = s w_t,$$

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where w_t is determined by equations (12), and the saving rate is given by (17). The law of motion equation of k_t becomes

$$k_{t+1} = \frac{\beta}{1+\beta}(1-\pi_t)\, f(k_t, \sigma).$$

The steady-state k^* satisfies

$$\frac{f(k^*, \sigma)}{k^*}[1 - \pi(k^*, \sigma)] = \frac{1+\beta}{\beta}. \tag{18}$$

We note that the steady state is locally stable if it satisfies

$$\frac{\beta}{1+\beta} w_1(k^*, \sigma) < 1, \tag{19}$$

which can be rewritten as

$$\frac{\pi(k^*, \sigma)}{\sigma} < 1. \tag{20}$$

Finally, assuming a locally stable steady state exists, direct differentiation of (18) yields

$$\frac{dk^*}{d\sigma} = \frac{k^*}{\left[1 - \dfrac{\pi(k^*, \sigma)}{\sigma}\right](\sigma - 1)^2}[\Phi_\sigma - \pi(k^*, \sigma)\Phi_k] > 0 \text{ if } k^* < \bar{k}.$$

This is the finding of Miyagiwa and Papageorgiou (2003).

3. STEADY-STATE ANALYSIS: A UNIFIED APPROACH

In this section, we provide a unified framework to study the influence of the elasticity of substitution on the steady-state value k^*. For the neoclassical one-sector growth models with constant savings rates that we examined in the last section, the steady-state per capita capital k^* can be obtained by solving the following equation:

$$\frac{f(k^*, \sigma)}{k^*}\{s^w[1 - \pi(k^*, \sigma)] + s^r \pi(k^*, \sigma)\} = g, \tag{21}$$

where s^w, s^r, and g are nonnegative constants. This is the unified framework that we will use for the steady-state analysis. An interpretation of the left-hand side (LHS) of (21) is that it is the savings per capita. Specifically, the two terms inside the bracket {.} represent the factor shares of savings, with the former from wage income and the latter from capital income. On the other hand, the right-hand side (RHS) is just the constant replacement investment per unit of capital. Before proceeding, we note that the LHS can be written as a function of k and σ. It consists of two parts, the average product of capital f/k and the weighted factor share $[s^w(1-\pi) + s^r\pi]$. Following the interpretation of Irmen and Klump (2009),

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the former denotes an efficiency effect whereas the latter represents a distribution effect.

First, we define

$$h(k^*, \sigma) \equiv \frac{f(k^*, \sigma)}{k^*}\{s^w[1 - \pi(k^*, \sigma)] + s^r\pi(k^*, \sigma)\}, \tag{22}$$

where

$$h_1 = \frac{\partial h}{\partial k^*} = -\frac{(1 - \pi^*)\, f\,(k^*, \sigma)}{(k^*)^2}\left[s^w + \frac{\pi^*}{\sigma}(s^r - s^w)\right], \tag{23}$$

$$h_2 = \frac{\partial h}{\partial \sigma} = [s^w(1 - \pi^*) + s^r\pi^*]\frac{f_2\,(k^*, \sigma)}{k^*} + (s^r - s^w)\frac{f\,(k^*, \sigma)}{k^*}\frac{(1 - \pi^*)\,\pi^*}{\sigma^2}\ln\frac{k^*}{\bar{k}}, \tag{24}$$

and $\pi^* = \pi(k^*, \sigma)$. In what follows, we apply the steady-state equation (21) to each of the growth models with constant savings rate. We adopt a graphical approach by plotting $h(k^*, \sigma)$ as a function of k so that h_1 provides us with information on its slope and h_2 tells us the direction of its shifts when σ changes.

3.1. The Solow Model

We recall the steady-state condition (13) and get

$$h(k^*, \sigma) = \frac{sf\,(k^*, \sigma)}{k^*}. \tag{25}$$

This is a special case of the general condition (21) with the following parameterization:

(1) $s^w = s^r = s > 0$,
(2) $g\,(k^*, \sigma) = \delta$.

The first parameterization highlights the fact that both capital and labor incomes are saved and channeled to investment in the steady state. Further, we know that

$$h_1 = -\frac{s\,(1 - \pi^*)\, f\,(k^*, \sigma)}{(k^*)^2} < 0.$$

The LHS of (13) is downward sloping in k^*. In addition, we have

$$h_2 = \frac{s^*}{k^*}f_2(k^*, \sigma) = \frac{1}{\sigma^2}\frac{1}{\psi^2}\frac{s^*y^*}{k^*}\Phi_\sigma > 0,$$

so that it shifts right as σ increases. Applying the implicit function theorem to (25) then yields

$$\frac{dk^*}{d\sigma} = -\frac{h_2\,(k^*, \sigma)}{h_1\,(k^*, \sigma)} = \frac{k^*\Phi_\sigma}{\sigma^2\psi^2\,(1 - \pi^*)} > 0.$$

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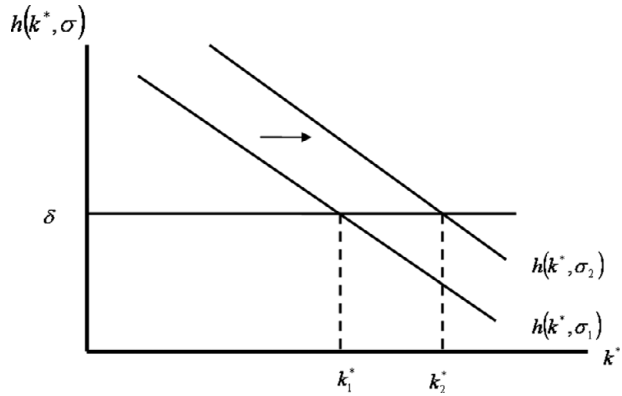


FIGURE 1. The influence of σ on k^* in the Solow model ($\sigma_1 > \sigma_2$).

The graphical representation is given in Figure 1. When σ increases from σ_1 to σ_2, $h_2 > 0$ implies that $h(k^*, \sigma)$ shifts to the right. The new steady-state k_2^* associated with a larger σ will be on the RHS of k_1^*. This is the finding of Klump and de La Grandville (2000). Because all incomes are saved in the Solow model, there is only the efficiency effect of factor substitution at work, so that $dk^*/d\sigma > 0$.

3.2. The Ramsey–Cass–Koopman Model

We recall the steady-state condition (16) and get

$$h(k^*, \sigma) = \frac{f(k^*, \sigma)}{k^*}\pi(k^*, \sigma). \tag{26}$$

It is a special case of the general condition (21) with the following parameterization:

(1) $s^w = 0$ and $s^r = 1$,
(2) $g(k^*, \sigma) = \rho + \delta$.

The first parameterization highlights the fact that only capital income is saved and channeled to investment in the steady state. Further, we know that

$$h_1 = -\frac{\pi^*(1-\pi^*) f(k^*, \sigma)}{\sigma (k^*)^2} < 0.$$

Again, the LHS of (16) is downward sloping in k^*. In addition, we have

$$h_2 = \frac{\pi^* f(k^*, \sigma)}{k^* (\sigma - 1)^2}[\Phi_\sigma + (1-\pi^*)\Phi_k], \tag{27}$$

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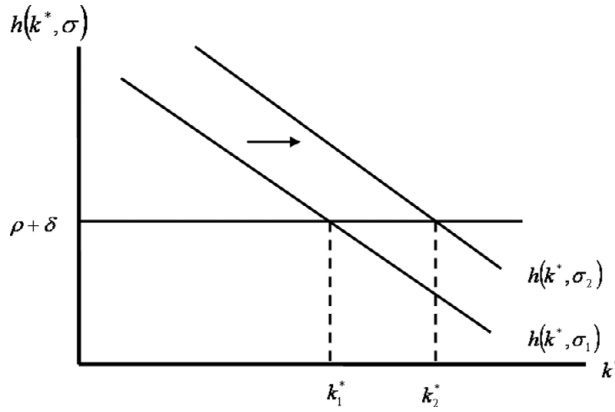


FIGURE 2. The influence of σ on k^* in the Ramsey model when $k^* > \bar{k}$ (where $\sigma_1 > \sigma_2$).

so that the LHS of (16) can shift in either direction as σ increases. Applying the implicit function theorem to (26) then yields

$$\frac{dk^*}{d\sigma} = -\frac{h_2(k^*,\sigma)}{h_1(k^*,\sigma)} = \frac{k^*}{\sigma\psi^2(1-\pi^*)}\{\Phi_\sigma + [1-\pi(k^*,\sigma)]\Phi_k\} > 0 \text{ if } k^* > \bar{k}.$$

The graphical representation is given in Figure 2. When σ increases from σ_1 to σ_2 and for $k^* > \bar{k}$, the $h(k^*,\sigma)$ locus shifts to the right. The new steady-state k_2^* associated with a larger σ will be on RHS of k_1^*. This is the finding of Klump (2001).

To shed further light on the comparative statics results, we take a closer examination of h_2. From (27), we note that the first term on the RHS represents the effect of σ on output (f) or the average product (f/k), whereas the second term is the effect on π. As we mention earlier, the former effect is the efficiency effect and the latter is the distribution effect. Thus, the comparative statics results on the influence of σ on k^*, i.e., $\text{sgn}(dk^*/d\sigma)$, depend on the working of these effects. The intuition can be explained as follows. When $k^* > \bar{k}$, capital is more abundant than labor in the economy. When σ increases so that there is an improvement in the factor substitution mechanism, more capital will be put into use. This in turn raises the capital share in production and hence its income share π. As savings come solely from capital income in the Ramsey model, an increase in factor substitution then also raises savings. As a result, the two effects on savings work in the same direction; thus we have $dk^*/d\sigma > 0$ and a higher elasticity of substitution leads to a higher level of per capita income in the steady state. If $k^* < \bar{k}$, then an improvement in factor substitution raises labor income share and lowers savings. Thus the two effects on savings are working in opposite directions, and we obtain the conclusion that $dk^*/d\sigma > 0$ when the efficiency effect dominates the distribution effect, i.e., $\Phi_\sigma + (1-\pi^*)\Phi_k > 0$.

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3.3. The Diamond Model with Logarithmic Utility Function

We recall the steady-state condition (18) and get

$$h(k^*, \sigma) = \frac{f(k^*, \sigma)}{k^*}[1 - \pi(k^*, \sigma)]. \tag{28}$$

It is a special case of the general condition (21) with the following parameterization:

(1) $s^w = 1$ and $s^r = 0$,
(2) $g = (1 + \beta)/\beta$.

The first parameterization highlights the fact that only labor income is saved and channeled to investment in the steady state. Further, we know that

$$h_1 = -\frac{(1 - \pi^*) f(k^*, \sigma)}{(k^*)^2}\left(1 - \frac{\pi^*}{\sigma}\right) < 0$$

under the local stability condition that $\sigma > \pi^*$. Again, the LHS of (18) is downward-sloping in k^*. In addition, we have

$$h_2 = \frac{1 + \beta}{\beta(\sigma - 1)^2}(\Phi_\sigma - \pi^* \Phi_k). \tag{29}$$

so that it can shift in either direction as σ increases. Applying the implicit function theorem to (28) then yields

$$\frac{dk^*}{d\sigma} = -\frac{h_2(k^*, \sigma)}{h_1(k^*, \sigma)}$$
$$= \frac{k^*}{\left[1 - \frac{\pi(k^*, \sigma)}{\sigma}\right](\sigma - 1)^2}[\Phi_\sigma - \pi(k^*, \sigma)\Phi_k] > 0 \text{ if } k^* < \bar{k}.$$

The graphical representation is given in Figure 3. When σ increases from σ_1 to σ_2 and for $k^* < \bar{k}$, the $h(k^*, \sigma)$ locus shifts to the right. The new steady-state k_2^* associated with a larger σ will be on the LHS of k_1^*. This is the finding of Miyagiwa and Papageorgiou (2003).

The intuition can be explained as follows. If $k^* < \bar{k}$, we have labor being relatively abundant than capital in the economy. When σ increases so that there is an improvement in the factor substitution mechanism, more labor will be put into use. This in turn raises the labor share in production and hence its income share $1 - \pi$. As savings come solely from labor income in the Diamond model, an increase in factor substitution then also raises savings. As a result, both the efficiency and distribution effects on savings work in the same direction, then we have the $h(k^*, \sigma)$ locus shifts right so that $dk^*/d\sigma > 0$; i.e., a higher elasticity of substitution leads to a higher level of per capita income in the steady state. On the other hand, if $k^* > \bar{k}$ then the two effects are working in opposite directions. We

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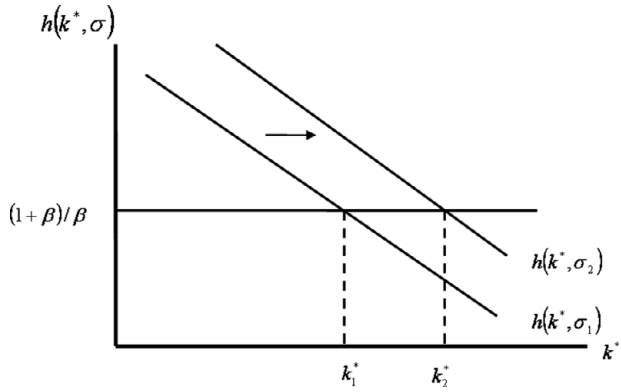


FIGURE 3. The influence of σ on k^* in the Diamond model with logarithmic utility when $k^* < \bar{k}$ and $\sigma > \pi^*$ (where $\sigma_1 > \sigma_2$).

obtain the conclusion that k^* increases as σ rises only when the efficiency effect dominates the distribution effect, i.e., $\Phi_\sigma - \pi^* \Phi_k > 0$.

3.4. Discussion

In summary, among the popular growth models, we have found that the effects of factor substitution on the steady-state levels of capital (hence output) depend on the amount of savings being channeled into replacement investment in the steady state. If an improvement in factor substitution raises the amount of savings in the steady state, then it leads to a higher equilibrium level of per capita income. We have shown that the factor-substitution mechanism works through two channels in affecting savings. We denote them as the efficiency and distribution effects of savings. The former captures the direct effects of factor substitution on savings via income which is always positive, whereas the latter captures the effects on savings via factor income shares which can go in either direction. It is shown that *as long as the efficiency effect dominates, a higher elasticity of substitution leads to a higher level of per capita income in the steady state*. In the Solow model, all incomes (both labor and capital incomes) are saved so that only the efficiency effect is at work in equilibrium. In the other growth models, only one type of incomes is saved so that there is this extra distribution effect at work. In the Ramsey model where only capital income is saved, the overall effect can be ambiguous. This is because if a higher elasticity of substitution reduces the capital income share, the distribution effect on savings becomes negative and works against the efficiency effect. Likewise, in the Diamond model, only wage income is saved. Thus if a higher elasticity of substitution reduces the wage income share, the distribution effect on savings again becomes negative and offsets the efficiency effect. Because The distribution effect depends on the relative magnitude between the steady-state and normalized levels of per capita capital, which explains why

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TABLE 1. Summary of the steady-state analysis

Model	$k^* < \bar{k}$	$k^* > \bar{k}$
Solow	+	+
Ramsey	?	+
Diamond (log U)	+	?

we obtain contrary conclusions in the literature when different growth models are adopted in the analysis.

To conclude the section, we summarize our findings on the effects of factor substitution on the steady-state per capita income for growth models with constant savings rates in the following proposition:

PROPOSITION 1. *The steady state of the growth models (Solow, Ramsey, and Diamond) with constant savings rates can be characterized by the equation*

$$h(k^*, \sigma) \equiv \frac{f(k^*, \sigma)}{k^*}\{s^w[1 - \pi(k^*, \sigma)] + s^r\pi(k^*, \sigma)\} = g.$$

The parameterization of the growth models is given by

(1) *The Solow model:* $s^w = s^r = s$, *and* $g = \delta$,
(2) *The Ramsey–Cass–Koopman model:* $s^w = 0$, $s^r = 1$, *and* $g = \rho + \delta$,
(3) *The Diamond model with logarithmic preferences:* $s^w = 1$, $s^r = 0$, *and* $g = \frac{1+\beta}{\beta}$.

There are two effects at work in determining $dk^*/d\sigma$*:*

(1) *The direct efficiency effect:* $\partial f(k^*, \sigma)/\partial\sigma$,
(2) *The indirect distribution effect:* $\partial\pi(k^*, \sigma)/\partial\sigma$.

As long as the efficiency effect dominates, then a higher elasticity of substitution leads to a higher level of per capita income in the steady state, i.e., $dk^*/d\sigma > 0$.

To close the section, we tabulate the effects of σ on the steady-state level of per capita capital k^* in different one-sector growth models for easy reference: see Table 1.

4. TRANSITIONAL DYNAMICS AND THE RATE OF CONVERGENCE

After examining the effects of factor substitution on the steady-state per capita variables, we move to looking at its transitional impact. Specifically, we study the effects of factor substitution on the speed of convergence, as it determines the relevance of the steady state relative to the transition path. In the Solow model with a Cobb–Douglas production technology, Sato (1963) found that the speed of convergence is very low, taking between 50 and 190 years to have a 90% completion of adjustment toward the long-run equilibrium. Ramanathan (1973, 1975) investigated the role of the elasticity of factor substitution in the determination of

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the speed of convergence in both the one- and two-sector Solow–Swan models. Recently, the empirical growth literature [e.g., Barro and Sala-i-Martin (1992), Sala-i-Martin (1994)] has established the benchmark rate of convergence, that is, the rate at which the gap between the current and long-run equilibrium values of a variable to be eliminated is about 2–3%. However, the standard one-sector Ramsey model predicts that the convergence rate is approximately 7%, which is much higher than the empirical benchmark.

Attempts have been made to systematically examine the rate of convergence in growth models. Turnovsky (2002) examined the effects of different elasticities of substitution on the rate of convergence in a standard one-sector Ramsey model of economic growth. He found that the rate of convergence is highly sensitive to both the intertemporal elasticity of substitution in consumption and the elasticity of factor substitution in production. In this section, we focus on the effect of factor substitution on the rates of convergence of the three different one-sector growth models with constant saving rates studied above, namely, the Solow model, the Ramsey model, and the Diamond model with logarithmic utility.

Our objective is to compare the influence of the elasticity of substitution on the asymptotic speed of convergence in the neoclassical one-sector growth models studied in the previous section. To compute the asymptotic speed of convergence, we first linearize the dynamical systems around the steady state,

$$\widehat{x}_{t+1} = J\widehat{x}_t,$$

where $\widehat{x}_t$ is the vector of the linearized state and control variables, i.e., $\widehat{x}_t = x_t - x^*$, and J is the Jacobian matrix. One remark here is that for the transitional dynamics we can normalize the CES production function around the steady state, i.e., $\overline{k} = k^*$. As a result, all the steady-state values as capital income share π^* and per capita capital k^* (and per capita consumption c^* in Ramsey model) will remain constant when the elasticity of substitution σ changes. In this note, we consider cases of the normalization both at the steady state and away from the steady state.

We then calculate the eigenvalues associated with the Jacobian matrix J and denote the largest among those within the unit circle by λ. Because k_t is the only predetermined variable in all the one-sector growth models discussed in the paper, λ is unique for the stable steady state. The asymptotic speed of convergence, denoted by γ, can be expressed as

$$\gamma = \left| \frac{\widehat{k}_{t+1} - \widehat{k}_t}{\widehat{k}_t} \right| = 1 - \lambda.$$

We can calculate the effect of the elasticity of substitution σ on the asymptotic speed of convergence γ as

$$\frac{d\gamma}{d\sigma} = -\frac{d\lambda}{d\sigma}.$$

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4.1. The Solow Model

In the Solow model, the fundamental accumulation equation for the motion of per capita capital is

$$k_{t+1} = sy_t + (1-\delta)\,k_t,$$

where $y_t = f\,(k_t, \sigma)$ is the technology describing by a normalized CES production function.

Linearizing the fundamental accumulation equation for the motion of per capita capital around the steady-state k^*, we have

$$\widehat{k}_{t+1} = [sf_1(k^*, \sigma) + 1 - \delta]\widehat{k}_t,$$

where $\widehat{k}_t = k_t - k^*$ and $f_1(k^*, \sigma) = (y^*/k^*)\pi^* = (\delta/s)\pi^*$. At the steady state, we have

$$\left(\frac{\overline{\pi}}{\pi^*}\right)^{\frac{\sigma}{\sigma-1}} = \frac{\delta}{s}\frac{\overline{k}}{\overline{y}}$$

and

$$\frac{d\pi^*}{d\sigma} = \frac{1}{\sigma^2}\pi^* \ln\left(\frac{\overline{y}/\overline{k}}{y^*/k^*}\right) \gtrless 0 \text{ as } k^* \gtrless \overline{k}.$$

Define the speed of convergence γ_S as

$$\gamma_S = \left|\frac{\widehat{k}_{t+1}}{\widehat{k}_t} - 1\right| = \delta - sf_1(k^*, \sigma) = \delta[1 - \pi(k^*, \sigma)] > 0 \tag{30}$$

and

$$\begin{aligned}\frac{d\gamma_S}{d\sigma} &= -s\frac{d}{d\sigma}[f_1(k^*, \sigma)] \\ &= -\delta\frac{d\pi^*}{d\sigma} \lessgtr 0 \text{ as } k^* \gtrless \overline{k}.\end{aligned} \tag{31}$$

Then, we conclude that a higher elasticity of substitution will decrease (increase) the speed of convergence if and only if the steady-state per capita capital is higher (lower) than the baseline level. If $\overline{k} = k^*$, i.e., we normalize the CES production function at the steady state, then $d\gamma_S/d\sigma = 0$ and σ does not affect the speed of convergence.[6]

4.2. The Ramsey–Cass–Koopman Model

The dynamical system of the Ramsey–Cass–Koopman model consists of the following difference equations:

$$k_{t+1} = f\,(k_t) + (1-\delta)\,k_t - c_t,$$

$$\frac{c_t^{-\theta}}{\beta c_{t+1}^{-\theta}} = 1 - \delta + f_1\,(k_{t+1}).$$

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At the steady state, we can solve out

$$\pi^* = \left\{\left[\frac{1}{\beta} - (1-\delta)\right]\frac{\overline{k}}{\overline{y}}\right\}^{1-\sigma}\overline{\pi}^{\sigma}$$

and hence

$$\frac{d\pi^*}{d\sigma} = \frac{1}{\sigma}\pi^* \ln\left(\frac{\overline{y}/\overline{k}}{y^*/k^*}\right) \gtrless 0 \text{ as } k^* \gtrless \overline{k}.$$

We linearize the system around the steady-state (k^*, c^*) and have the following Jacobian matrix J:

$$J = \begin{pmatrix} \frac{1}{\beta} & -1 \\ c^* f_{11}/\theta & 1 - \beta c^* f_{11}/\theta \end{pmatrix}.$$

The trace (T_R) and determinant (D_R) of the Jacobian matrix J are

$$T_R = \frac{1}{\beta} + 1 - \frac{\beta c^* f_{11}^*}{\theta} > 2, \tag{32}$$

$$D_R = \frac{1}{\beta} > 1.$$

The characteristic equation is given by

$$p(\lambda) = \lambda^2 - T_R\lambda + D_R,$$

where the stable root is

$$\lambda_1 = \frac{T_R}{2} - \sqrt{\frac{T_R^2}{4} - \frac{1}{\beta}}. \tag{33}$$

Because $T_R^2 - \frac{4}{\beta} > 0$ and $T_R > \sqrt{T_R^2 - \frac{4}{\beta}}$, we get

$$\frac{d\lambda_1}{d\sigma} = -\frac{1}{2}\left(\frac{T_R}{\sqrt{T_R^2 - \frac{4}{\beta}}} - 1\right)\frac{dT_R}{d\sigma} \equiv -\eta_0\frac{dT_R}{d\sigma}.$$

From the definition of the speed of convergence, we have

$$\gamma_R = \left|\frac{\widehat{k}_{t+1}}{\widehat{k}_t} - 1\right| = 1 - \lambda_1 > 0, \tag{34}$$

so that

$$\frac{d\gamma_R}{d\sigma} = -\frac{d\lambda_1}{d\sigma} = -\eta_1\left(\eta_2 + \eta_3\frac{d\pi^*}{d\sigma}\right), \tag{35}$$

where $\eta_1 = \eta_0[\beta f_1(k^*)/\sigma\theta] > 0$, $\eta_2 = (1-\pi^*)c^*/\sigma k^* > 0$ and $\eta_3 = f_1(k^*)/\pi^{*2} - \delta > 0$.

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In summary, if the initial level of per capita capital is at least as large as the normalized level ($k^* \geq \overline{k}$), the economy with the higher elasticity of substitution will have a lower speed of convergence because $d\pi^*/d\sigma \geq 0$; i.e., $d\gamma_R/d\sigma < 0$.

4.3. The Diamond Model

Suppose that the utility function is logarithmic so that we have a constant saving rate as $s(r_{t+1}) = \beta/(1+\beta)$. It corresponds to the model discussed by Miyagiwa and Papageorgiou (2003), where the law of motion equation of k_t is given by

$$k_{t+1} = \frac{\beta}{1+\beta} w_t = \frac{\beta}{1+\beta} (1-\pi_t)\, f\,(k_t, \sigma)\,. \tag{36}$$

At the steady state, we have

$$\left(\frac{\overline{\pi}}{\pi^*}\right)^{\frac{\sigma}{\sigma-1}} (1-\pi^*) = \frac{1+\beta}{\beta}\frac{\overline{k}}{\overline{y}}$$

and hence

$$\frac{d\pi^*}{d\sigma} = \frac{1-\pi^*}{\sigma-\pi^*}\frac{\pi^*}{\sigma} \ln \frac{\overline{y}/\overline{k}}{y^*/k^*} \gtrless 0 \text{ as } k^* \gtrless \overline{k} \text{ and } \sigma > \pi^*.$$

By linearization around the steady state, we have

$$\widehat{k}_{t+1} = -\frac{\beta}{1+\beta} k^* f_{11}(k^*) \widehat{k}_t,$$

where the stability of the system requires that

$$\left| -\frac{\beta}{1+\beta} k^* f_{11}(k^*) \right| = -\frac{\beta}{1+\beta} k^* f_{11}(k^*) < 1.$$

Define the speed of convergence (γ_D) in the Diamond model as

$$\gamma_D = \left| \frac{\widehat{k}_{t+1}}{\widehat{k}_t} - 1 \right| = 1 + \frac{\beta}{1+\beta} k^* f_{11}(k^*) > 0 \text{ as } k^* \text{ is stable,} \tag{37}$$

and the effect of σ on it is

$$\frac{d\gamma_D}{d\sigma} = -\frac{1}{\sigma}\left(\frac{d\pi^*}{d\sigma} - \frac{\pi^*}{\sigma}\right) = \frac{1}{\sigma}\left(1 - \frac{1-\pi^*}{\sigma-\pi^*} \ln \frac{\overline{y}/\overline{k}}{y^*/k^*}\right)\frac{\pi^*}{\sigma}. \tag{38}$$

If the initial level of per capita capital is not larger than the normalized level ($k^* \leq \overline{k}$), then the economy with the higher elasticity of substitution will have a higher speed of convergence because $d\pi^*/d\sigma \leq 0$, i.e., $d\gamma_D/d\sigma > 0$.

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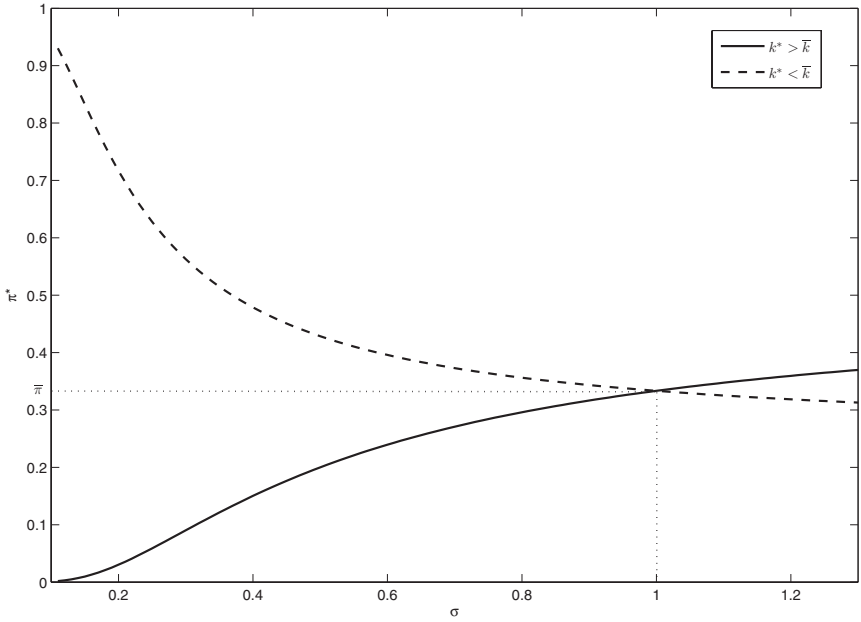


FIGURE 4. The relation between steady-state capital income share π^* and elasticity of substitution σ.

4.4. A Comparison

We first consider the effect of factor substitution on the rate of convergence in the Solow model. According to (30), we have

$$\gamma_S = \delta[1 - \pi(k^*, \sigma)], \tag{39}$$

so that the effects of σ on γ_S mainly work through the steady-state capital income share. Figure 4 provides a graphical representation of the relation between the elasticity of factor substitution and the capital income share. Depending on the relative magnitude between the steady-state per capita capital (k^*) and its normalized level ($\overline{k}$), the effects of factor substitution on the capital income share are in opposite directions. Specifically, the capital income share rises (falls) with the elasticity of factor substitution when $k^* > (<)\overline{k}$. Then (39) implies that the rate of convergence falls (rises) with the elasticity of factor substitution when $k^* > (<)\overline{k}$. When $k^* = \overline{k}$, we have $\pi^* = \overline{\pi}$ so that the capital income share and hence the rate of convergence remain unchanged as σ varies. To understand the intuition, we need to identify the baseline or normalized capital intensity as the benchmark of relative capital scarcity in the steady state. When $k^* < \overline{k}$ so that capital is relatively scarce in the steady state, an improvement in factor substitution allows further substitution of labor for capital to take place. As a result, the labor share rises (or equivalently π falls) so that the rate of convergence increases as

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given by (39). This corroborates the finding of Klump and Preissler (2000) in their continuous-time version of the Solow model: "a higher elasticity of substitution can only increase the speed of adjustment if the capital stock in the steady state is relatively more scarce than the input of labour, with both factor inputs being compared to their baseline values" (p. 50).

In the optimizing models of Ramsey and Diamond, we note that the determination of the steady-state level of per capita capital involves its marginal product.[7] As discussed in the previous section on steady-state analysis, this is due to the fact that savings in these optimizing models consist of only one type of factor incomes. This in turn implies that the effects of factor substitution on the rates of convergence depend on how it affects diminishing returns (f_{11}). It is straightforward to show that

$$f_{11} = -\frac{f_1}{\sigma k}(1-\pi),$$

so that there exists a direct effect of σ (i.e., $\partial f_{11}/\partial\sigma$) on the rates of convergence which is absent in the Solow model. Let us examine the effects of factor substitution on the rate of convergence in these models in order. In the Ramsey model, we recall (35) that the effects of σ on the rate of convergence depend on its effects on the trace of the Jacobian matrix. We rewrite (32) as

$$T_R = \frac{1}{\beta} + 1 - \beta c^* f_{11}^* = \frac{1}{\beta} + 1 + \beta f_1^* \left(\frac{f_1^*}{\pi^*} - \delta\right)\left(\frac{1-\pi^*}{\sigma}\right).$$

But we know that the steady-state marginal product of capital is given by $f_1^* = 1/\beta - 1 + \delta$ and is independent of σ in the Ramsey model. So the effects of σ on the rate of convergence work only through the direct channel and through the relative scarcity channel via π^*. Thus, we get (35)

$$\frac{d\gamma_R}{d\sigma} = -\eta_1\left(\eta_2 + \eta_3\frac{d\pi^*}{d\sigma}\right),$$

where the direct effect of σ is captured by the constant term $-\eta_1\eta_2 < 0$. On the other hand, the indirect effect that works through relative factor scarcity is captured by the term $-\eta_1\eta_3(d\pi^*/d\sigma)$. If capital is relatively scarce in the steady state, so that $k^* < \bar{k}$, then an improvement in factor substitution allows substitution of labor for capital and the labor share rises. Thus we have $d\pi^*/d\sigma < 0$ and the two effects work in opposite directions. On the other hand, when $k^* \geq \bar{k}$, the two effects go together so that a higher σ reduces the rate of convergence in the Ramsey model.

In the Diamond model with a constant savings rate, we rewrite (37) as

$$\gamma_D = 1 + \frac{\beta}{1+\beta}k^* f_{11}(k^*) = 1 - \frac{\beta}{1+\beta}\frac{w^*}{k^*}\frac{\pi^*}{\sigma}.$$

From (36), we have $w^*/k^* = (1+\beta)/\beta$ so that the wage–capital ratio is independent of σ in the Diamond model. Thus again the effects of σ on the rate of

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convergence work only through the direct channel and through the relative scarcity channel via π^*. We then recall (38) to get

$$\frac{d\gamma_D}{d\sigma} = -\frac{1}{\sigma}\left(\frac{d\pi^*}{d\sigma} - \frac{\pi^*}{\sigma}\right).$$

In this case, the direct effect of σ is given by π^*/σ^2 and is positive. If capital is relatively scarce in the steady state so that $k^* < \bar{k}$, then an improvement in factor substitution allows substitution of labor for capital and the labor share rises. Thus we have $d\pi^*/d\sigma < 0$ and the two effects work in the same direction. A higher σ raises the rate of convergence in the Diamond model. On the other hand, when $k^* > \bar{k}$, the two effects work against each other so that a higher σ has an ambiguous effect on the rate of convergence.

We can summarize these findings in the following expression:

$$\frac{d\gamma_i}{d\sigma} = \Gamma_i + \Psi_i \frac{d\pi_i^*}{d\sigma},$$

where $i = S, R, D$ denotes the different growth models under consideration. Specifically, we have

(1) the Solow model:

$$\Gamma_S = 0, \Psi_S = -\delta < 0, d\pi_S^*/d\sigma = (\pi_S^*/\sigma^2)\ln[(\bar{y}/\bar{k})/(y_S^*/k_S^*)] \gtrless 0 \text{ as } k_S^* \gtrless \bar{k},$$

(2) the Ramsey model:

$$\Gamma_R = -\eta_1\eta_2 < 0, \Psi_R = -\eta_1\eta_3 < 0,$$
$$d\pi_R^*/d\sigma = (\pi_R^*/\sigma)\ln[(\bar{y}/\bar{k})/(y_R^*/k_R^*)] \gtrless 0 \text{ as } k_R^* \gtrless \bar{k}, \quad \text{and}$$

(3) the Diamond model:

$$\Gamma_D = \pi_D^*/\sigma^2 > 0, \Psi_D = -1/\sigma < 0,$$
$$d\pi_D^*/d\sigma = [(1-\pi_D^*)/(\sigma - \pi_D^*)](\pi_D^*/\sigma)\ln[(\bar{y}/\bar{k})/(y_D^*/k_D^*)] \gtrless 0 \text{ as } k_D^* \gtrless \bar{k}.$$

The following proposition then concludes our findings on rate of convergence:

PROPOSITION 2. *In all three one-sector models, there is a relative factor scarcity effect of factor substitution on the rate of convergence that works through the capital income share π. It depends on the relative magnitude between the steady-state per capita capital (k^*) and its normalized level ($\bar{k}$). In the optimizing one-sector growth models of Ramsey and Diamond, there is another direct effect of σ on the rate of convergence due to the fact that savings come from only one type of factor incomes. The net outcome then depends on the interaction of these effects in the steady state.*

To close the section, we summarize the effects of σ on the rate of convergence in different one-sector growth models as follows:

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TABLE 2. Summary of the transitional dynamics

Model	$k^* < \overline{k}$	$k^* > \overline{k}$	$k^* = \overline{k}$ (direct effect)
Solow	+	−	0
Ramsey	?	−	−
Diamond (log U)	+	?	+

5. A PARAMETERIZED EXAMPLE

Following the two-step calibration of Klump and Saam (2008), we first choose the values for the economically relevant point, k_i, y_i, π_i, and ψ_i.[8] Taking the Cobb–Douglas case with $\psi_i = 0$ (or $\sigma_i = 1$) as the economically relevant point, we obtain the parameters for the initial CES production function as

$$A_i = y_i \lim_{\psi_i \to 0} e^{\frac{1}{\psi_i} \ln\left[\pi_i k_i^{-\psi_i} + (1-\pi_i)\right]} = y_i k_i^{-\pi_i},$$

$$\alpha_i = \pi_i,$$

and

$$A_i = y_i k_i^{-\alpha_i}.$$

On the basis of the literature, we choose $\alpha_i = 1/3$ and $k_i = 10$ for numerical convenience. For y_i, as we like to have it as the initial steady-state level of output, it has to be determined below by the steady-state conditions of different models.

Next, we choose a baseline point $\overline{k}$, and the corresponding values of per capita output $\overline{y}$ and capital share $\overline{\pi}$ are

$$\overline{y} = A_i \overline{k}^{\alpha_i},$$

$$\overline{\pi} = \alpha_i.$$

The parameters for the CES production function associated with a new σ_j are

$$A_j = \overline{y}[\overline{\pi}\overline{k}^{-\psi_j} + (1 - \overline{\pi})]^{\frac{1}{\psi_j}}$$

and

$$\alpha_j = \frac{\overline{\pi}\overline{k}^{-\psi_j}}{\overline{\pi}\overline{k}^{-\psi_j} + (1 - \overline{\pi})}.$$

The normalized CES function and its marginal product are given as follows:

$$y_j = A_j[\alpha_j k^{\psi_j} + (1 - \alpha)]^{\frac{1}{\psi_j}}$$

and

$$f_1 = A_j[\alpha_j k_j^{\psi_j} + (1 - \alpha_j)]^{\frac{1}{\psi_j}-1} \alpha_j k_j^{\psi_j - 1}.$$

We summarize our choice of the parameterization in Table 3.

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TABLE 3. Parameter values

	Value
Parameters for all models	
Discount factor (β)	0.99
Depreciation rate of capital (δ)	0.04
Values for the economically relevant point	
Per capita capital (k_i)	10
Capital share (π_i)	$\frac{1}{3}$
Elasticity of substitution (σ_i)	1

5.1. The Steady-State Effects

For the Solow model, we choose $s = 0.25$. For $\sigma_i = 1$, we know that

$$y_i = \frac{\delta}{s} k_i,$$

and then we solve for

$$\pi^* = \left(\frac{\delta}{s} \frac{\overline{k}}{\overline{y}} \right)^{\frac{1-\sigma_j}{\sigma_j}} \overline{\pi}$$

and

$$k^* = \left(\frac{1-\overline{\pi}}{\overline{\pi}} \frac{\pi^*}{1-\pi^*} \right)^{\frac{1}{\psi_j}} \overline{k}.$$

The steady-state effects of σ on k^* is depicted in Figure 5. It is obvious that the effects are always positive regardless of the relative magnitude between the steady-state and normalized levels of per capita capital.

In the Ramsey model, we assume the intertemporal elasticity of substitution in consumption to be unity, i.e., $\theta = 1$. Because k_i and y_i are the steady-state values for $\sigma_i = 1$, we know that

$$r_i = \alpha_i \frac{y_i}{k_i} = \frac{1}{\beta} - (1-\delta),$$

where we could solve for $y_i = [1/\beta - (1-\delta)]k_i/\alpha_i$. We can calculate the steady-state per capita capital and capital income share for $\sigma_j \neq \sigma_i$ as

$$\pi^* = \left\{ \left[\frac{1}{\beta} - (1-\delta) \right] \frac{\overline{k}}{\overline{y}} \right\}^{1-\sigma_j} \overline{\pi}^{\sigma_j},$$

$$k^* = \left[\frac{1}{1-\alpha_j} \left(\frac{\frac{1}{\beta} - (1-\delta)}{\alpha_j A_j} \right)^{\frac{\psi_j}{1-\psi_j}} - \frac{\alpha_j}{1-\alpha_j} \right]^{-\frac{1}{\psi_j}}.$$

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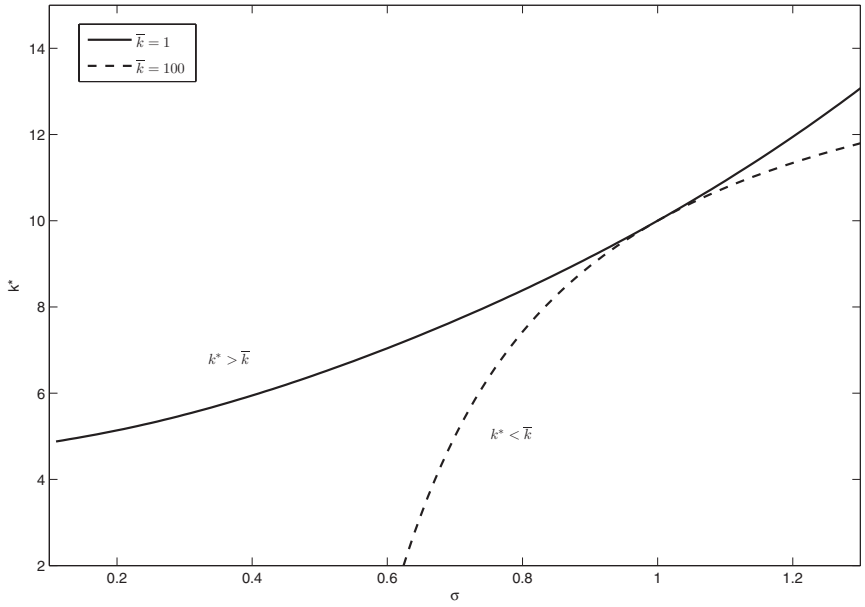


FIGURE 5. Relation between k^* and σ in the Solow model.

Figure 6 shows the steady-state effects of σ on k^*. As explained in the previous section, when $k^* > \overline{k}$, the efficiency effect and the distribution effect work in the same direction, so we have an overall positive steady-state effect of σ on k^*. However, when $k^* < \overline{k}$, the efficiency effect and the distribution effect work in opposite directions, so that the overall steady-state effect of σ on k^* becomes ambiguous.

Finally, in the Diamond model, where the saving rate is constant, we set $\pi_i = \pi^* = 1/3$ to get

$$y_i = \frac{k_i\,(1+\beta)}{\beta\,(1-\pi_i)}.$$

We can solve for k^* and π^* from the equations

$$G(k^*, \sigma_j) = \frac{\beta}{1+\beta}(1-\alpha_j)A_j\left[\alpha_j (k^*)^{\psi_j} + (1-\alpha_j)\right]^{\frac{1}{\psi_j}-1} k^* - 1 = 0$$

and

$$\left(\frac{\overline{\pi}}{\pi^*}\right)^{\frac{\sigma_j}{\sigma_j - 1}} (1-\pi^*) = \frac{1+\beta}{\beta}\frac{\overline{k}}{\overline{y}}.$$

Figure 7 depicts the steady-state effects of σ on k^*. In the Diamond model with logarithmic preferences, when $k^* < \overline{k}$, the efficiency effect and the distribution effect work in the same direction so that we have an overall positive steady-state effect of σ on k^*. However, when $k^* > \overline{k}$, the efficiency effect and the distribution

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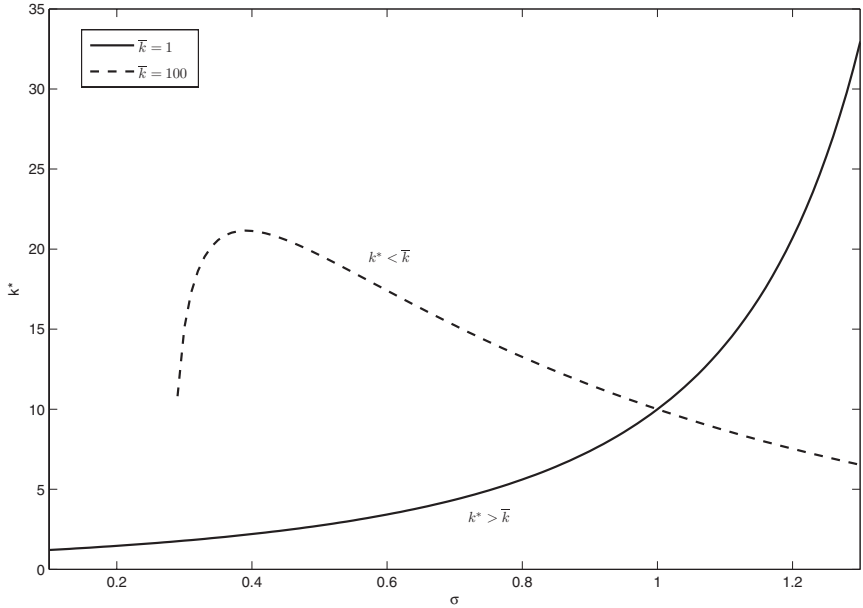


FIGURE 6. Relation between k^* and σ in the Ramsey model.

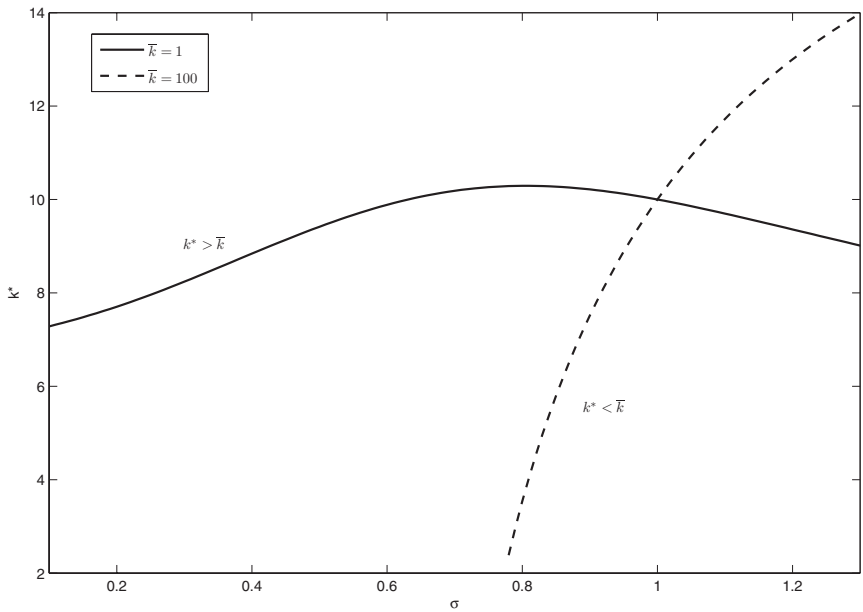


FIGURE 7. Relation between k^* and σ in the Diamond model.

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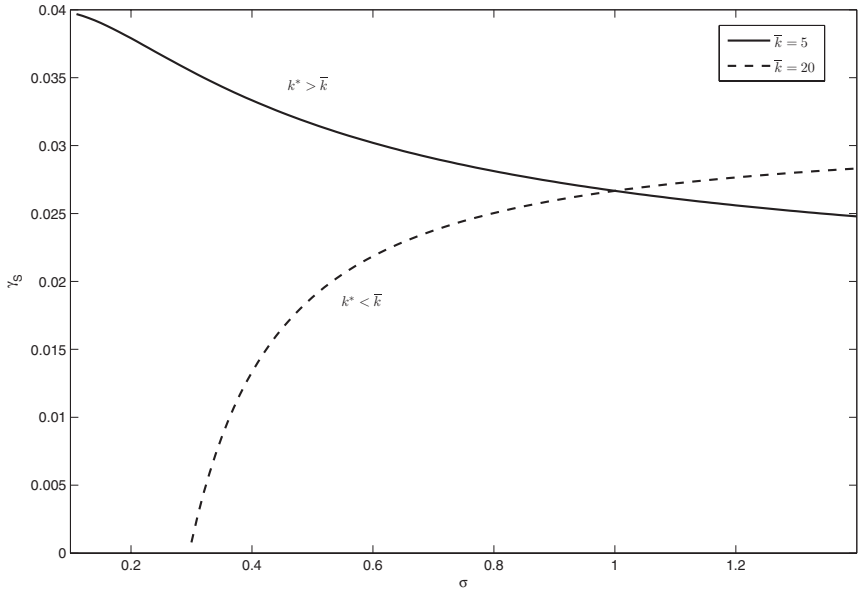


FIGURE 8. Rate of convergence in the Solow model.

effect work against each other, so the overall steady-state effect of σ on k^* becomes ambiguous.

5.2. The Rates of Convergence

To study the effects of factor substitution on the rates of convergence, we recall their expressions in different growth models as follows. In the Solow model, we have

$$\gamma_S = \delta(1 - \pi^*).$$

So the effects of σ on γ_S work through the capital income share π^*. They are depicted in Figure 8 and depend crucially on the relative magnitude between the steady-state and normalized levels of per capita capital. Specifically, when $k^* < \overline{k}$, the effect is positive. When $k^* > \overline{k}$, we have a negative effect.

In the Ramsey model, we have

$$\gamma_R = 1 - \frac{T_R}{2} + \sqrt{\frac{T_R^2}{4} - \frac{1}{\beta}},$$

where

$$T_R = \frac{1}{\beta} + 1 + \frac{[1 - \beta(1 - \delta)]^2}{\beta}\left[\frac{1}{\pi^*} - \frac{\beta\delta}{1 - \beta(1 - \delta)}\right]\frac{(1 - \pi^*)}{\sigma}.$$

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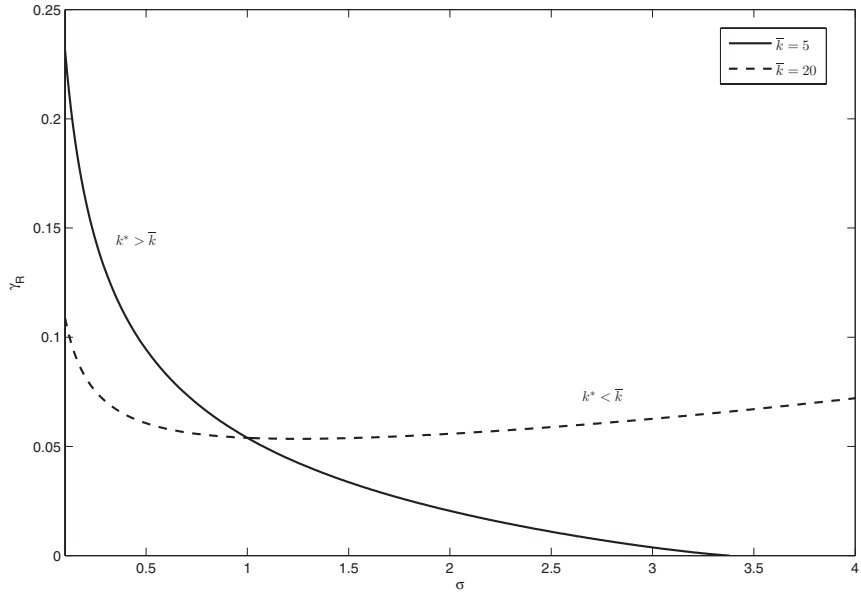


FIGURE 9. Rate of convergence in the Ramsey model.

As explained in the preceding section, now we have two effects at work: a direct effect and the profit-share effect. It is discussed that the direct effect on the rate of convergence is negative in the Ramsey model. The net outcome of these effects again depends on the relative magnitude between the steady-state and normalized levels of per capita capital. Specifically, when $k^* > \overline{k}$, the two effects work in the same direction so that the overall effect is negative. When $k^* < \overline{k}$, the two effects work against each other and the net outcome is ambiguous. Figure 9 is a graphical representation of these cases.

Finally, in the Diamond model with constant saving rates, the rate of convergence is given by

$$\gamma_D = 1 - \frac{\pi^*}{\sigma}.$$

Again the direct effect and the profit-share effect are at work in determining the net effect of factor substitution on the rate of convergence. As explained before, the direct effect is positive in the Diamond model. Figure 10 shows the outcomes of the cases. When $k^* < \overline{k}$, the two effects work in the same direction, so that the overall effect is positive. When $k^* > \overline{k}$, the two effects work against each other and the net outcome becomes ambiguous.

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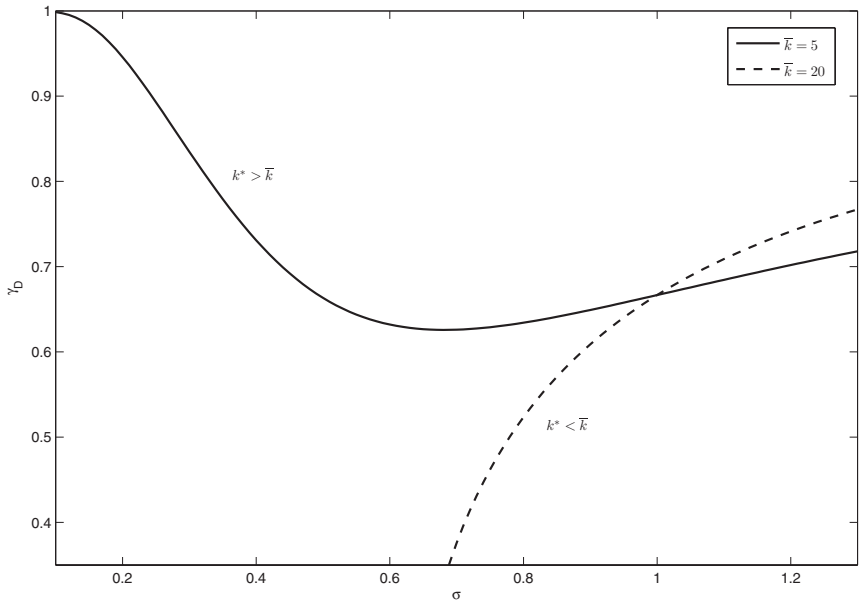


FIGURE 10. Rate of convergence in the Diamond model.

6. CONCLUSION

This paper has examined the effects of factor substitution on economic growth for different one-sector models of economic growth, including the Solow model, the Ramsey–Cass–Koopman model, and the Diamond overlapping-generations model. Our main finding is that, if an improvement in factor substitution raises the amount of savings in the steady state, then it leads to a higher equilibrium level of per capita income. The effects of factor substitution on savings can be classified as the efficiency and distribution effects. Specifically, the efficiency effect of factor substitution on savings captures the direct impact via income, which is always positive, whereas the distribution effect identifies the effects on savings via factor income shares which can go in either direction. It is shown that if the efficiency effect dominates, then a higher elasticity of substitution leads to a higher level of per capita steady-state income. In the Solow model, all incomes (both labor and capital incomes) are saved so that the distribution effect is absent in equilibrium. In the Ramsey and Diamond models, only one type of factor incomes is saved so that there is a distribution effect at work when factor substitution adjusts. Then the choice of the normalized level of per-capita capital is no longer innocuous, because it determines the direction of the distribution effect. Specifically, in the Ramsey model, only capital income is saved. If a higher elasticity of substitution reduces the capital income share, the distribution effect on savings is negative and works against the efficiency effect. Likewise, in the Diamond model, only wage

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income is saved. If a higher elasticity of substitution reduces the wage income share, the distribution effect on savings again is negative and offsets the efficiency effect. Because the distribution effect depends on the relative magnitude between the steady-state and normalized levels of per capita capital, this explains why we obtain contrary conclusions in the literature when different growth models are adopted in the analysis.

The above steady-state findings of factor substitution concerning the interaction of the efficiency and distribution effects are derived within growth models with constant savings rates. In the general setting of the Diamond overlapping-generations model where the savings rate is endogenous, factor substitution affects savings also through the saving-rate channel. In this case, we have found that the efficiency effects are consistent with those from models of constant saving rates. An improvement of factor substitution raises both the saving rate and income, so that savings must rise. If the efficiency effect dominates, then a higher elasticity of substitution still leads to a higher level of per capita income in the steady state. On the other hand, if the distribution effects dominate, then the overall effect of factor substitution on the steady-state level of income becomes ambiguous. This is because the distribution effects of factor substitution on the saving rate and income (the saving base) are in opposite directions. This is the complication that arises from the endogenous rate of savings.

Finally, in transition, we have examined the effects of substitution on the rate of convergence. Among all three models of economic growth with constant savings rate, the equilibrium capital income share is the channel through which factor substitution affects the rate of convergence. This channel reflects the relative scarcity of capital and its direction depends on the relative magnitude between the equilibrium and normalized levels of capital. When only a share of total income is saved in the optimizing growth models, we have an additional direct effect of factor substitution on the rate of convergence. This direct effect arises due to the fact that the steady-state level of capital depends on its marginal product. It captures the positive impact of the elasticity of substitution on the diminishing marginal product of capital. Thus, depending on the interaction of the direct effect and the relative scarcity effect, the net outcomes are characterized according to the choice of the normalized level of per capita capital.

NOTES

1. We note that the assumption of a fixed savings rate is preserved in the Diamond model of Miyagiwa and Papageorgiou (2003).

2. Numerical subscripts are used to denote partial derivatives; i.e., $f_i(.)$ is the partial derivative of f with respect to its ith argument.

3. Without loss of generality, the size of the labor force is assumed to be constant over time, so that we abstract from population growth.

4. The steady-state analysis of the general Diamond model with variable saving rates is conducted in the Appendix.

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5. We describe the Solow model and the Ramsey model in a discrete-time framework to facilitate comparison with the overlapping-generation model.

6. Guo and Lansing (2009) have adopted this special case of normalization in their analysis.

7. Specifically, in the Ramsey model, the steady-state level of per capita capital is given by $f_1^* = 1/\beta - 1 + \delta$. In the Diamond model with a constant savings rate, k^* is given by $f^*/k^* - f_1^* = (1 + \beta)/\beta$. But in the Solow model, the steady state is determined by $sf^* = \delta k^*$, so that f_1^* is not involved.

8. The subscript i indicates the "initial" choice of the economically relevant point.

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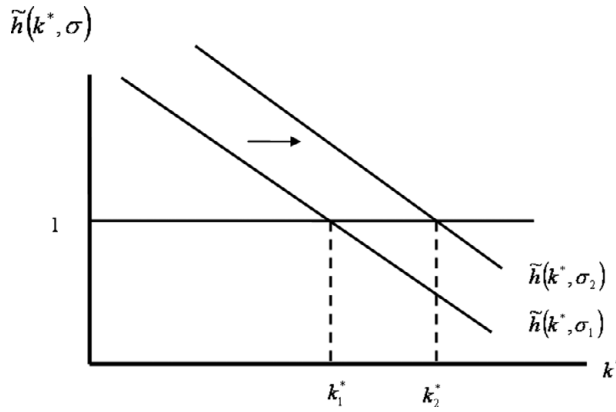


FIGURE A.1. The influence of σ on k^* in the general Diamond model when the efficiency effect dominates (where $\sigma_1 > \sigma_2$).

APPENDIX: THE GENERAL DIAMOND MODEL

In the general case of the Diamond (1965) model, the law of motion for per capita capital is

$$k_{t+1} = s\,(r_{t+1})\,w_t = s\,(r_{t+1})\,(1-\pi_t)\,f\,(k_t,\sigma)\,.$$

In the steady state, we have

$$k^* = s\,(r^*)\,[1-\pi^*\,(k^*,\sigma)]f\,(k^*,\sigma)$$

or

$$\tilde{h}\,(k^*,\sigma) \equiv s\,(f_1\,(k^*,\sigma))\,h\,(k^*,\sigma) = 1, \tag{A.1}$$

because $r^* = f_1(k^*,\sigma)$. Comparing the steady-state equation (A.1) with those from the growth models of constant saving rates [e.g., (13), (16), and (18)], we obtain an extra effect from changing σ through the saving rate $s[f_1(k^*,\sigma)]$. According to Galor and Ryder (1988), $s' \equiv \partial s/\partial r \geq 0$ is a necessary condition for the uniqueness of a nontrivial steady-state equilibrium. This uniqueness condition, together with the local stability condition (20), implies that the LHS of (A.1) is downward-sloping in k:

$$\tilde{h}_1 = \frac{\partial \tilde{h}}{\partial k^*} = h\,(k^*,\sigma)\,f_{11}\,(k^*,\sigma)\,s'\,(r^*) - \frac{(1-\pi^*)\,f\,(k^*,\sigma)\,s\,(r^*)}{(k^*)^2}\left(1-\frac{\pi^*}{\sigma}\right) < 0,$$

where $s' \geq 0$ and $\sigma > \pi^*$. This is illustrated in Figure A.1.

Next, we are going to examine the relation between $\tilde{h}$ and σ. Direct differentiation of (A.1) yields

$$\tilde{h}_2 = s\,(r^*)\,h_2\,(k^*,\sigma) + h\,(k^*,\sigma)\,s'\,(r^*)\,f_{12}\,(k^*,\sigma). \tag{A.2}$$

There are two effects of σ on savings $\tilde{h}$ according to (A.2): one works through the saving base as before (the first term on the RHS) and a new one works through the saving rate (the

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second term on the RHS). If we take a closer look into each of these saving-related terms, we can further separate them into two effects, an efficiency effect versus a distribution effect:

$$\tilde{h}_2 = \frac{h\,(k^*, \sigma)}{(\sigma - 1)^2}\{(\Phi_\sigma - \pi^* \Phi_k) + r^* s'\,(r^*)\,[\Phi_\sigma + (1 - \pi^*)\,\Phi_k]\}. \tag{A.3}$$

We recall that $\Phi_\sigma = -[\pi^* \ln(\frac{\overline{\pi}}{\pi^*}) + (1 - \pi^*) \ln(\frac{1-\overline{\pi}}{1-\pi^*})] > 0$ represents the efficiency effect on savings through income, whereas $\Phi_k = \psi^2 \ln(k^*/\bar{k})$ denotes the distribution effect on saving via factor shares. Thus, the effect of factor substitution on the steady-state per capita capital is given by

$$\frac{dk^*}{d\sigma} = -\frac{\tilde{h}_2\,(k^*, \sigma)}{\tilde{h}_1\,(k^*, \sigma)} \text{ or } \operatorname{sgn}\left(\frac{dk^*}{d\sigma}\right) = \operatorname{sgn}\left(\tilde{h}_2\right).$$

According to (A.2), the efficiency effects on savings work in the same direction. As before, an improvement of factor substitution given by an increase in σ raises income (f/k). For a given saving rate, more savings are obtained, as the saving base has increased. Given π, an increase in f/k leads to a higher r, so that the saving rate also increases, given the uniqueness condition that $s' \geq 0$. As a result, savings must rise when σ increases, from the perspective of the efficiency effect. For the distribution effects, suppose we have $k^* < \bar{k}$, so that an increase in σ reduces the capital share π. This then increases labor income so that more savings are obtained in the Diamond model. This is the distribution effect that works through the saving base. However, as π decreases, r is reduced for a given level of income (f/k). Given $s' \geq 0$, this leads to a lower saving rate, so that we get a negative distribution effect on savings via the saving rate. The net distribution effect on savings is therefore ambiguous.

In summary, when the saving rate is endogenous in the general Diamond model, the efficiency effects are consistent with those from models of constant saving rates. An improvement of factor substitution raises both the saving rate and income, so that savings must rise. If the efficiency effect dominates, then a higher elasticity of substitution leads to a higher level of per capita income in the steady state (see Figure A.1). On the other hand, if the distribution effects dominate, then the overall effect of σ on the steady-state level of income is ambiguous. This is because the distribution effects of σ on the saving rate and income (the saving base) are of opposite directions. This is the complication that arises from the general Diamond model.

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