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A counterexample to the composition condition conjecture for polynomial Abel differential equations

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Abstract. Polynomial Abel differential equations are considered a model problem for the classical Poincaré center–focus problem for planar polynomial systems of ordinary differential equations. In the last few decades, several works pointed out that all centers of the polynomial Abel differential equations satisfied the composition conditions (also called universal centers). In this work we provide a simple counterexample to this conjecture.

1. *Introduction*

In the last few decades, some authors consider polynomial Abel differential equations as a model to tackle the center problem for a trigonometric Abel differential equation coming from a planar polynomial system of ordinary differential equations, see [**6–8**]. We denote as a *polynomial Abel differential equation* an ordinary differential equation of the form

$$\frac{dy}{dx} = p(x)y^2 + q(x)y^3, \tag{1}$$

where y is real, x is a real independent variable considered in a real interval $[a, b]$ and $p(x)$ and $q(x)$ are real polynomials in $\mathbb{R}[x]$. We fix two real values $a < b$ and the *center problem for a polynomial Abel differential equation* (1) is to characterize when all the solutions in a neighborhood of the solution $y = 0$ take the same value when $x = a$ and $x = b$, i.e. $y(a) = y(b)$. In this case, we say that the differential equation (1) has a *center* in $[a, b]$. In this framework, given any real continuous function $c(x)$, we denote by $\tilde{c}(x) := \int_a^x c(\sigma)\, d\sigma$ and we will say that a real continuous function $w(x)$ is *periodic in* $[a, b]$ if $w(a) = w(b)$.

Alwash and Lloyd in [**4**] provided a sufficient condition for an Abel trigonometric equation

$$\frac{d\rho}{d\theta} = a_1(\theta)\rho^2 + a_2(\theta)\rho^3, \tag{2}$$

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where ρ is real, θ is a real and periodic independent variable with $\theta \in [0, 2\pi]$, and $a_1(\theta)$ and $a_2(\theta)$ are real trigonometric polynomials, to have a center in $[0, 2\pi]$. We recall that the center problem for equation (2) is to characterize when all the solutions in a neighborhood of the solution $\rho = 0$ are periodic of period 2π. Inspired by this work, Briskin, Françoise and Yomdin in [**6**] provided the following sufficient condition for the polynomial Abel differential equation (1).

THEOREM 1. [**6**] *If there exists a real differentiable function w periodic in $[a, b]$ and such that*

$$\tilde{p}(x) = p_1(w(x)) \quad \text{and} \quad \tilde{q}(x) = q_1(w(x))$$

for some real differentiable functions p_1 and q_1, then the polynomial Abel differential equation (1) *has a center in $[a, b]$.*

In [**15**] it is shown that if the sufficient condition stated in Theorem 1 is satisfied then there is a countable set of definite integrals which need to vanish. In [**15**] it is also shown that this is equivalent to the existence of a real polynomial $w(x)$ with $w(a) = w(b)$ and two real polynomials $p_1(x)$ and $q_1(x)$ such that $\tilde{p}(x) = p_1(w(x))$ and $\tilde{q}(x) = q_1(w(x))$. This sufficient condition is known as the *composition condition*.

To see that the composition condition implies that equation (1) has a center in $[a, b]$ one can consider the transformation $y(x) = Y(w(x))$ in equation (1) in order to obtain the following Abel differential equation

$$\frac{dY}{dw} = p_1'(w)Y^2 + q_1'(w)Y^3. \tag{3}$$

Hence, there is a bijection between the solutions $Y = Y(w)$ of equation (3) and the solutions $y = Y(w(x))$ of equation (1). Since w is periodic in $[a, b]$, we get that equation (1) has a center in $[a, b]$ because $y(a) = Y(w(a)) = Y(w(b)) = y(b)$.

It turns out that, until the present note, all the known polynomial Abel differential equations which have a center in $[a, b]$ satisfied the composition condition. Hence, in several works what is known as the *composition conjecture* was established, see [**2**, **19**] and references therein. This conjecture says that the sufficient condition given in Theorem 1 is also necessary. That is, if a polynomial Abel differential equation (1) has a center in $[a, b]$, the conjecture states that the composition condition is satisfied, that is there exists a real polynomial $w(x)$ with $w(a) = w(b)$ and two real polynomials $p_1(x)$ and $q_1(x)$ such that $\tilde{p}(x) = p_1(w(x))$ and $\tilde{q}(x) = q_1(w(x))$. In fact in [**19**] this conjecture was proved for lower degrees of the polynomials $p(x)$ and $q(x)$ of equation (1). Moreover, this conjecture is satisfied under certain restrictions on the coefficients of the polynomial Abel differential equation, see for instance [**9**], [**3**, Theorem 2], [**5**, Theorem 2] and [**19**, Theorem 7].

For a trigonometric Abel differential equation (2), Alwash in [**1**] showed that this conjecture is not true, see also [**3**, **13**, **14**, **17**, **18**]. The *composition condition* for a trigonometric Abel differential equation (2) is that there exist real polynomials $p_1(x)$, $p_2(x) \in \mathbb{R}[x]$ and a trigonometric polynomial $\omega(\theta)$ such that $\tilde{a}_i(\theta) = p_i(\omega(\theta))$, for $i = 1, 2$. Recall that $\tilde{a}_i(\theta) := \int_0^\theta a_i(s)\, ds$. The fact that $\omega(\theta)$ and p_1, p_2 can be taken to be polynomials is proved in [**15**, **16**]. There exist several counterexamples of the fact that the composition conjecture is not satisfied in the trigonometric case. The authors of

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[**1**, **3**, **13**] provide examples of trigonometric polynomials $a_1(\theta)$ and $a_2(\theta)$ for which the corresponding trigonometric Abel differential equation (2) has a center and does not satisfy the composition condition. A survey of the last results for polynomial and trigonometric Abel differential equations is given in [**19**].

The main result of this note is the following.

THEOREM 2. *The polynomial Abel differential equation* (1) *with*

$$p(x) = 40x^4 - 30x^2 + 2, \quad q(x) = 75x^9 - 150x^7 + 88x^5 - 10x^3 - 3x,$$

has a center in $[-1, 1]$ *and does not satisfy the composition condition.*

Remark that the polynomials $p(x)$ and $q(x)$ satisfy $p(-1) = p(1)$ and $q(-1) = q(1)$, that is, they are periodic in $[-1, 1]$.

In the following section we prove the main result of this note.

2. *Proof of Theorem 2*

We first provide a couple of well-known definitions. An *invariant algebraic curve* of an Abel differential equation (1) is an algebraic curve $f = 0$, where f is a real non-constant polynomial, such that there exists a real polynomial $k(x, y)$, called the cofactor, satisfying

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}(p(x)y^2 + q(x)y^3) = k(x, y)\, f(x, y).$$

Note that, in this case, if $p \in \mathbb{R}^2$ is such that $f(p) = 0$, then the orbit of equation (1) starting at p is contained in the curve $f = 0$.

A *first integral* of the Abel differential equation (1) is a non-locally constant function H of class $\mathcal{C}^1$ defined in an open set $\mathcal{U}$ such that

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y}(p(x)y^2 + q(x)y^3) = 0$$

in $\mathcal{U}$. Note that the level curves $H = c$ with c constant, contain the orbits of the Abel differential equation (1). The knowledge of an explicit first integral $H(x, y)$ of the differential equation (1) may allow it to be proven that it has a center in $[a, b]$ provided that the equation $H(a, y_a) = H(b, y_b)$ implies that $y_a = y_b$. The Darboux theory of integrability provides a technique to find a first integral given one or several invariant algebraic curves of the differential equation.

System (1), with $p(x)$ and $q(x)$ given by the statement of the theorem, has the invariant algebraic curves

$$\begin{aligned} f_1 &= (2 + 8xy - 24x^3y + 16x^5y + y^2 + 2x^2y^2 - 34x^4y^2 \\ &\quad + 88x^6y^2 - 87x^8y^2 + 30x^{10}y^2)/2, \\ f_2 &= (3 + 12xy - 42x^3y + 30x^5y + 2y^2 + 3x^2y^2 - 72x^4y^2 \\ &\quad + 202x^6y^2 - 210x^8y^2 + 75x^{10}y^2)/3, \\ f_3 &= 1 + 3xy - 8x^3y + 5x^5y, \end{aligned}$$

and the rational first integral $H(x, y) = y^2 f_1^3/f_2^4$. This first integral satisfies that $H(1, y) = H(-1, y) = y^2$ consequently this Abel trigonometric equation has a center in

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$[-1, 1]$. This is due to the fact that any solution in a neighborhood of the straight line $y = 0$ is continuous and differentiable in $[-1, 1]$. We also know that none of these solutions can cross the straight line $y = 0$. Therefore, the equation $H(1, y_1) = H(-1, y_{-1})$ which reads for $y_1^2 = y_{-1}^2$ implies that $y_1 = y_{-1}$. Besides, attending to the obtained first integral, system (1) with $p(x)$ and $q(x)$ given by the statement of the theorem admits a type of first integral studied in [**20**] for Abel differential equations. In particular, it corresponds to a case with five solutions, that is, $n = 5$.

If an Abel differential equation satisfies the composition condition, that is there exists a real polynomial $w(x)$ with $w(a) = w(b)$ and two real polynomials $p_1(x)$ and $q_1(x)$ such that $\tilde{p}(x) = p_1(w(x))$ and $\tilde{q}(x) = q_1(w(x))$, then we have that

$$p(x) = p_1'(w(x))w'(x) \quad \text{and} \quad q(x) = q_1'(w(x))w'(x).$$

It is clear that $w(x)$ is a polynomial of degree at least 2 because $w(a) = w(b)$ with $a < b$ and that $w'(x)$ has at least one real root in the interval (a, b). We deduce that, if the composition condition is satisfied, then p and q have at least one real root in common in the interval (a, b). In order to prove that this Abel differential equation given in Theorem 1 does not satisfy the composition condition, we only check whether p and q have a real root in common. The resultant of this two polynomials is 7080837120000 and, thus, different from zero. Hence, the Abel differential equation (1) does not satisfy the composition condition.

Another way to prove that the Abel differential equation given in Theorem 1 does not satisfy the composition condition is to consider the integrals of $p(x)$ and $q(x)$, since one is of degree 5 and the other of degree 10, then the only common factors must be of degree 1 or degree 5. Both cases are easily dismissed.

We can also prove that this Abel differential equation does not satisfy the composition condition by using the equivalence between composition condition and the existence of a universal center, see [**16**].

We consider a more general framework. Consider an ordinary differential equation

$$\frac{dy}{dx} = \sum_{i \geq 1} a_i(x) y^{i+1}, \tag{4}$$

where the right-hand side of the equation is an analytic real function in a neighborhood of the straight line $y = 0$ and the solution of the equation is considered on a fixed real interval $[a, b]$. The definition of the center problem and the composition condition are analogous to the particular case of an Abel differential equation. An explicit expression for the first return map of a differential equation (4) was given in [**10**], see also [**12**]. This expression is given in terms of the following iterated integrals, of order k,

$$I_{i_1,\ldots,i_k} := \int \cdots \int_{a \leq s_1 \leq \cdots \leq s_k \leq b} a_{i_k}(s_k) \cdots a_{i_1}(s_1)\, ds_k \cdots ds_1,$$

where, by convention, for $k = 0$ we assume that this equals 1. Let $y(x;\, y_0)$, $x \in [a, b]$, be the Lipschitz solution of the differential equation (4) with initial value $y(a;\, y_0) = y_0$. Then $P(y_0) := y(b;\, y_0)$ is the first return map of this differential equation, and in [**10**, **12**] the following theorem is proved.

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THEOREM 3. *For sufficiently small initial values y_0 the first return map $P(y_0)$ is an absolute convergent power series $P(y_0) = y_0 + \sum_{n=1}^{\infty} c_n y_0^{n+1}$, where*

$$c_n = \sum_{i_1+\cdots+i_k=n} c_{i_1,\ldots,i_k} I_{i_1,\ldots,i_k},$$

and

$$c_{i_1,\ldots,i_k} = (n - i_1 + 1) \cdot (n - i_1 - i_2 + 1) \cdot (n - i_1 - i_2 - i_3 + 1) \cdots 1.$$

The following definition is given in [**11**]. Equation (4) determines a *universal center* if for all positive integers $i_1, \ldots, i_k$ with $k \geq 1$ the iterated integral $I_{i_1,\ldots,i_k} = 0$. Moreover, in [**16**] it was proved that equation (4) has a universal center if and only if the composition condition is satisfied.

System (1) with $p(x)$ and $q(x)$ given by the statement of the theorem has the iterated integral

$$I_{122} = \int_{-1}^{1} \int_{-1}^{x_3} \int_{-1}^{x_2} p(x_1)q(x_2)q(x_3)\, dx_1\, dx_2\, dx_3 = -\frac{131072}{6235515}.$$

Hence, we have a non-universal center and this completes the proof.

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